From MELL Proof-Nets to Explicit Substitution Calculi

Agenda for Today

Agenda

Motivations

- Give a computational meaning to contraction and weakening.
- Understand the behaviour of contraction and weakening when implementing lambda-calculus.
- Study properties of the underlying calculus.

Agenda

The Untyped $\lambda\mathtt{lxr}$-calculus

$(Terms)$	$t, u ::=$	x	*variable*
	$\mid$	$\lambda x.t$	*abstraction*
	$\mid$	$t\ u$	*application*
	$\mid$	$t[x/u]$	*substitution*
	$\mid$	$W_x(t)$	*weakening*
	$\mid$	$C_x^{y,z}(t)$	*contraction*

Free Variables:

$$
\begin{array}{lclclcl}
\mathbf{fv}(x) & := & \{x\} & \qquad & \mathbf{fv}(\lambda x.t) & := & \mathbf{fv}(t) \setminus \{x\} \\
\mathbf{fv}(tu) & := & \mathbf{fv}(t) \cup \mathbf{fv}(u) & & \mathbf{fv}(t[x/u]) & := & (\mathbf{fv}(t) \setminus \{x\}) \cup \mathbf{fv}(u) \\
\mathbf{fv}(W_x(t)) & := & \mathbf{fv}(t) \cup \{x\} & & \mathbf{fv}(C_x^{y,z}(t)) & := & (\mathbf{fv}(t) \setminus \{y, z\}) \cup \{x\}
\end{array}
$$

We only consider *well-formed* terms:

- Linearity
- Compulsory presence
- Barendregt's convention

Notation Given $\Phi \subseteq \mathbf{fv}(t)$, $R_\Delta^\Phi(t)$ denotes the renaming of Φ by Δ. Example: $R_{y_1 y_2}^{x_1 x_2}(x_1 x_2 x_3) = y_1 y_2 x_3$.

Congruence I

AC of contraction:

$$
\begin{array}{lcll}
C_w^{x,v}(C_x^{z,y}(t)) & \equiv & C_w^{x,y}(C_x^{z,v}(t)) & \text{if } x \neq y, v \\
C_x^{y,z}(t) & \equiv & C_x^{z,y}(t) & \\
C_{x'}^{y',z'}(C_x^{y,z}(t)) & \equiv & C_x^{y,z}(C_{x'}^{y',z'}(t)) & \text{if } x \neq y', z' \ \& \ x' \neq y, z
\end{array}
$$

C of weakening:

$$W_x(W_y(t)) \equiv W_y(W_x(t))$$

Congruence II

Commutativity of substitutions:

$$t[x/u][y/v] \equiv t[y/v][x/u] \text{ if } y \notin \mathbf{fv}(u) \;\&\; x \notin \mathbf{fv}(v)$$

Contraction and substitution have the same status:

$$C_w^{y,z}(t)[x/u] \equiv C_w^{y,z}(t[x/u]) \text{ if } x \neq w \;\&\; y, z \notin \mathbf{fv}(u)$$

Reduction Rules for the $\lambda\mathtt{lxr}$-calculus

$$(B) \qquad (\lambda x.t)\ u \rightarrow \ t[x/u]$$

$$
\begin{array}{llcll}
(Abs) & (\lambda y.t)[x/u] & \to & \lambda y.t[x/u] & \\
(App1) & (t\ v)[x/u] & \to & t[x/v]\ v & \text{if } x \in \mathbf{fv}(t) \\
(App2) & (t\ v)[x/u] & \to & t\ v[x/u] & \text{if } x \in \mathbf{fv}(v) \\
(Var) & x[x/u] & \to & u & \\
(Weak1) & W_x(t)[x/u] & \to & W_{\mathbf{fv}(u)}(t) & \\
(Weak2) & W_y(t)[x/u] & \to & W_y(t[x/u]) & \text{if } x \neq y \\
(Cont1) & C_x^{y,z}(t)[x/u] & \to & C_\Phi^{\Delta,\Pi}(t[y/u_1][z/u_2]) & \\
 & & & \text{where } \Phi := \mathbf{fv}(u) & \\
 & & & u_1 = R_\Delta^\Phi(u) \text{ and } u_2 = R_\Pi^\Phi(u) & \\
(Comp) & t[y/v][x/u] & \to & t[y/v[x/u]] \text{ if } x \in \mathbf{fv}(v) &
\end{array}
$$

SubSystem t

$$
\begin{array}{lllll}
(WAbs) & \lambda x.W_y(t) & \rightarrow & W_y(\lambda x.t) & x \neq y \\
(WApp1) & W_y(u)\ v & \rightarrow & W_y(u\ v) & \\
(WApp2) & u\ W_y(v) & \rightarrow & W_y(u\ v) & \\
(WSubs) & t[x/W_y(u)] & \rightarrow & W_y(t[x/u]) & \\
(Merge) & C^{y,z}_w(W_y(t)) & \rightarrow & R^z_w(t) & \\
(Cross) & C^{y,z}_w(W_x(t)) & \rightarrow & W_x(C^{y,z}_w(t)) & x \neq y,\quad x \neq z \\
(CAbs) & C^{y,z}_w(\lambda x.t) & \rightarrow & \lambda x.C^{y,z}_w(t) & \\
(CApp1) & C^{y,z}_w(t\ u) & \rightarrow & C^{y,z}_w(t)\ u & y,z \in \mathbf{fv}(t) \\
(CApp2) & C^{y,z}_w(t\ u) & \rightarrow & t\ C^{y,z}_w(u) & y,z \in \mathbf{fv}(u) \\
(CSubs) & C^{y,z}_w(t[x/u]) & \rightarrow & t[x/C^{y,z}_w(u)] & y,z \in \mathbf{fv}(u)
\end{array}
$$

The reduction relation $\lambda\mathtt{lxr}$

The reduction relation is generated by the previous rewriting rules and congruence axioms:

$$t \rightarrow_{\lambda\mathtt{lxr}} t' \text{ iff } \exists t_1, t_2 \; t \equiv t_1 \rightarrow_{B+\mathtt{x}+\mathtt{t}} t_2 \equiv t'$$

Example

$$
\begin{array}{ll}
(\lambda x.W_u(C_x^{y,z}(y\ z)))\ w & \to \\
W_u(C_x^{y,z}(y\ z))[x/w] & \to \\
W_u(C_x^{y,z}(y\ z)[x/w]) & \to \\
W_u(C_w^{w_1,w_2}((y\ z)[y/w_1][z/w_2])) & \to \\
W_u(C_w^{w_1,w_2}((y[y/w_1]\ z)[z/w_2])) & \to \\
W_u(C_w^{w_1,w_2}(y[y/w_1]\ z[z/w_2])) & \to \\
W_u(C_w^{w_1,w_2}(w_1\ z[z/w_2])) & \to \\
W_u(C_w^{w_1,w_2}(w_1\ w_2)) &
\end{array}
$$

Agenda

Properties of the $\lambda\mathtt{lxr}$-calculus

1. (Full composition) $t[x/v] \to^* \ t\{x = v\}$
   for an *appropiate* notion of meta-substitution (acting on terms containing in principle non-evaluated substitutions)
2. (Preservation of free variables) If $t \to_{\lambda\mathtt{lxr}} \ t'$, then $\mathbf{fv}(t) = \mathbf{fv}(t')$
3. (Convergence) $\mathtt{xt} = \mathtt{x} \cup \mathtt{t}$ is convergent (terminating and confluent).
   Which is the form of a term in $\mathtt{xt}$-normal form?

Connexion with λ-calculus

$\mathcal{B}()$ hides resource control

$$\lambda\mathtt{lxr} \quad \overset{\mathcal{B}()}{\longrightarrow} \quad \lambda \quad \overset{\mathcal{A}()}{\longleftarrow}$$

$\mathcal{A}()$ introduces resource operators

From λlxr to λ

$$
\begin{array}{lcl}
\mathcal{B}(x) & = & x \\
\mathcal{B}(\lambda x.t) & = & \lambda x.\mathcal{B}(t) \\
\mathcal{B}(W_x(t)) & = & \mathcal{B}(t) \\
\mathcal{B}(C_x^{y,z}(t)) & = & \mathcal{B}(t)\{y\backslash x\}\{z\backslash x\} \\
\mathcal{B}(t\ u) & = & \mathcal{B}(t)\ \mathcal{B}(u) \\
\mathcal{B}(t[x/u]) & = & \mathcal{B}(t)\{x\backslash \mathcal{B}(u)\}
\end{array}
$$

Relating $\lambda\texttt{lxr}$ and λ

Lemma

1. *If* $M \equiv N$, *then* $\mathcal{B}(M) = \mathcal{B}(N)$.
2. *If* $M \rightarrow_{B} N$, *then* $\mathcal{B}(M) \rightarrow_{\beta}^{*} \mathcal{B}(N)$.
3. *If* $M \rightarrow_{\texttt{xt}} N$, *then* $\mathcal{B}(M) = \mathcal{B}(N)$.

Proposition [Projecting $\lambda\texttt{lxr}$-reductions] $M \rightarrow_{\lambda\texttt{lxr}} N$, then $\mathcal{B}(M) \rightarrow_{\beta}^{*} \mathcal{B}(N)$.

$$
\begin{array}{llll}
\mathcal{A}(x) & := & x & \\
\mathcal{A}(\lambda x.t) & := & \lambda x.\mathcal{A}(t) & \text{if } x \in \mathbf{fv}(t) \\
\mathcal{A}(\lambda x.t) & := & \lambda x.W_x(\mathcal{A}(t)) & \text{if } x \notin \mathbf{fv}(t) \\
\mathcal{A}(tu) & := & C_{\Phi}^{\Delta,\Pi}(R_{\Delta}^{\Phi}(\mathcal{A}(t))\ R_{\Pi}^{\Phi}(\mathcal{A}(u))) & \text{where } \Phi := \mathbf{fv}(t) \cap \mathbf{fv}(u)
\end{array}
$$

Example: $\mathcal{A}(\lambda x.y\ y) = \lambda x.W_x(C_y^{z,z'}(z\ z'))$

Lemma

For all λ-terms t and u such that $x \in \mathbf{fv}(t)$, we have

$$C^{\Delta,\Pi}_{\Phi}(R^{\Phi}_{\Delta}(\mathcal{A}(t))[x/R^{\Phi}_{\Pi}(\mathcal{A}(u))]) \to^{*}_{\mathtt{xt}} \mathcal{A}(t\{x\backslash u\})$$

where $\Phi := (\mathbf{fv}(t) \setminus \{x\}) \cap \mathbf{fv}(u)$.

Proposition [Simulating β-reductions]

If $t \to_{\beta} t'$, then $\mathcal{A}(t) {\to^{+}}_{\lambda\mathtt{lxr}} W_{\mathbf{fv}(t)\setminus\mathbf{fv}(t')}(\mathcal{A}(t'))$.

Exemple $t = (\lambda x.y)z \to_{\beta} y = t'$ and $\mathcal{A}(t) = (\lambda x.W_x(y))z {\to^{+}}_{\lambda\mathtt{lxr}} W_z(\mathcal{A}(y))$.

Connexion with λ-calculus (Summary)

$\mathcal{B}()$ hides resource control

$$t \rightarrow^*_{\texttt{xt}} W_\Pi(\mathcal{A}(\mathcal{B}(t))) \qquad \lambda\texttt{lxr} \quad \overset{\mathcal{B}()}{\longrightarrow} \quad \underset{\mathcal{A}()}{\longleftarrow} \quad \lambda \qquad t = \mathcal{B}(\mathcal{A}(t))$$

$\mathcal{A}()$ introduces resource operators

Example: $t = (\lambda x.W_x(y))W_z(z') \rightarrow^*_{\texttt{xt}} W_z((\lambda x.W_x(y))z') = W_z(\mathcal{A}(\mathcal{B}(t)))$.

Confluence

Lemma

The $\mathtt{xt}$*-normal form of* t *is* $W_{\mathbf{fv}(t)\setminus\mathbf{fv}(\mathcal{B}(t))}(\mathcal{A}(\mathcal{B}(t)))$*.*

Exemple Let $t = C_x^{x_1,x_2}((\lambda y.x_1\ (x_2\ W_y(z)))\ W_k(w))$. Then $\mathtt{xt}(t) = W_k((\lambda y.W_y(C_x^{x_1,x_2}(x_1\ (x_2\ z))))\ w)$.

Theorem (Confluence modulo)

The reduction relation $\lambda\mathtt{lxr}$ *is confluent (even on terms with meta-variables).*

Preservation of Strong Normalization

Theorem

(PSN) If $M \in SN^{\beta}$ then $\mathcal{A}(M) \in SN^{\lambda \mathtt{lxr}}$.

The $\lambda\mathtt{lxr}$-calculus breaks Melliès' counter-example of non-termination.

Agenda

Typing Rules for the $\lambda\mathtt{lxr}$-calculus

$$\frac{}{x : A \vdash x : A}\ (\text{ax}) \qquad \frac{\Delta \vdash u : B \quad \Gamma, x : B \vdash t : A}{\Gamma, \Delta \vdash t[x/u] : A}\ (\text{cut})$$

$$\frac{\Gamma \vdash t : A \to B \quad \Delta \vdash u : A}{\Gamma, \Delta \vdash (t\ u) : B}\ (\to \text{e}) \qquad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \to B}\ (\to \text{i})$$

$$\frac{\Gamma, x : B, y : B \vdash t : A}{\Gamma, z : B \vdash C_z^{x,y}(t) : A}\ (\text{c}) \qquad \frac{\Gamma \vdash t : A}{\Gamma, x : B \vdash W_x(t) : A}\ (\text{w})$$

where Γ, Δ is only defined if Γ and Δ do not share variables.

Notation We write $\Gamma \vdash_{\lambda\mathtt{lxr}} t : A$ if $\Gamma \vdash t : A$ is derivable in this system.

Remark If $\Gamma \vdash_{\lambda\mathtt{lxr}} t : A$, then $\mathbf{fv}(t) = \mathbf{dom}(\Gamma)$.

Agenda

Some Relevant Typed Properties

1. (Relating $\lambda\mathtt{lxr}$-typing and λ-typing)
   1. If $\Gamma \vdash_{\mathtt{a}} t : A$ then $\Gamma \vdash_{\lambda\mathtt{lxr}} W_{\Gamma\setminus\mathbf{fv}(t)}(\mathcal{A}(t)) : A$
   2. If $\Gamma \vdash_{\lambda\mathtt{lxr}} t : A$ then $\Gamma \vdash_{\mathtt{a}} \mathcal{B}(t) : A$
2. (Subject reduction) If $\Gamma \vdash_{\lambda\mathtt{lxr}} t : A$ et $t \rightarrow_{\lambda\mathtt{lxr}} t'$, then $\Gamma \vdash_{\lambda\mathtt{lxr}} t' : A$.

(Call-by-Name) Translation of Formulae

$$
\begin{array}{lcl}
\iota^+ & := & \iota \\
(A \to B)^+ & := & ?(A^-) \parr B^+ \\
A^- & := & (A^+)^\perp
\end{array}
$$

Translation of Derivations

Let $\Gamma = x_1 : B_1, \ldots, x_n : B_n$. Then $\Gamma \vdash_{\lambda \mathtt{lxr}} t : A$ translates to a MELL Proof-Net written $(\Gamma \vdash t : A)^\circ$ with interface $?\Gamma^-, A^+$, where $?\Gamma^-$ means $?B_1^-, \ldots, ?B_n^-$

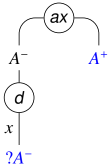
Original derivation:

$$\frac{}{x : A \vdash x : A}\ (\text{ax})$$

Sequent Translation:

$$\frac{\overline{\vdash A^-, A^+}}{\vdash ?A^-, A^+}$$

Proof-Net Translation:



Translating (→ e)

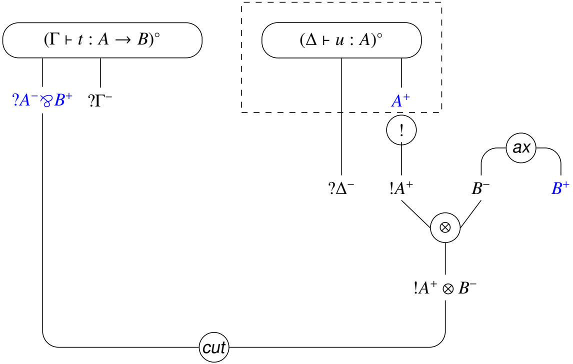
Original derivation:

$$\frac{\Gamma \vdash t : A \to B \qquad \Delta \vdash u : A}{\Gamma, \Delta \vdash tu : B}\ (\to\ \mathrm{e})$$

Sequent Translation:

$$\dfrac{\dfrac{i.h.}{\vdash ?\Gamma^-, ?A^- \parr B^+} \qquad \dfrac{\dfrac{\dfrac{i.h.}{\vdash ?\Delta^-, A^+}}{\vdash ?\Delta^-, !A^+} \qquad \dfrac{}{\vdash B^-, B^+}}{\vdash ?\Delta^-, !A^+ \otimes B^-, A^+}}{\vdash ?\Gamma^-, ?\Delta^-, B^+}$$

Proof-Net Translation:



Translating $(\rightarrow \mathrm{i})$

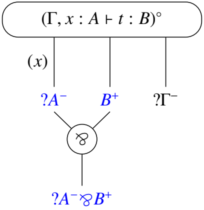
Original derivation:

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \rightarrow B}\,(\rightarrow \mathrm{i})$$

Sequent Translation:

$$\frac{\dfrac{i.h.}{\vdash ?\Gamma^{-}, ?A^{-}, B^{+}}}{\vdash ?\Gamma^{-}, ?A^{-} \parr B^{+}}$$

Proof-Net Translation:



Translating (cut)

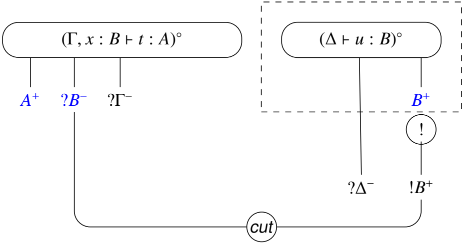
Original derivation:

$$\frac{\Delta \vdash u : B \qquad \Gamma, x : B \vdash t : A}{\Delta, \Gamma \vdash t[x \backslash u] : A} \text{(cut)}$$

Sequent Translation:

$$\dfrac{\dfrac{\dfrac{i.h.}{\vdash ?\Delta^-, B^+}}{\vdash ?\Delta^-, !B^+} \qquad \dfrac{i.h.}{\vdash ?\Gamma^-, ?B^-, A^+}}{\vdash ?\Delta^-, ?\Gamma^-, A^+}$$

Proof-Net Translation:



Translating (c)

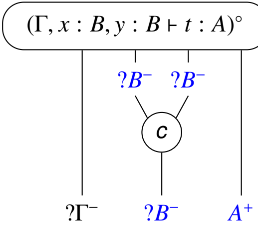
Original derivation:

$$\frac{\Gamma, x : B, y : B \vdash t : A}{\Gamma, z : B \vdash C_z^{x,y}(t) : A} \text{ (c)}$$

Sequent Translation:

$$\frac{\vdash ?\Gamma^-, ?B^-, ?B^-, A^+}{\vdash ?\Gamma^-, ?B^-, A^+}$$

Proof-Net Translation:



Translating (w)

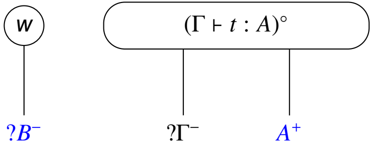
Original derivation:

$$\frac{\Gamma \vdash t : A}{\Gamma, x : B \vdash W_x(t) : A}\ (\mathrm{w})$$

Sequent Translation:

$$\frac{\vdash ?\Gamma^-, A^+}{\vdash ?\Gamma^-, ?B^-, A^+}$$

Proof-Net Translation:



Simulating λlxr with Proof Nets

Theorem (Soundness)

λlxr is sound w.r.t proof-nets:
If $\Gamma \vdash_{\lambda\texttt{lxr}} t : A$, then $t \to_{\lambda\texttt{lxr}} u$ implies $(\Gamma \vdash_{\lambda\texttt{lxr}} t : A)^\circ \to^*_{\mathcal{R}/\mathcal{E}} (\Gamma \vdash_{\lambda\texttt{lxr}} u : A)^\circ$.

The proof uses the following property:

Lemma

Let t, u be a λlxr-typed terms s.t. $\Gamma \vdash_{\lambda\texttt{lxr}} t : A$ and $\Gamma \vdash_{\lambda\texttt{lxr}} u : A$.

- *If $t \equiv u$, then $(\Gamma \vdash t : A)^\circ \simeq_{\mathcal{E}} (\Gamma \vdash u : A)^\circ$.*
- *If $t \to_B u$, then $(\Gamma \vdash t : A)^\circ {\to^+}_{\mathcal{R}/\mathcal{E}} (\Gamma \vdash u : A)^\circ$.*
- *If $t \to_{\texttt{xt}} u$, then $(\Gamma \vdash t : A)^\circ \to^*_{\mathcal{R}/\mathcal{E}} (\Gamma \vdash u : A)^\circ$.*

Corollary of the Translation

Theorem (Strong Normalisation)

The relation $\rightarrow_{\lambda\texttt{lxr}}$ is strongly normalising on well-typed $\lambda\texttt{lxr}$-terms:
if $\Gamma \vdash_{\lambda\texttt{lxr}} t : A$, then $t \in SN(\lambda\texttt{lxr})$.

Proof.

Using the previous lemma, the termination property of the relation `xt` and SN of $\rightarrow_{\mathcal{R}/\mathcal{E}}$. □

Towards completeness

- Define a congruence $\approx$ for proof-nets.
- Define a congruence $\cong$ for $\lambda\mathtt{lxr}$-terms.
- Show that $(\Gamma \vdash t_1 : A)^\circ \approx (\Gamma' \vdash t_2 : A')^\circ$ implies $t_1 \cong t_2$.

Agenda

Summary

The $\lambda\mathtt{lxr}$-calculus is a computational interpretation of natural deduction plus cut and structural rules enjoying the following properties:

- Confluence on all the terms.
- Simulation of one-step β-reduction.
- Preservation of β-strong normalization.
- Strong normalization of well-typed terms.
- Full and safe composition.
- Sound and complete with respect to proof-nets.
- Explicit operators for implementation issues.