Linear Logic and Intersection Types

Agenda for Today

Agenda

Simply versus Intersection Types

- Simply typed terms $\subseteq$ strongly-normalizing terms.
- Simply typed terms $\not\supseteq$ strongly-normalizing terms. (e.g. $\lambda x.xx$).
- Finite polymorphism

Simply types	No	
Intersection Types	Yes	$x : (\sigma \to \sigma) \wedge \sigma \vdash xx : \sigma$

- Intersection typed terms = strongly-normalizing terms.
- Intersection type systems are undecidable.

Intersection Types

- System $\mathcal{C}$ (Coppo-Dezani).

  Contains a constant type ω to type ALL terms.

- System $\mathcal{P}$ (Pottinger).

  No constant type.

- Types in both systems enjoy ACI Axioms.

$$
\begin{array}{lccc}
\textit{Associativity} & (\sigma \wedge \rho) \wedge \tau & \sim & \sigma \wedge (\rho \wedge \tau) \\
\textit{Commutativity} & \sigma \wedge \rho & \sim & \rho \wedge \sigma \\
\textit{Idempotence} & \sigma \wedge \sigma & \sim & \sigma
\end{array}
$$

Some Typical Results in λ-Calculus by means of Intersection Types

Definition

Let $\mathcal{R}$ be any reduction sequence.

- A term $\mathtt{t}$ is an $\mathcal{R}$-normal form iff there is no p such that $\mathtt{t} \rightarrow_{\mathcal{R}} \mathtt{p}$.
- A term $\mathtt{t}$ is $\mathcal{R}$-weakly normalizing iff there is some reduction sequence $\mathtt{t} \rightarrow^*_{\mathcal{R}} \mathtt{p}$, where $\mathtt{p}$ is a normal form.
- A term $\mathtt{t}$ is $\mathcal{R}$-strongly normalizing iff there is no infinite reduction sequence $\mathtt{t} \rightarrow_{\mathcal{R}} \rightarrow_{\mathcal{R}} \ldots$.

The previous three notions will be considered for different reduction relations, notably for β-reduction.

- A term $\mathtt{t}$ is β-head normalizing iff $\mathtt{t} \rightarrow^*_{\beta} \lambda \mathtt{x}_1 \ldots \mathtt{x}_n . y \mathtt{t}_1 \ldots \mathtt{t}_m (n, m \geq 0)$.
- A head context is a context of the shape $(\lambda \mathtt{x}_1 \ldots \mathtt{x}_n . \Box) \mathtt{t}_1 \ldots \mathtt{t}_m \ (n, m \geq 0)$
- A term $\mathtt{t}$ is solvable iff there is a head context $\mathtt{C}$ s.t. $\mathtt{C}[\mathtt{t}] \rightarrow^*_{\beta} \lambda \mathtt{x}.\mathtt{x}$

Some Typical Results by means of Intersection Types

Theorem

A term $\mathtt{t}$ is **solvable** iff it is β**-head normalizing** iff $\Gamma \vdash \mathtt{t} : \sigma$ in system $\mathcal{C}$ for some Γ and σ such that $\sigma \nsim \omega$.

Theorem

A term $\mathtt{t}$ is β**-weakly normalizing** iff $\Gamma \vdash \mathtt{t} : \sigma$ in system $\mathcal{C}$ for some Γ and σ such that $\omega \notin \mathtt{pos}^{+}([\Gamma, \sigma])$.

Theorem

A term $\mathtt{t}$ is **strongly-normalizing** iff $\Gamma \vdash \mathtt{t} : \sigma$ in system $\mathcal{P}$ for some Γ and σ.

Comments

- Only qualitative (and not quantitative) information is provided in the previous theorems.
- No relation between types and consumption of resources.

Towards Non-Idempotent Intersection Types

- The idempotence axiom is ruled out, i.e. $\sigma \wedge \sigma \nsim \sigma$
- The non-idempotent intersection operator $\wedge$ can be seen as the multiplicative linear logic connective $\otimes$.
- Quantitave models for λ-calculi (De Carvalho, Ehrhard).
- Head-Normalization, Solvability, Weak-Normalization and Strong-Normalization can be proved by combinatorial arguments (weighted subject reduction properties).
- Implicit Complexity $\Rightarrow$ use of ressources (e.g. substitution) can be measured.
- Partial substitution (c.f. Proof-Nets) can be seen as a measured operation of substitution.

Agenda

Towards non-idempotent types

- Pioneers: P. Gardner, A. Kfoury, D. de Carvalho.
- Several typing systems sharing the same principles.
- Intuitionistic calculi, classical calculi, etc.
- Called also quantitative types.
- We choose here to present the main principles of non-idempotent types on a language with partial substitution.

The Linear Substitution Calculus

- Inspired from Milner's calculus with partial substitution and the structural lambda calculus.
- Terms: λ-terms with explicit substitutions.
- Contexts are terms with one whole $\square$.
- Reduction Relation $\rightarrow_{\mathrm{LSC}}$ is at a distance and performs partial substitution:

$$
\begin{array}{lll l}
(\lambda x.t)[y_1/v_1]\ldots[y_n/v_n]u & \mapsto_{\mathtt{B}} & t[x/u][y_1/v_1]\ldots[y_n/v_n] & \\
\mathtt{C}[\![x]\!][x/u] & \mapsto_{\mathtt{c}} & \mathtt{C}[\![u]\!][x/u] & \\
t[x/u] & \mapsto_{\mathtt{Gc}} & t & \text{if } |t|_x = 0
\end{array}
$$

Example

$(xx)[x/I] \rightarrow_{\mathtt{c}} (xI)[x/I] \rightarrow_{\mathtt{c}} (II)[x/u] \rightarrow_{\mathtt{Gc}} II \rightarrow_{\mathtt{B}} x[x/I] \rightarrow_{\mathtt{c}} I[x/I] \rightarrow_{\mathtt{Gc}} I$

Linear-Head Reduction

- Inspired from Milner's calculus.
- With Linear-Logic Proof-Nets flavour.
- Conexion with Krivine's Abstract Machine.
- Linear-Head Reduction is a standard strategy.
- Linear-Head Reduction (lineary) performs reduction only on head-positions.
- Linear-Head Reduction does not erase terms.

Linear-Head Reduction Formally

- Linear-Head Contexts: $L_H ::= \Box \mid \lambda x.L_H \mid L_H t \mid L_H[x/t]$.
- Linear-Head Reduction (written $\rightarrow_{\mathtt{lhr}}$): rules $\{\mathtt{B}, \mathtt{hc}\}$ closed by head-linear contexts.

$$
\begin{array}{lll}
(\lambda x.t)[y_1/v_1]\ldots[y_n/v_n]u & \mapsto_{\mathtt{B}} & t[x/u][y_1/v_1]\ldots[y_n/v_n] \\
L_H[\![x]\!][x/u] & \mapsto_{\mathtt{hc}} & L_H[\![u]\!][x/u]
\end{array}
$$

Remark: $\rightarrow_{\mathtt{lhr}} \subseteq \rightarrow_{\mathtt{LSC}}$.

Example

$(xx)[x/I] \rightarrow_{\mathtt{hc}} (Ix)[x/I] \rightarrow_{\mathtt{B}} y[y/x][x/I] \rightarrow_{\mathtt{hc}} x[y/x][x/I] \rightarrow_{\mathtt{hc}} I[y/x][x/I]$

is a linear-head reduction

$(xx)[x/I] \rightarrow_{\mathtt{c}} (xI)[x/I] \rightarrow_{\mathtt{c}} (II)[x/I] \rightarrow_{\mathtt{Gc}} II \rightarrow_{\mathtt{B}} y[y/I] \rightarrow_{\mathtt{c}} I$

is not

Types for Milner's calculus

Types:

- Strict Types
- Non-idempotent intersection
- Intersection types are represented by multisets ($\sigma \wedge \sigma$ is $\{\!\{\sigma, \sigma\}\!\}$).

$$\begin{array}{lll} \sigma & ::= & \alpha \mid \mathtt{A} \rightarrow \sigma & \text{(linear types)} \\ \mathtt{A} & ::= & \{\!\{\ \}\!\} \mid \mathtt{B} & \text{(multiset types)} \\ \mathtt{B} & ::= & \{\!\{\sigma_k\}\!\}_{k \in K}\ (I \neq \emptyset) & \text{(non-empty multiset types)} \end{array}$$

Typing Rules:

- Relevance (no weakening)
- Multiplicative rules
- Syntax-directed typing rules
- Typed terms may contain untyped subterms ($I = \emptyset$)

The Type System $\mathcal{HW}$

$$\frac{}{\mathtt{x} : \{\!\{\sigma\}\!\} \vdash \mathtt{x} : \sigma}\ (\mathtt{var}) \qquad \frac{\Gamma \vdash \mathtt{t} : \sigma}{\Gamma \setminus \mathtt{x} \vdash \lambda \mathtt{x}.\mathtt{t} : \Gamma(\mathtt{x}) \to \sigma}\ (\to \mathtt{I})$$

$$\frac{\Gamma \vdash \mathtt{t} : \{\!\{\sigma_k\}\!\}_{k \in K} \to \sigma \qquad (\Delta_k \vdash \mathtt{u} : \sigma_k)_{k \in K}}{\Gamma +_{k \in K} \Delta_k \vdash \mathtt{tu} : \sigma}\ (\to \mathtt{E})$$

$$\frac{\mathtt{x} : \{\!\{\sigma_k\}\!\}_{k \in K}; \Gamma \vdash \mathtt{t} : \sigma \qquad (\Delta_k \vdash \mathtt{u} : \sigma_k)_{k \in K}}{\Gamma +_{k \in K} \Delta_k \vdash \mathtt{t}[\mathtt{x}/\mathtt{u}] : \sigma}\ (\mathtt{Cut})$$

Example

$$\dfrac{\dfrac{\dfrac{}{\mathtt{x} : \{\!\{\{\!\{\ \}\!\} \to \sigma\}\!\} \vdash \mathtt{x} : \{\!\{\ \}\!\} \to \sigma}\ (\mathtt{var})}{\mathtt{x} : \{\!\{\{\!\{\ \}\!\} \to \sigma\}\!\} \vdash \mathtt{x}\mathtt{v} : \sigma}\ (\to \mathtt{E})}{\mathtt{x} : \{\!\{\{\!\{\ \}\!\} \to \sigma\}\!\} \vdash (\mathtt{x}\mathtt{v})[\mathtt{y}/\mathtt{u}] : \sigma}\ (\mathtt{Cut})$$

Typed terms may contain (pink) untyped subterms.

Measuring Typing Derivations

Definition

Given a type derivation Φ, we define the measure $\mathtt{M}(\Phi)$ as the number of occurrences of typing rules $\mathtt{var}$, $\rightarrow \mathtt{I}$, $\rightarrow \mathtt{E}$ and $\mathtt{Cut}$ in Φ.

Example

$$\dfrac{\dfrac{\dfrac{}{\mathtt{x} : \{\!\{\{\!\{\ \}\!\} \rightarrow \sigma\}\!\} \vdash \mathtt{x} : \{\!\{\ \}\!\} \rightarrow \sigma}\,(\mathtt{var})}{\mathtt{x} : \{\!\{\{\!\{\ \}\!\} \rightarrow \sigma\}\!\} \vdash \mathtt{x}\mathtt{v} : \sigma}\,(\rightarrow \mathtt{E})}{\mathtt{x} : \{\!\{\{\!\{\ \}\!\} \rightarrow \sigma\}\!\} \vdash (\mathtt{x}\mathtt{v})[\mathtt{y}/\mathtt{u}] : \sigma}\,(\mathtt{Cut})$$

The measure of this type derivation is 3.

Relevance

Lemma (Weak Relevance)

If $\Gamma \vdash_{\mathcal{HW}} t : \sigma$*, then* $\mathtt{dom}(\Gamma) \subseteq \mathtt{fv}(t)$.

Technical Tools for Characterizing Head-Linear Normalization

- Position of terms are finite words on the alphabet $\{0, 1\}$.
- A position $p \in \texttt{pos}(\texttt{t})$ is a typed occurrence of Φ if either $p = \epsilon$, or $p = ip'$ $(i = 0, 1)$ and $p' \in \texttt{pos}(\texttt{t}|_{\texttt{i}})$ is a typed occurrence of *some* of their corresponding subderivations.
- A redex occurrence of t which is also a typed occurrence of Φ is a typed redex occurrence of t in Φ.

$$
\dfrac{\dfrac{}{x{:}\{\{\{\{\tau,\tau\}\}\to\tau\}\}\vdash x{:}\{\{\tau,\tau\}\}\to\tau} \quad \dfrac{y{:}\{\{\{\{\ \}\}\to\tau\}\}\vdash y{:}\{\{\ \}\}\to\tau}{y{:}\{\{\{\{\ \}\}\to\tau\}\}\vdash yz{:}\tau} \quad \dfrac{y{:}\{\{\{\{\tau\}\}\to\tau\}\}\vdash y{:}\{\{\tau\}\}\to\tau \qquad z{:}\{\{\tau\}\}\vdash z{:}\tau}{y{:}\{\{\{\{\tau\}\}\to\tau\}\}, z{:}\{\{\tau\}\}\vdash yz{:}\tau}}{x{:}\{\{\{\{\tau,\tau\}\}\to\tau\}\}, y{:}\{\{\{\{\ \}\}\to\tau, \{\{\tau\}\}\to\tau\}\}, z{:}\{\{\tau\}\}\vdash x(yz){:}\tau}
$$

$$
\dfrac{\dfrac{}{x{:}\{\{\{\{\tau,\tau\}\}\to\tau\}\}\vdash x{:}\{\{\tau,\tau\}\}\to\tau} \quad \dfrac{y{:}\{\{\{\{\ \}\}\to\tau\}\}\vdash y{:}\{\{\ \}\}\to\tau}{y{:}\{\{\{\{\ \}\}\to\tau\}\}\vdash yz{:}\tau} \quad \dfrac{y{:}\{\{\{\{\ \}\}\to\tau\}\}\vdash y{:}\{\{\ \}\}\to\tau}{y{:}\{\{\{\{\ \}\}\to\tau\}\}\vdash yz{:}\tau}}{x{:}\{\{\{\{\tau,\tau\}\}\to\tau\}\}, y{:}\{\{\{\{\ \}\}\to\tau, \{\{\ \}\}\to\tau\}\}\vdash x(yz){:}\tau}
$$

Results: towards characterization of head-linear normalization

Lemma (Weighted Subject Reduction for Typed Redex Occurrences)

Whenever $\Pi : \Gamma \vdash_{\mathcal{HW}} \mathtt{t} : \sigma$ *and* $\mathtt{t} \rightarrow_{\mathtt{LSC}} \mathtt{t}'$ *reduces a typed redex occurrence, then* $\Pi' : \Gamma \vdash_{\mathcal{HW}} \mathtt{t}' : \sigma$ *and* $\mathtt{M}(\Pi) > \mathtt{M}(\Pi')$.

Corollary (Weighted Subject Reduction for Linear-Head Reduction)

Whenever $\Pi : \Gamma \vdash_{\mathcal{HW}} \mathtt{t} : \sigma$ *and* $\mathtt{t} \rightarrow_{\mathtt{lhr}} \mathtt{t}'$, *then* $\Pi' : \Gamma \vdash_{\mathcal{HW}} \mathtt{t}' : \sigma$ *and* $\mathtt{M}(\Pi) > \mathtt{M}(\Pi')$.

Lemma (Subject Expansion for the Linear Substitution Calculus)

If $\Pi : \Gamma \vdash_{\mathcal{HW}} \mathtt{t} : \sigma$ *and* $\mathtt{t}' \rightarrow_{\mathtt{LSC}} \mathtt{t}$, *then* $\Pi' : \Gamma \vdash_{\mathcal{HW}} \mathtt{t}' : \sigma$.

Corollary (Subject Reduction for Linear-Head Reduction)

If $\Pi : \Gamma \vdash_{\mathcal{HW}} \mathtt{t} : \sigma$ *and* $\mathtt{t}' \rightarrow_{\mathtt{lhr}} \mathtt{t}$, *then* $\Pi' : \Gamma \vdash_{\mathcal{HW}} \mathtt{t}' : \sigma$.

Results: characterization of head-linear normalization

Theorem

The following statements are equivalent:

- `t` *is* `lhr`**-weakly normalizing**.
- `t` *is* $\mathcal{HW}$*-typable.*

Proof. uses, between others,

- Weighted Subject Reduction (combinatorial argument for non-idempotent types).
- Subject Expansion.

Results: towards characterization of weak normalization

Positive and Negative occurrences of types are defined as follows.

- $\mathtt{A} \in \mathtt{pos}^{+}(\mathtt{A})$.
- $\mathtt{A} \in \mathtt{pos}^{+}(\{\!\{\sigma_{\mathrm{k}}\}\!\}_{\mathrm{k}\in\mathrm{K}})$ if $\exists k\ \mathtt{A} \in \mathtt{pos}^{+}(\sigma_{\mathrm{k}})$; $\mathtt{A} \in \mathtt{pos}^{-}(\{\!\{\sigma_{\mathrm{k}}\}\!\}_{\mathrm{k}\in\mathrm{K}})$ if $\exists k\ \mathtt{A} \in \mathtt{pos}^{-}(\sigma_{\mathrm{k}})$.
- $\mathtt{A} \in \mathtt{pos}^{+}(\mathtt{E} \to \tau)$ if $\mathtt{A} \in \mathtt{pos}^{-}(\mathtt{E})$ or $\mathtt{A} \in \mathtt{pos}^{+}(\tau)$; $\mathtt{A} \in \mathtt{pos}^{-}(\mathtt{E} \to \tau)$ if $\mathtt{A} \in \mathtt{pos}^{+}(\mathtt{E})$ or $\mathtt{A} \in \mathtt{pos}^{-}(\tau)$.
- $\mathtt{A} \in \mathtt{pos}^{+}(\Gamma)$ if $\exists\ y \in \mathrm{dom}(\Gamma)$ s.t. $\mathtt{A} \in \mathtt{pos}^{-}(\Gamma(\mathrm{y}))$; $\mathtt{A} \in \mathtt{pos}^{-}(\Gamma)$ if $\exists\ y \in \mathrm{dom}(\Gamma)$ s.t. $\mathtt{A} \in \mathtt{pos}^{+}(\Gamma(\mathrm{y}))$.
- $\mathtt{A} \in \mathtt{pos}^{+}([\Gamma, \tau])$ if $\mathtt{A} \in \mathtt{pos}^{+}(\Gamma)$ or $\mathtt{A} \in \mathtt{pos}^{+}(\tau)$; $\mathtt{A} \in \mathtt{pos}^{-}([\Gamma, \tau])$ if $\mathtt{A} \in \mathtt{pos}^{-}(\Gamma)$ or $\mathtt{A} \in \mathtt{pos}^{-}(\tau)$.

As an example, $\{\!\{\ \}\!\} \in \mathtt{pos}^{+}(\{\!\{\ \}\!\})$, so that $\{\!\{\ \}\!\} \in \mathtt{pos}^{-}(\{\!\{\ \}\!\} \to \sigma)$, $\{\!\{\ \}\!\} \in \mathtt{pos}^{+}(\mathtt{x}{:}\{\!\{\{\!\{\ \}\!\} \to \sigma\}\!\})$ and $\{\!\{\ \}\!\} \in \mathtt{pos}^{+}([\mathtt{x}{:}\{\!\{\{\!\{\ \}\!\} \to \sigma\}\!\}, \sigma])$.

Results: characterization of weak normalization

Theorem (Weak-Normalization)

A term $\mathtt{t}$ *is* $\mathtt{LSC}$**-weakly normalizing** *iff* $\Gamma \vdash \mathtt{t} : \sigma$ *in system* $\mathcal{HW}$ *for some* Γ *and* σ *such that* $\{\!\{\,\}\!\} \notin \mathtt{pos}^{+}([\Gamma, \sigma])$.

The following term is $\mathtt{LSC}$-weakly normalizing:

$$\mathtt{z} : \alpha \vdash (\lambda \mathtt{x}.\mathtt{z})(\Delta\Delta) : \alpha$$

The following term is not $\mathtt{LSC}$-weakly normalizing:

$$\mathtt{z} : \{\!\{\{\!\{\,\}\!\} \to \alpha\}\!\} \vdash \mathtt{z}(\Delta\Delta) : \alpha$$

The Type System $\mathcal{S}$

$$\frac{}{\mathtt{x} : \{\!\{\sigma\}\!\} \vdash \mathtt{x} : \sigma}\ (\mathtt{var}) \qquad \frac{\Gamma \vdash \mathtt{t} : \sigma}{\Gamma \setminus \mathtt{x} \vdash \lambda\mathtt{x}.\mathtt{t} : \Gamma(\mathtt{x}) \to \sigma}\ (\to \mathtt{I})$$

$$\frac{\Gamma \vdash \mathtt{t} : \{\!\{\sigma_k\}\!\}_{k\in K} \to \sigma \qquad (\Delta_k \vdash \mathtt{u} : \sigma_k)_{k\in K} \qquad I \neq \emptyset}{\Gamma +_{k\in K} \Delta_k \vdash \mathtt{tu} : \sigma}\ (\to \mathtt{E1})$$

$$\frac{\mathtt{x} : \{\!\{\sigma_k\}\!\}_{k\in K}; \Gamma \vdash \mathtt{t} : \sigma \qquad (\Delta_k \vdash \mathtt{u} : \sigma_k)_{k\in K} \qquad I \neq \emptyset}{\Gamma +_{k\in K} \Delta_k \vdash \mathtt{t}[\mathtt{x}/\mathtt{u}] : \sigma}\ (\mathtt{Cut1})$$

$$\frac{\Gamma \vdash \mathtt{t} : \{\!\{\ \}\!\} \to \sigma \qquad \Delta \vdash \mathtt{u} : \tau}{\Gamma + \Delta \vdash \mathtt{tu} : \sigma}\ (\to \mathtt{E2}) \qquad \frac{\Gamma \vdash \mathtt{t} : \sigma \qquad \Delta \vdash \mathtt{u} : \tau \qquad x \notin \mathrm{dom}(\Gamma)}{\Gamma + \Delta \vdash \mathtt{t}[\mathtt{x}/\mathtt{u}] : \sigma}\ (\mathtt{Cut2})$$

Relevance

Lemma (Strong Relevance)

If $\Gamma \vdash_{\mathcal{S}} t : \sigma$*, then* $\mathrm{dom}(\Gamma) = \mathtt{fv}(t)$*.*

Results: characterization of strong normalization

Theorem (Strong-Normalization)

A term `t` *is* `LSC`**-strongly normalizing** *iff* `t` *is* $\mathcal{S}$*-typable.*

Proof. uses

- Postponement of erasing steps (which does not hold in λ-calculus).
- Weighted Subject Reduction for system $\mathcal{S}$.
- Subject Expansion for system $\mathcal{S}$.

Agenda

Turning the inhabitation problem into a decidable question

Typing and Inhabitation Problems

	Typing	**Inhabitation**
	$? \vdash t :\ ?$	$\Gamma \vdash ? : \mathtt{A}$
Simple Types :	*Decidable*	*Decidable*
Idempotent Types :	*Undecidable*	*Undecidable*
		(Urzyczyn)
Non-idempotent Types :	*Undecidable*	*Decidable*
		(Bucciarelli-K.-Ronchi Della Rocca)

Decidability of restricted classes of idempotent types (Kurata & Takahashi), (Urzyczyn), (Bunder), (Kuśmierek), (Dudenhefner & Rehof), . . .

The Typing System for the Lambda-Calculus

$$\frac{}{\mathtt{x} : \{\!\{\sigma\}\!\} \vdash \mathtt{x} : \sigma}\ (\mathtt{var}) \qquad \frac{\Gamma \vdash \mathtt{t} : \sigma}{\Gamma \setminus \mathtt{x} \vdash \lambda \mathtt{x}.\mathtt{t} : \Gamma(\mathtt{x}) \to \sigma}\ (\to \mathtt{I})$$

$$\frac{\Gamma \vdash \mathtt{t} : \{\!\{\sigma_i\}\!\}_{i \in I} \to \sigma \qquad (\Delta_i \vdash \mathtt{u} : \sigma_i)_{i \in I}}{\Gamma +_{i \in I} \Delta_i \vdash \mathtt{tu} : \sigma}\ (\to \mathtt{E})$$

Approximants

How to restrict the search space of the algorithm?

We use the normalization property to restrict the search space to approximants:

$$a, b, c \ ::= \ \Omega \mid \mathcal{N} \qquad \mathcal{N} \ ::= \ \lambda x.\mathcal{N} \mid \mathcal{L} \qquad \mathcal{L} \ ::= \ \mathtt{x} \mid \mathcal{L}a$$

The Algorithm

$$\frac{\mathtt{a} \in \mathtt{T}(\Gamma + (\mathtt{x} : \mathtt{A}), \tau) \qquad \mathtt{x} \notin \mathrm{dom}(\Gamma)}{\lambda \mathtt{x}.\mathtt{a} \in \mathtt{T}(\Gamma, \mathtt{A} \to \tau)} \; (\mathtt{Abs})$$

$$\frac{\Gamma = +_{i=1\ldots n}\Gamma_i \qquad (\mathtt{b}_i \in \mathtt{TI}(\Gamma_i, \sigma_i))_{i=1\ldots n}}{\mathtt{x}\mathtt{b}_1 \ldots \mathtt{b}_n \in \mathtt{T}(\Gamma + (\mathtt{x} : \{\!\{\mathtt{A}_1 \to \ldots \to \mathtt{A}_n \to \tau\}\!\}), \tau)} \; (\mathtt{Head})$$

$$\frac{(\mathtt{a}_i \in \mathtt{T}(\Gamma_i, \sigma_i))_{i \in I} \qquad \uparrow_{i \in I} \mathtt{a}_i}{\bigvee_{i \in I} \mathtt{a}_i \in \mathtt{TI}(+_{i \in I}\Gamma_i, \{\!\{\sigma_i\}\!\}_{i \in I})} \; (\mathtt{Union})$$

Soundness and Completeness

Theorem

The algorithm terminates, is sound and complete.

Some Final Remarks

- Non-idempotent intersection types are particularly pertinent for calculi with ressources.
- Arithmetical arguments for terminating proofs.
- New logical characterization of linear-head normalization.
- The inhabitation problem for intersection types has been proved to be undecidable, but breaking the idempotency of intersection types makes inhabitation decidable.