Linear Logic and Intersection Types

Agenda

1. Some Historical Remarks
2. Non-Idempotent Types
3. Inhabitation

Simply versus Intersection Types

- Simply typed terms $\subseteq$ strongly-normalizing terms.
- Simply typed terms $\not\supseteq$ strongly-normalizing terms.

  (e.g. $\lambda x.xx$).
- Intersection Types: The judgment $\Gamma \vdash t : \sigma \wedge \tau$ means that t has both types σ and τ.
- Finite polymorphism

Simply types	No	
Intersection Types	Yes	$x : (\sigma \to \sigma) \wedge \sigma \vdash xx : \sigma$

- Intersection typed terms = strongly-normalizing terms.
- Intersection type systems are undecidable.

Intersection Types - Some Historical References

- System $\mathcal{C}$ (Coppo-Dezani) and System $\mathcal{P}$ (Pottinger).
- Types in both systems enjoy ACI Axioms.

$$
\begin{array}{lccc}
Associativity & (\sigma \wedge \rho) \wedge \tau & \sim & \sigma \wedge (\rho \wedge \tau) \\
Commutativity & \sigma \wedge \rho & \sim & \rho \wedge \sigma \\
Idempotence & \sigma \wedge \sigma & \sim & \sigma
\end{array}
$$

- ACI intersection types can be seen as sets: $(\sigma \wedge \sigma) \wedge \tau$ is represented by $\{\sigma, \tau\}$.
- Systems $\mathcal{C}$ and $\mathcal{P}$ were used to characterize different normalization properties of λ-calculus.

Basic Definitions of Normalization for λ-Calculus

Definition

Let $\mathcal{R}$ be any reduction sequence.

- A term $\mathtt{t}$ is an $\mathcal{R}$-normal form iff there is no $\mathtt{p}$ such that $\mathtt{t} \rightarrow_{\mathcal{R}} \mathtt{p}$.
- A term $\mathtt{t}$ is $\mathcal{R}$-weakly normalizing iff there is some reduction sequence $\mathtt{t} \rightarrow^{*}_{\mathcal{R}} \mathtt{p}$, where $\mathtt{p}$ is an $\mathcal{R}$-normal form.
- A term $\mathtt{t}$ is $\mathcal{R}$-strongly normalizing iff there is no infinite reduction sequence $\mathtt{t} \rightarrow_{\mathcal{R}} \rightarrow_{\mathcal{R}} \ldots$.

The previous three notions will be considered for different reduction relations, notably for β-reduction.

Examples

- The terms $\mathtt{I} = \lambda x.x$ and $\Delta = \lambda x.xx$ are β-normal forms. They are β-weakly and β-strongly normalizing.
- The term $(\lambda x.\mathtt{I})(\Delta\Delta)$ is not a β-normal form. It is β-weakly normalizing, but not β-strongly normalizing.
- The term $\Delta\Delta$ is not a β-normal form. It is neither β-weakly normalizing, nor β-strongly normalizing.

Some Typical Results in λ-Calculus by means of Intersection Types

Definition

- A term $\mathtt{t}$ is in β-head normal form iff $\mathtt{t} = \lambda \mathtt{x}_1 \ldots \mathtt{x}_n . \mathtt{y} \mathtt{t}_1 \ldots \mathtt{t}_m (n, m \geq 0)$.
- A term $\mathtt{t}$ is β-head normalizing iff there is some reduction $\mathtt{t} \rightarrow^*_\beta \mathtt{p}$, where $\mathtt{p}$ is a β-head normal form.
- A head context is a context of the shape $(\lambda \mathtt{x}_1 \ldots \mathtt{x}_n . \Box) \mathtt{t}_1 \ldots \mathtt{t}_m \ (n, m \geq 0)$
- A term $\mathtt{t}$ is solvable iff there is a head context $\mathtt{C}$ s.t. $\mathtt{C}[\mathtt{t}] \rightarrow^*_\beta \lambda \mathtt{x}.\mathtt{x}$

Example

The term $\mathtt{I}$ is solvable, while $\Delta\Delta$ is not solvable.

Some Typical Results by means of Intersection Types

Theorem

A term $\mathtt{t}$ is **solvable** iff it is β**-head normalizing** iff $\Gamma \vdash \mathtt{t} : \sigma$ in <u>system $\mathcal{C}$</u> for some Γ and σ such that $\sigma \nsim \omega$.

Theorem

A term $\mathtt{t}$ is β**-weakly normalizing** iff $\Gamma \vdash \mathtt{t} : \sigma$ in <u>system $\mathcal{C}$</u> for some Γ and σ such that $\omega \notin \mathtt{pos}^{+}([\Gamma, \sigma])$.

Theorem

A term $\mathtt{t}$ is **strongly-normalizing** iff $\Gamma \vdash \mathtt{t} : \sigma$ in <u>system $\mathcal{P}$</u> for some Γ and σ.

Comments

- Only qualitative information is provided in the previous theorems.
- No relation between types and (quantitative) consumption of resources.

Towards Non-Idempotent Intersection Types

- The idempotence axiom is ruled out, i.e. $\sigma \wedge \sigma \nsim \sigma$
- The non-idempotent intersection operator $\wedge$ can be seen as the multiplicative linear logic connective $\otimes$.
- Quantitave models for λ-calculi (De Carvalho, Ehrhard).
- Head-Normalization, Solvability, Weak-Normalization and Strong-Normalization can be proved by combinatorial arguments (weighted subject reduction properties).
- Implicit Complexity $\Rightarrow$ use of ressources (e.g. substitution) can be measured.
- Partial substitution (c.f. Proof-Nets) can be seen as a measured operation of substitution.

Agenda

Towards non-idempotent types

- Pioneers: P. Gardner, A. Kfoury, D. de Carvalho.
- Today there are several typing systems sharing the same principles.
- Intuitionistic calculi (call-by-name, call-by-value, call-by-need, mixing strategies), classical calculi, etc.
- Called also quantitative types.
- We choose here to present the main principles of non-idempotent types on a language with partial substitution (instead of the λ-calculus).

The Linear Substitution Calculus

- Inspired from Milner's calculus with partial substitution and the structural lambda calculus.
- Terms: λ-terms with explicit substitutions.
- Contexts are terms with one whole $\Box$.
- Reduction Relation $\rightarrow_{\mathtt{M}}$ is at a distance and performs partial substitution:

$$
\begin{array}{llll}
(\lambda x.t)[y_1/v_1]\ldots[y_n/v_n]u & \mapsto_{\mathtt{dB}} & t[x/u][y_1/v_1]\ldots[y_n/v_n] & \\
\mathsf{C}[\![x]\!][x/u] & \mapsto_{\mathtt{ls}} & \mathsf{C}[\![u]\!][x/u] & \\
t[x/u] & \mapsto_{\mathtt{gc}} & t & \text{if } |t|_x = 0
\end{array}
$$

Example

$(xx)[x/I] \rightarrow_{\mathtt{ls}} (xI)[x/I] \rightarrow_{\mathtt{ls}} (II)[x/I] \rightarrow_{\mathtt{gc}} II \rightarrow_{\mathtt{dB}} x[x/I] \rightarrow_{\mathtt{ls}} I[x/I] \rightarrow_{\mathtt{gc}} I$

Linear-Head Reduction

- Inspired from Milner's calculus.
- With Linear-Logic Proof-Nets flavour.
- Conexion with Krivine's Abstract Machine.
- Linear-Head Reduction is a standard strategy.
- Linear-Head Reduction (lineary) performs reduction only on head-positions.
- Linear-Head Reduction does not erase terms.

Linear-Head Reduction Formally

- Linear-Head Contexts: $\mathtt{L_H} ::= \Box \mid \lambda x.\mathtt{L_H} \mid \mathtt{L_H}t \mid \mathtt{L_H}[x/t]$.
- Linear-Head Reduction (written $\rightarrow_{\mathtt{LHR}}$): rules $\{\mathtt{dB}, \mathtt{hls}\}$ closed by head-linear contexts.

$$\begin{array}{lll} (\lambda x.t)[y_1/v_1]\ldots[y_n/v_n]u & \mapsto_{\mathtt{dB}} & t[x/u][y_1/v_1]\ldots[y_n/v_n] \\ \mathtt{L_H}[\![x]\!][x/u] & \mapsto_{\mathtt{hls}} & \mathtt{L_H}[\![u]\!][x/u] \end{array}$$

Remark: $\rightarrow_{\mathtt{LHR}} \subseteq \rightarrow_{\mathtt{M}}$.

Example

$(xx)[x/I] \rightarrow_{\mathtt{hls}} (Ix)[x/I] \rightarrow_{\mathtt{dB}} y[y/x][x/I] \rightarrow_{\mathtt{hls}} x[y/x][x/I] \rightarrow_{\mathtt{hls}} I[y/x][x/I]$

is a linear-head reduction

$(xx)[x/I] \rightarrow_{\mathtt{ls}} (xI)[x/I] \rightarrow_{\mathtt{ls}} (II)[x/I] \rightarrow_{\mathtt{gc}} II \rightarrow_{\mathtt{dB}} y[y/I] \rightarrow_{\mathtt{ls}} I$

is not

Types for Milner's calculus

Types:

- Strict Types
- Non-idempotent intersection
- Intersection types are represented by multisets ($\sigma \wedge \sigma$ is $[\sigma, \sigma]$).

$$
\begin{array}{lcll}
\sigma & ::= & \alpha \mid \mathtt{A} \to \sigma & \text{(linear types)} \\
\mathtt{A} & ::= & [\,] \mid \mathtt{B} & \text{(multiset types)} \\
\mathtt{B} & ::= & [\sigma_k]_{k \in K} \ (K \neq \emptyset) & \text{(non-empty multiset types)}
\end{array}
$$

Typing Rules:

- Relevance (no weakening)
- Multiplicative rules
- Syntax-directed typing rules
- Typed terms may contain untyped subterms ($I = \emptyset$)

The Type System $\mathcal{HW}$

$$\frac{}{\mathtt{x} : [\sigma] \vdash \mathtt{x} : \sigma}\ (\mathtt{var}) \qquad \frac{\Gamma \vdash \mathtt{t} : \sigma}{\Gamma \setminus \mathtt{x} \vdash \lambda \mathtt{x}.\mathtt{t} : \Gamma(\mathtt{x}) \to \sigma}\ (\to \mathtt{I})$$

$$\frac{\Gamma \vdash \mathtt{t} : [\sigma_k]_{k \in K} \to \sigma \qquad (\Delta_k \vdash \mathtt{u} : \sigma_k)_{k \in K}}{\Gamma +_{k \in K} \Delta_k \vdash \mathtt{tu} : \sigma}\ (\to \mathtt{E})$$

$$\frac{\mathtt{x} : [\sigma_k]_{k \in K}; \Gamma \vdash \mathtt{t} : \sigma \qquad (\Delta_k \vdash \mathtt{u} : \sigma_k)_{k \in K}}{\Gamma +_{k \in K} \Delta_k \vdash \mathtt{t}[\mathtt{x}/\mathtt{u}] : \sigma}\ (\mathtt{Cut})$$

Example

$$\dfrac{\dfrac{\dfrac{}{\mathtt{x} : [[\,] \to \sigma] \vdash \mathtt{x} : [\,] \to \sigma}\,(\mathtt{var})}{\mathtt{x} : [[\,] \to \sigma] \vdash \mathtt{x}\mathtt{v} : \sigma}\,(\to \mathtt{E})}{\mathtt{x} : [[\,] \to \sigma] \vdash (\mathtt{x}\mathtt{v})[\mathtt{y}/\mathtt{u}] : \sigma}\,(\mathtt{Cut})$$

Typed terms may contain (pink) untyped subterms.

Measuring Typing Derivations

Definition

Given a type derivation Φ, we define its measure $\mathtt{sz}(\Phi)$ as the number of occurrences of typing rules $\mathtt{var}$, $\to \mathtt{I}$, $\to \mathtt{E}$ and $\mathtt{Cut}$ in Φ.

Example

$$\dfrac{\dfrac{\dfrac{}{\mathtt{x} : [[\,] \to \sigma] \vdash \mathtt{x} : [\,] \to \sigma}\,(\mathtt{var})}{\mathtt{x} : [[\,] \to \sigma] \vdash \mathtt{x}\mathtt{v} : \sigma}\,(\to \mathtt{E})}{\mathtt{x} : [[\,] \to \sigma] \vdash (\mathtt{x}\mathtt{v})[\mathtt{y}/\mathtt{u}] : \sigma}\,(\mathtt{Cut})$$

The measure of this type derivation is 3.

Relevance

Lemma (Weak Relevance)

If $\Gamma \vdash_{\mathcal{HW}} t : \sigma$*, then* $\mathtt{dom}(\Gamma) \subseteq \mathtt{fv}(t)$*.*

Technical Tools for Characterizing Head-Linear Normalization

- Position of terms are finite words on the alphabet $\{0, 1\}$.
- A position $p \in \texttt{pos(t)}$ is a typed occurrence of Φ if either $p = \epsilon$, or $p = ip'$ $(i = 0, 1)$ and $p' \in \texttt{pos(t}|_{\texttt{i}}\texttt{)}$ is a typed occurrence of *some* of their corresponding subderivations.
- A redex occurrence of t which is also a typed occurrence of Φ is a typed redex occurrence of t in Φ.

$$\dfrac{\dfrac{}{x{:}[[\tau,\tau]\to\tau] \vdash x{:}[\tau,\tau]\to\tau} \quad \dfrac{y{:}[[\,]\to\tau] \vdash y{:}[\,]\to\tau}{y{:}[[\,]\to\tau] \vdash yz{:}\tau} \quad \dfrac{y{:}[[\tau]\to\tau] \vdash y{:}[\tau]\to\tau \qquad z{:}[\tau] \vdash z{:}\tau}{y{:}[[\tau]\to\tau], z{:}[\tau] \vdash yz{:}\tau}}{x{:}[[\tau,\tau]\to\tau], y{:}[[\,]\to\tau, [\tau]\to\tau], z{:}[\tau] \vdash x(yz){:}\tau}$$

$$\dfrac{\dfrac{}{x{:}[[\tau,\tau]\to\tau] \vdash x{:}[\tau,\tau]\to\tau} \quad \dfrac{y{:}[[\,]\to\tau] \vdash y{:}[\,]\to\tau}{y{:}[[\,]\to\tau] \vdash yz{:}\tau} \quad \dfrac{y{:}[[\,]\to\tau] \vdash y{:}[\,]\to\tau}{y{:}[[\,]\to\tau] \vdash yz{:}\tau}}{x{:}[[\tau,\tau]\to\tau], y{:}[[\,]\to\tau, [\,]\to\tau] \vdash x(yz){:}\tau}$$

Results: towards characterization of head-linear normalization

Lemma (Weighted Subject Reduction for Typed Redex Occurrences)

Whenever $\Pi : \Gamma \vdash_{\mathcal{HW}} \mathtt{t} : \sigma$ *and* $\mathtt{t} \rightarrow_{\mathtt{M}} \mathtt{t}'$ *reduces a typed redex occurrence, then* $\Pi' : \Gamma \vdash_{\mathcal{HW}} \mathtt{t}' : \sigma$ *and* $\mathtt{sz}(\Pi) > \mathtt{sz}(\Pi')$.

Corollary (Weighted Subject Reduction for Linear-Head Reduction)

Whenever $\Pi : \Gamma \vdash_{\mathcal{HW}} \mathtt{t} : \sigma$ *and* $\mathtt{t} \rightarrow_{\mathtt{LHR}} \mathtt{t}'$, *then* $\Pi' : \Gamma \vdash_{\mathcal{HW}} \mathtt{t}' : \sigma$ *and* $\mathtt{sz}(\Pi) > \mathtt{sz}(\Pi')$.

Lemma (Subject Expansion for the Linear Substitution Calculus)

If $\Pi : \Gamma \vdash_{\mathcal{HW}} \mathtt{t} : \sigma$ *and* $\mathtt{t}' \rightarrow_{\mathtt{M}} \mathtt{t}$, *then* $\Pi' : \Gamma \vdash_{\mathcal{HW}} \mathtt{t}' : \sigma$.

Corollary (Subject Reduction for Linear-Head Reduction)

If $\Pi : \Gamma \vdash_{\mathcal{HW}} \mathtt{t} : \sigma$ *and* $\mathtt{t}' \rightarrow_{\mathtt{LHR}} \mathtt{t}$, *then* $\Pi' : \Gamma \vdash_{\mathcal{HW}} \mathtt{t}' : \sigma$.

Results: characterization of head-linear normalization

Theorem

The following statements are equivalent:

- `t` *is* `LHR`**-weakly normalizing***.*
- `t` *is* $\mathcal{HW}$*-typable.*

Proof. uses, between others,

- Weighted Subject Reduction (combinatorial argument for non-idempotent types).
- Subject Expansion.

Results: towards characterization of weak normalization

Positive and Negative occurrences of types are defined as follows.

- $\mathtt{A} \in \mathtt{pos}^{+}(\mathtt{A})$.
- $\mathtt{A} \in \mathtt{pos}^{+}([\sigma_{\mathrm{k}}]_{\mathrm{k}\in\mathrm{K}})$ if $\exists k\ \mathtt{A} \in \mathtt{pos}^{+}(\sigma_{\mathrm{k}})$; $\mathtt{A} \in \mathtt{pos}^{-}([\sigma_{\mathrm{k}}]_{\mathrm{k}\in\mathrm{K}})$ if $\exists k\ \mathtt{A} \in \mathtt{pos}^{-}(\sigma_{\mathrm{k}})$.
- $\mathtt{A} \in \mathtt{pos}^{+}(\mathtt{E} \to \tau)$ if $\mathtt{A} \in \mathtt{pos}^{-}(\mathtt{E})$ or $\mathtt{A} \in \mathtt{pos}^{+}(\tau)$; $\mathtt{A} \in \mathtt{pos}^{-}(\mathtt{E} \to \tau)$ if $\mathtt{A} \in \mathtt{pos}^{+}(\mathtt{E})$ or $\mathtt{A} \in \mathtt{pos}^{-}(\tau)$.
- $\mathtt{A} \in \mathtt{pos}^{+}(\Gamma)$ if $\exists\ y \in \mathrm{dom}(\Gamma)$ s.t. $\mathtt{A} \in \mathtt{pos}^{-}(\Gamma(\mathrm{y}))$; $\mathtt{A} \in \mathtt{pos}^{-}(\Gamma)$ if $\exists\ y \in \mathrm{dom}(\Gamma)$ s.t. $\mathtt{A} \in \mathtt{pos}^{+}(\Gamma(\mathrm{y}))$.
- $\mathtt{A} \in \mathtt{pos}^{+}([\Gamma, \tau])$ if $\mathtt{A} \in \mathtt{pos}^{+}(\Gamma)$ or $\mathtt{A} \in \mathtt{pos}^{+}(\tau)$; $\mathtt{A} \in \mathtt{pos}^{-}([\Gamma, \tau])$ if $\mathtt{A} \in \mathtt{pos}^{-}(\Gamma)$ or $\mathtt{A} \in \mathtt{pos}^{-}(\tau)$.

As an example, $[\,] \in \mathtt{pos}^{+}([\,])$, so that $[\,] \in \mathtt{pos}^{-}([\,] \to \sigma)$, $[\,] \in \mathtt{pos}^{+}(\mathrm{x}{:}[[\,] \to \sigma])$ and $[\,] \in \mathtt{pos}^{+}([\mathrm{x}{:}[[\,] \to \sigma], \sigma])$.

Results: characterization of weak normalization

Theorem (Weak-Normalization)

A term $\mathtt{t}$ *is* $\mathtt{M}$**-weakly normalizing** *iff* $\Gamma \vdash \mathtt{t} : \sigma$ *in system* $\mathcal{HW}$ *for some* Γ *and* σ *such that* $[\,] \notin \mathtt{pos}^{+}([\Gamma, \sigma])$.

The following term is $\mathtt{M}$-weakly normalizing:

$$\mathtt{z} : \alpha \vdash (\lambda \mathtt{x}.\mathtt{z})(\Delta\Delta) : \alpha$$

The following term is not $\mathtt{M}$-weakly normalizing:

$$\mathtt{z} : [[\,] \to \alpha] \vdash \mathtt{z}(\Delta\Delta) : \alpha$$

The Type System $\mathcal{S}$

$$\frac{}{\mathtt{x} : [\sigma] \vdash \mathtt{x} : \sigma}\ (\mathtt{var}) \qquad \frac{\Gamma \vdash \mathtt{t} : \sigma}{\Gamma \setminus \mathtt{x} \vdash \lambda \mathtt{x}.\mathtt{t} : \Gamma(\mathtt{x}) \to \sigma}\ (\to \mathtt{I})$$

$$\frac{\Gamma \vdash \mathtt{t} : [\sigma_k]_{k \in K} \to \sigma \qquad (\Delta_k \vdash \mathtt{u} : \sigma_k)_{k \in K} \qquad K \neq \emptyset}{\Gamma +_{k \in K} \Delta_k \vdash \mathtt{tu} : \sigma}\ (\to \mathtt{E1})$$

$$\frac{\mathtt{x} : [\sigma_k]_{k \in K}; \Gamma \vdash \mathtt{t} : \sigma \qquad (\Delta_k \vdash \mathtt{u} : \sigma_k)_{k \in K} \qquad K \neq \emptyset}{\Gamma +_{k \in K} \Delta_k \vdash \mathtt{t}[\mathtt{x}/\mathtt{u}] : \sigma}\ (\mathtt{Cut1})$$

$$\frac{\Gamma \vdash \mathtt{t} : [\,] \to \sigma \qquad \Delta \vdash \mathtt{u} : \tau}{\Gamma + \Delta \vdash \mathtt{tu} : \sigma}\ (\to \mathtt{E2}) \qquad \frac{\Gamma \vdash \mathtt{t} : \sigma \qquad \Delta \vdash \mathtt{u} : \tau \qquad x \notin \mathrm{dom}(\Gamma)}{\Gamma + \Delta \vdash \mathtt{t}[\mathtt{x}/\mathtt{u}] : \sigma}\ (\mathtt{Cut2})$$

Relevance

Lemma (Strong Relevance)

If $\Gamma \vdash_{\mathcal{S}} t : \sigma$, *then* $\mathtt{dom}(\Gamma) = \mathtt{fv}(t)$.

Results: characterization of strong normalization

Theorem (Strong-Normalization)

A term $\mathtt{t}$ *is* $\mathtt{M}$**-strongly normalizing** *iff* $\mathtt{t}$ *is* $\mathcal{S}$*-typable.*

Proof. uses

- Postponement of erasing steps (which does not hold in λ-calculus).
- Weighted Subject Reduction for system $\mathcal{S}$.
- Subject Expansion for system $\mathcal{S}$.

Summary

- Type system $\mathcal{HW}$ characterizes $\mathtt{LHR}$-weak normalization.
- Type system $\mathcal{HW}$ with positive constraints characterizes $\mathtt{M}$-weak normalization.
- Type system $\mathcal{S}$ characterizes $\mathtt{M}$-weak normalization.

Agenda

Turning the inhabitation problem into a decidable question

Typing and Inhabitation Problems

	Typing	**Inhabitation**
	$? \vdash t : \ ?$	$\Gamma \vdash ? : \mathtt{A}$
Simple Types :	*Decidable*	*Decidable*
Idempotent Types :	*Undecidable*	*Undecidable* **(Urzyczyn)**
Non-idempotent Types :	*Undecidable*	*Decidable* **(Bucciarelli-K.-Ronchi Della Rocca)**

Decidability of restricted classes of idempotent types (Kurata & Takahashi), (Urzyczyn), (Bunder), (Kuśmierek), (Dudenhefner & Rehof), . . .

The Typing System for the Lambda-Calculus

$$\frac{}{\mathtt{x} : [\sigma] \vdash \mathtt{x} : \sigma}\ (\mathtt{var}) \qquad \frac{\Gamma \vdash \mathtt{t} : \sigma}{\Gamma \setminus \mathtt{x} \vdash \lambda \mathtt{x}.\mathtt{t} : \Gamma(\mathtt{x}) \to \sigma}\ (\to \mathtt{I})$$

$$\frac{\Gamma \vdash \mathtt{t} : [\sigma_i]_{i \in I} \to \sigma \qquad (\Delta_i \vdash \mathtt{u} : \sigma_i)_{i \in I}}{\Gamma +_{i \in I} \Delta_i \vdash \mathtt{tu} : \sigma}\ (\to \mathtt{E})$$

Approximants

How to restrict the search space of the algorithm?

We use the normalization property to restrict the search space to approximants:

$$\mathrm{a}, \mathrm{b}, \mathrm{c} \ ::= \ \Omega \mid \mathcal{N} \qquad \mathcal{N} \ ::= \ \lambda \mathrm{x}.\mathcal{N} \mid \mathcal{L} \qquad \mathcal{L} \ ::= \ \mathtt{x} \mid \mathcal{L}\mathrm{a}$$

$$\frac{\mathtt{a} \in \mathtt{T}(\Gamma + (\mathtt{x} : \mathtt{A}), \tau) \qquad \mathtt{x} \notin \mathrm{dom}(\Gamma)}{\lambda \mathtt{x}.\mathtt{a} \in \mathtt{T}(\Gamma, \mathtt{A} \to \tau)} \ (\mathtt{Abs})$$

$$\frac{\Gamma = +_{i=1\ldots n}\Gamma_i \qquad (\mathtt{b}_i \in \mathtt{TI}(\Gamma_i, \mathtt{A}_i))_{i=1\ldots n}}{\mathtt{x}\mathtt{b}_1 \ldots \mathtt{b}_n \in \mathtt{T}(\Gamma + (\mathtt{x} : [\mathtt{A}_1 \to \ldots \to \mathtt{A}_n \to \tau]), \tau)} \ (\mathtt{Head})$$

$$\frac{\Gamma = +_{i=1\ldots n}\Gamma_i \qquad (\mathtt{a}_i \in \mathtt{T}(\Gamma_i, \sigma_i))_{i \in I} \qquad \uparrow_{i \in I} \mathtt{a}_i}{\bigvee_{i \in I} \mathtt{a}_i \in \mathtt{TI}(\Gamma, [\sigma_i]_{i \in I})} \ (\mathtt{Union})$$

Soundness and Completeness

Theorem

The algorithm terminates, is sound and complete.

Some Final Remarks

- Non-idempotent intersection types are particularly pertinent for calculi with ressources.
- Arithmetical arguments for terminating proofs.
- New logical characterizations of different notions of normalization for higher-order languages.
- The inhabitation problem for intersection types has been proved to be undecidable, but breaking the idempotency of intersection types makes inhabitation decidable.