Reduction Rules and Equations for the λs-calculus

Equations :			
$t[x/u][y/v]$	$=_{\mathtt{C}}$	$t[y/v][x/u]$	if $y \notin \mathtt{fv}(u)$ & $x \notin \mathtt{fv}(v)$
Reduction Rules :			
$(\lambda x.t)\ u$	$\to_{\mathtt{B}}$	$t[x/u]$	
$x[x/u]$	$\to_{\mathtt{Var}}$	u	
$t[x/u]$	$\to_{\mathtt{Gc}}$	t	if $x \notin \mathtt{fv}(t)$
$(t\ u)[x/v]$	$\to_{\mathtt{App}_1}$	$(t[x/v]\ u[x/v])$	if $x \in \mathtt{fv}(t)$ & $x \in \mathtt{fv}(u)$
$(t\ u)[x/v]$	$\to_{\mathtt{App}_2}$	$(t\ u[x/v])$	if $x \notin \mathtt{fv}(t)$ & $x \in \mathtt{fv}(u)$
$(t\ u)[x/v]$	$\to_{\mathtt{App}_3}$	$(t[x/v]\ u)$	if $x \in \mathtt{fv}(t)$ & $x \notin \mathtt{fv}(u)$
$(\lambda y.t)[x/v]$	$\to_{\mathtt{Lamb}}$	$\lambda y.t[x/v]$	if $y \notin \mathtt{fv}(v)$
$t[x/u][y/v]$	$\to_{\mathtt{Comp}_1}$	$t[y/v][x/u[y/v]]$	if $y \in \mathtt{fv}(u)$ & $y \in \mathtt{fv}(t)$
$t[x/u][y/v]$	$\to_{\mathtt{Comp}_2}$	$t[x/u[y/v]]$	if $y \in \mathtt{fv}(u)$ & $y \notin \mathtt{fv}(t)$

Let $\mathtt{s} = \{\mathtt{Var}, \mathtt{Gc}, \mathtt{App}_1, \mathtt{App}_2, \mathtt{App}_3, \mathtt{Lamb}, \mathtt{Comp}_1, \mathtt{Comp}_2\}$.

From λs-calculus to MELL Proof-Nets

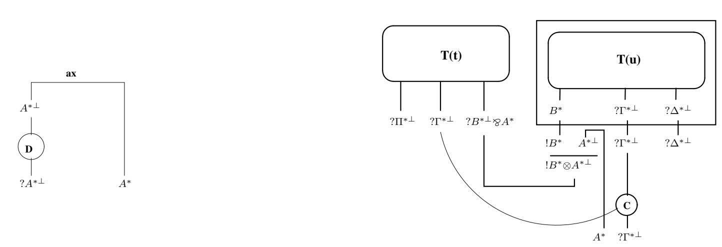
Translating types

$$A^* \;:=\; A \qquad \text{if } A \text{ is atomic}$$
$$(A \to B)^* \;:=\; ?((A^*)^{\perp}) \parr B^*$$

Translating terms

$T(x : A \vdash x : A)$

$T(\Pi, \Gamma, \Delta \vdash t\ u : A)$ where $T(\Pi, \Gamma \vdash t : B \to A)$ & $T(\Gamma, \Delta \vdash u : B)$

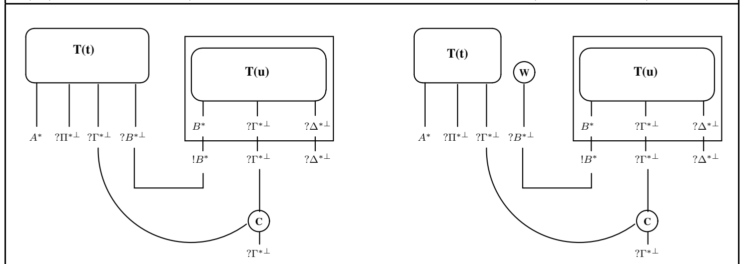


$T(\Gamma \vdash \lambda x.t : B \to C)$ where $\Gamma, x : B \vdash t : C$

$T(\Gamma \vdash \lambda x.t : B \to C)$ where $\Gamma \vdash t : C$

$T(\Pi, \Gamma, \Delta \vdash t[x/u] : A)$ where $\Pi, \Gamma, x : B \vdash t : A$ & $\Gamma, \Delta \vdash u : B$

$T(\Pi, \Gamma, \Delta \vdash t[x/u] : A)$ where $\Pi, \Gamma \vdash t : A$ & $\Gamma, \Delta \vdash u : B$



Translating reduction

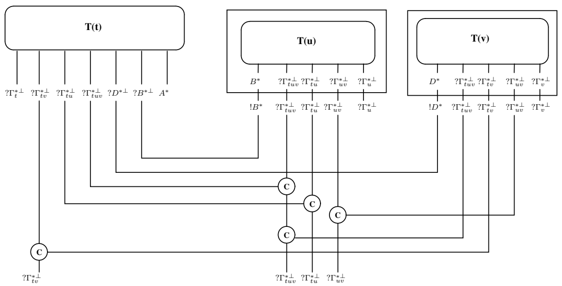
Theorem 0.1 *Let s be a λ*s*-typed term.*

1. *If $s =_{\mathtt{C}} s'$, then $T(s) \sim_E T(s')$.*
2. *If $s \to_{\mathtt{App}_3,\mathtt{Lamb}} s'$, then $T(s) \sim_E T(s')$.*
3. *If $s \to_{(\mathtt{B},\mathtt{s})\setminus\{\mathtt{App}_3,\mathtt{Lamb}\}} s'$, then $T(s) \to^{+}_{R/E} C[T(s')]$.*

Proof. The proof proceeds by induction on $\to_{\lambda\mathtt{s}}$. We first show that cases where $s \to_{\lambda\mathtt{s}} s'$ is an external reduction step, for which we consider all the root reduction/equivalence cases.

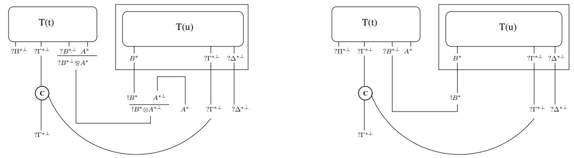
- For $s = t[x/u][y/v] =_{\mathtt{C}} t[y/v][x/u] = s'$, where $y \notin \mathtt{fv}(u)$ & $x \notin \mathtt{fv}(u)$, we show here the case $x \in \mathtt{fv}(t)$ & $y \in \mathtt{fv}(t)$, all the other ones being similar. Thus $\Gamma \vdash s : A$ comes from $\Gamma_{tuv}, \Gamma_{tu}, \Gamma_{tv}, \Gamma_t, x : B, y : D \vdash t : A$ and $\Gamma_{tuv}, \Gamma_{tu}, \Gamma_{uv}, \Gamma_u \vdash u : B$ and $\Gamma_{tuv}, \Gamma_{tv}, \Gamma_{uv}, \Gamma_v \vdash v : D$, where $\Gamma_{tuv} := \mathtt{fv}(t) \cap \mathtt{fv}(u) \cap \mathtt{fv}(u)$, $\Gamma_{tu} := \mathtt{fv}(t) \cap \mathtt{fv}(u) \setminus \mathtt{fv}(v)$, $\Gamma_{tv} := \mathtt{fv}(t) \cap \mathtt{fv}(v) \setminus \mathtt{fv}(u)$, $\Gamma_{uv} := \mathtt{fv}(u) \cap \mathtt{fv}(v) \setminus \mathtt{fv}(t)$, $\Gamma_t := \mathtt{fv}(t) \setminus y \setminus x \setminus \mathtt{fv}(u) \setminus \mathtt{fv}(v)$, $\Gamma_u := \mathtt{fv}(u) \setminus \mathtt{fv}(t) \setminus \mathtt{fv}(v)$ and $\Gamma_v := \mathtt{fv}(v) \setminus \mathtt{fv}(t) \setminus \mathtt{fv}(u)$.

  The proof-net $T(s) \sim_E T(s')$ is given by

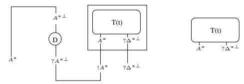


- For $s = (\lambda x.t)\ u \to_{\mathtt{B}} t[x/u] = s'$ with $\Pi, \Gamma, \Delta \vdash (\lambda x.t)\ u : A$ coming from $\Pi, \Gamma \vdash \lambda x.t : B \to A$ and $\Gamma, \Delta \vdash u : B$, where $\Gamma := \mathtt{fv}(\lambda x.t) \cap \mathtt{fv}(u)$, $\Pi := \mathtt{fv}(\lambda x.t) \setminus \mathtt{fv}(u)$ and $\Delta := \mathtt{fv}(u) \setminus \mathtt{fv}(\lambda x.t)$. We show here the case $x \in \mathtt{fv}(t)$, the case $x \notin \mathtt{fv}(t)$ being similar.

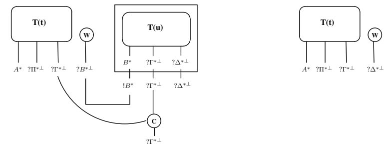
  We can verify that $T(s)$ (on the left) reduces to $T(s')$ (on the right) in exactly two steps so that $C[\_]$ is empty, i.e. $T(s) \to_{\invamp\text{-}\otimes} \to_{\mathtt{ax\text{-}cut}} T(s')$.



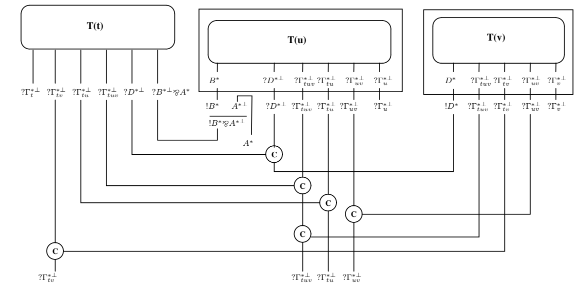
- For $s = x[x/u] \to_{\mathtt{Var}} u = s'$, coming from $x : A \vdash x : A$ and $\Delta \vdash u : A$ where $\Delta := \mathtt{fv}(u)$. We can verify that $T(s)$ (on the left) reduces to $T(s')$ (on the right) in exactly two steps so that $C[\_]$ is empty, i.e. $T(s) \to^{*}_{\mathtt{d\text{-}b,ax\text{-}cut}} T(s')$.



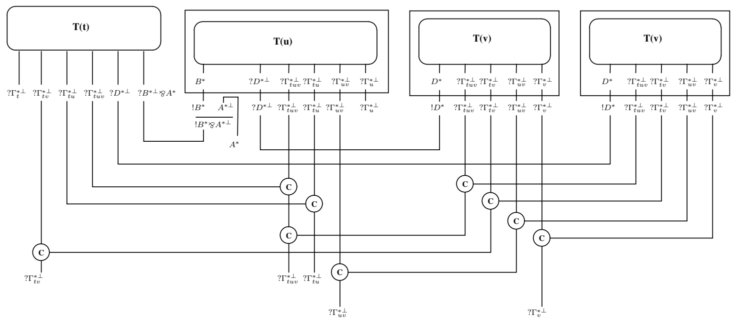
- For $s = t[x/u] \to_{\mathtt{Gc}} t$, with $x \notin \mathtt{fv}(t)$, coming from $\Pi, \Gamma \vdash t : A$ and $\Gamma, \Delta \vdash u : B$, where $\Gamma := \mathtt{fv}(t) \cap \mathtt{fv}(u)$, $\Pi := \mathtt{fv}(t) \setminus \mathtt{fv}(u)$ and $\Delta := \mathtt{fv}(u) \setminus \mathtt{fv}(t)$. We can verify that $T(s) \to^{*}_{\mathtt{w\text{-}b,U}} C[T(s')]$, where $C[\_]$ contains all the weakenings wires for $?\Delta^{*\perp}$.



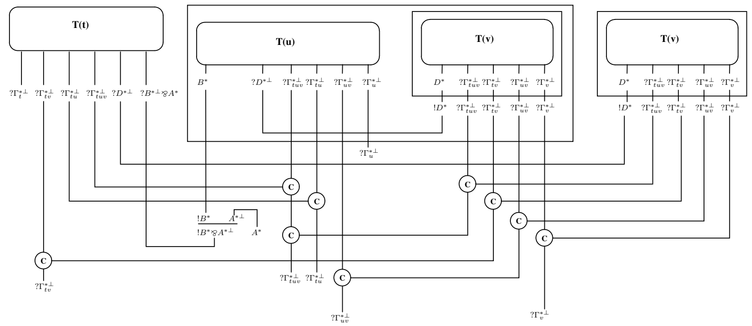
- For $s = (t\ u)[x/v] \to_{\mathtt{App}_1} (t[x/v]\ u[x/v]) = s'$, with $x \in \mathtt{fv}(t)$ & $x \in \mathtt{fv}(u)$, coming from $\Gamma_{tuv}, \Gamma_{tu}, \Gamma_{tv}, \Gamma_t, x : D \vdash t : B \to A$ and $\Gamma_{tuv}, \Gamma_{tu}, \Gamma_{uv}, \Gamma_u, x : D \vdash u : B$ and $\Gamma_{tuv}, \Gamma_{tv}, \Gamma_{uv}, \Gamma_v \vdash v : D$, where $\Gamma_{tuv} := \mathtt{fv}(t) \cap \mathtt{fv}(u) \cap \mathtt{fv}(u)$, $\Gamma_{tu} := \mathtt{fv}(t) \cap \mathtt{fv}(u) \setminus x \setminus \mathtt{fv}(v)$, $\Gamma_{tv} := \mathtt{fv}(t) \cap \mathtt{fv}(v) \setminus \mathtt{fv}(u)$, $\Gamma_{uv} := \mathtt{fv}(u) \cap \mathtt{fv}(v) \setminus \mathtt{fv}(t)$, $\Gamma_t := \mathtt{fv}(t) \setminus \mathtt{fv}(u) \setminus \mathtt{fv}(v)$, $\Gamma_u := \mathtt{fv}(u) \setminus \mathtt{fv}(t) \setminus \mathtt{fv}(v)$ and $\Gamma_v := \mathtt{fv}(v) \setminus \mathtt{fv}(t) \setminus \mathtt{fv}(u)$. The proof-net $T(s)$ is given by



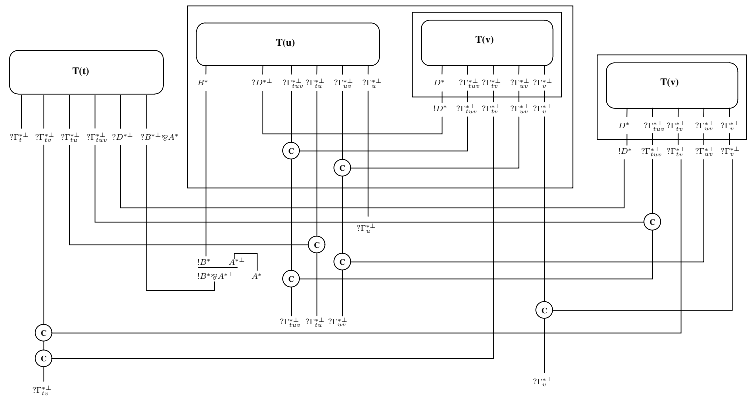
which reduces by $\rightarrow_{\texttt{c-b}}$ to the proof-net



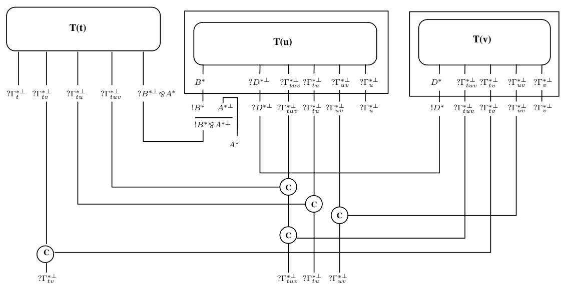
which reduces by $\rightarrow_{\texttt{b-b}}$ to the proof-net



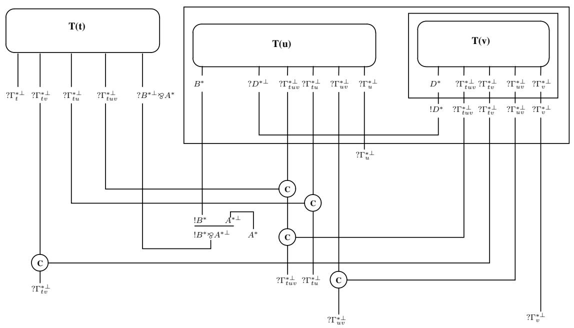
which is equivalent via $\sim_E$ to the proof-net $T(s')$



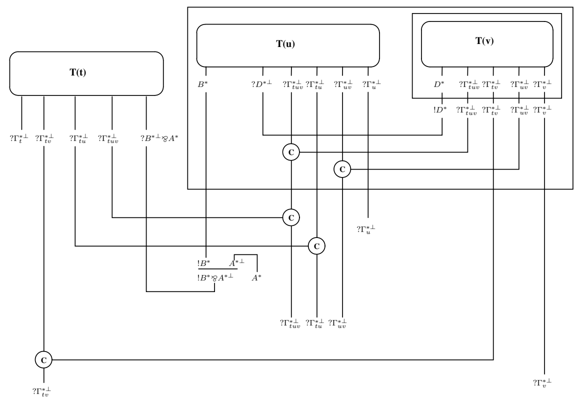
- For $s = (t\ u)[x/v] \rightarrow_{\texttt{App}_2} (t\ u[x/v]) = s'$, with $x \notin \texttt{fv}(t)$ & $x \in \texttt{fv}(u)$, coming from $\Gamma_{tuv}, \Gamma_{tu}, \Gamma_{tv}, \Gamma_t \vdash t : B \rightarrow A$ and $\Gamma_{tuv}, \Gamma_{tu}, \Gamma_{uv}, \Gamma_u, x : D \vdash u : B$ and $\Gamma_{tuv}, \Gamma_{tv}, \Gamma_{uv}, \Gamma_v \vdash v : D$, where $\Gamma_{tuv} := \texttt{fv}(t) \cap \texttt{fv}(u) \cap \texttt{fv}(u)$, $\Gamma_{tu} := \texttt{fv}(t) \cap \texttt{fv}(u) \setminus x \setminus \texttt{fv}(v)$, $\Gamma_{tv} := \texttt{fv}(t) \cap \texttt{fv}(v) \setminus \texttt{fv}(u)$, $\Gamma_{uv} := \texttt{fv}(u) \cap \texttt{fv}(v) \setminus \texttt{fv}(t)$, $\Gamma_t := \texttt{fv}(t) \setminus \texttt{fv}(u) \setminus \texttt{fv}(v)$, $\Gamma_u := \texttt{fv}(u) \setminus \texttt{fv}(t) \setminus \texttt{fv}(v)$ and $\Gamma_v := \texttt{fv}(v) \setminus \texttt{fv}(t) \setminus \texttt{fv}(u)$. The proof-net $T(s)$ is given by



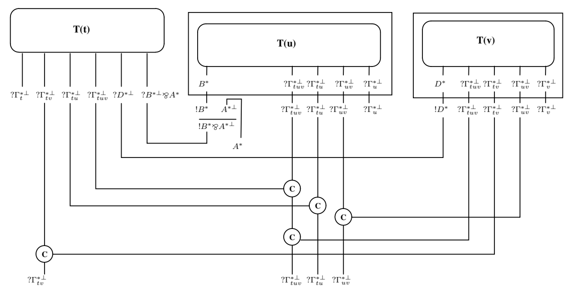
which reduces by $\rightarrow_{\texttt{b-b}}$ to the proof-net



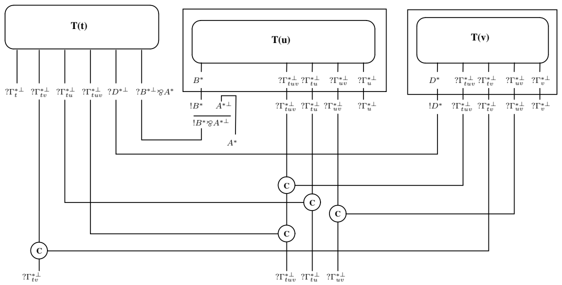
which is equivalent via $\sim_E$ to the proof-net $T(s')$



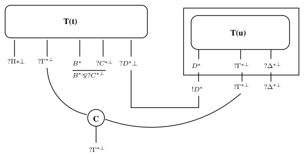
- For $s = (t\ u)[x/v] \rightarrow_{\texttt{App}_3} (t[x/v]\ u) = s'$, with $x \in \texttt{fv}(t)$ & $x \notin \texttt{fv}(u)$, coming from $\Gamma_{tuv}, \Gamma_{tu}, \Gamma_{tv}, \Gamma_t, x : D \vdash t : B \rightarrow A$ and $\Gamma_{tuv}, \Gamma_{tu}, \Gamma_{uv}, \Gamma_u \vdash u : B$ and $\Gamma_{tuv}, \Gamma_{tv}, \Gamma_{uv}, \Gamma_v \vdash v : D$, where $\Gamma_{tuv} := \texttt{fv}(t) \cap \texttt{fv}(u) \cap \texttt{fv}(u)$, $\Gamma_{tu} := \texttt{fv}(t) \cap \texttt{fv}(u) \setminus x \setminus \texttt{fv}(v)$, $\Gamma_{tv} := \texttt{fv}(t) \cap \texttt{fv}(v) \setminus \texttt{fv}(u)$, $\Gamma_{uv} := \texttt{fv}(u) \cap \texttt{fv}(v) \setminus \texttt{fv}(t)$, $\Gamma_t := \texttt{fv}(t) \setminus \texttt{fv}(u) \setminus \texttt{fv}(v)$, $\Gamma_u := \texttt{fv}(u) \setminus \texttt{fv}(t) \setminus \texttt{fv}(v)$ and $\Gamma_v := \texttt{fv}(v) \setminus \texttt{fv}(t) \setminus \texttt{fv}(u)$. The proof-net $T(s)$ is given by



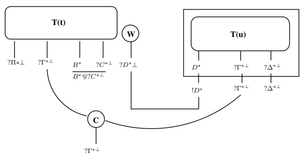
which is equivalent via $\sim_E$ to the proof-net $T(s')$



- For $s = (\lambda y.t)[x/u] \to_{\mathtt{Lamb}} \lambda y.t[x/u] = s'$, with $x \in \mathtt{fv}(\lambda y.t)$, coming from $\Pi, \Gamma, x : D \vdash \lambda y.t : B \to C$ and $\Gamma, \Delta \vdash u : D$ where $\Gamma := \mathtt{fv}(\lambda y.t) \cap \mathtt{fv}(u)$ and $\Pi := \mathtt{fv}(\lambda y.t) \setminus x \setminus \mathtt{fv}(u)$ and $\Delta := \mathtt{fv}(u) \setminus \mathtt{fv}(\lambda y.t)$. We show here the case $y \in \mathtt{fv}(t)$, the case $y \notin \mathtt{fv}(t)$ being similar. We have exactly the same interpretation $T(\_)$ for both terms s and s' which is given by the proof-net:

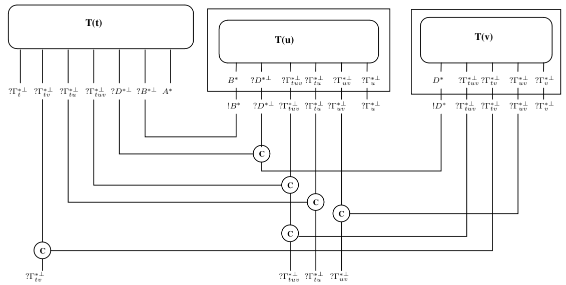


- For $s = (\lambda y.t)[x/u] \to_{\mathtt{Lamb}} \lambda y.t[x/u] = s'$, where $x \notin \mathtt{fv}(\lambda y.t)$, coming from $\Pi, \Gamma \vdash \lambda y.t : B \to C$ and $\Gamma, \Delta \vdash u : D$ where $\Gamma := \mathtt{fv}(\lambda y.t) \cap \mathtt{fv}(u)$ and $\Pi := \mathtt{fv}(\lambda y.t) \setminus \mathtt{fv}(u)$ and $\Delta := \mathtt{fv}(u) \setminus \mathtt{fv}(\lambda y.t)$. We show here the case $y \in \mathtt{fv}(t)$, the case $y \notin \mathtt{fv}(t)$ being similar. We have exactly the same interpretation $T(\_)$ for both terms s and s' which is given by the following proof-net.

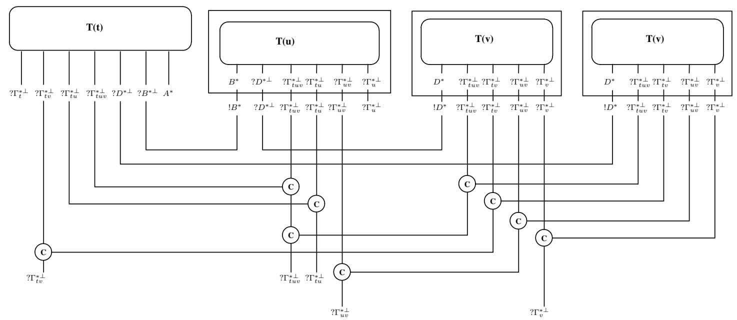


- For $s = t[x/u][y/v] \to_{\mathtt{Comp}_1} t[y/v][x/u[y/v]] = s'$, with $y \in \mathtt{fv}(t)$ & $y \in \mathtt{fv}(u)$. We show here the case $x \in \mathtt{fv}(t)$, the case $x \notin \mathtt{fv}(t)$ being similar. Thus, $\Gamma \vdash s : A$ comes from $\Gamma_{tuv}, \Gamma_{tu}, \Gamma_{tv}, \Gamma_t, x : B, y : D \vdash t : A$ and $\Gamma_{tuv}, \Gamma_{tu}, \Gamma_{uv}, \Gamma_u, y : D \vdash u : B$ and $\Gamma_{tuv}, \Gamma_{tv}, \Gamma_{uv}, \Gamma_v \vdash v : D$, where $\Gamma_{tuv} := \mathtt{fv}(t) \cap \mathtt{fv}(u) \cap \mathtt{fv}(u)$, $\Gamma_{tu} := \mathtt{fv}(t) \cap \mathtt{fv}(u) \setminus y \setminus \mathtt{fv}(v)$, $\Gamma_{tv} := \mathtt{fv}(t) \cap \mathtt{fv}(v) \setminus \mathtt{fv}(u)$, $\Gamma_{uv} := \mathtt{fv}(u) \cap \mathtt{fv}(v) \setminus \mathtt{fv}(t)$, $\Gamma_t := \mathtt{fv}(t) \setminus \mathtt{fv}(u) \setminus \mathtt{fv}(v)$, $\Gamma_u := \mathtt{fv}(u) \setminus \mathtt{fv}(t) \setminus \mathtt{fv}(v)$ and $\Gamma_v := \mathtt{fv}(v) \setminus \mathtt{fv}(t) \setminus \mathtt{fv}(u)$.

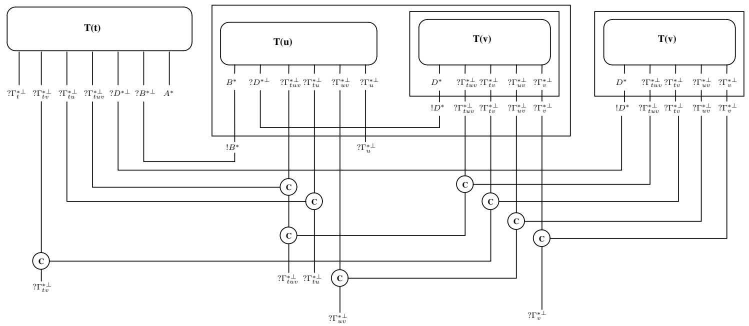
This case is similar to $\mathtt{App}_1$. The proof-net $T(s)$ is given by



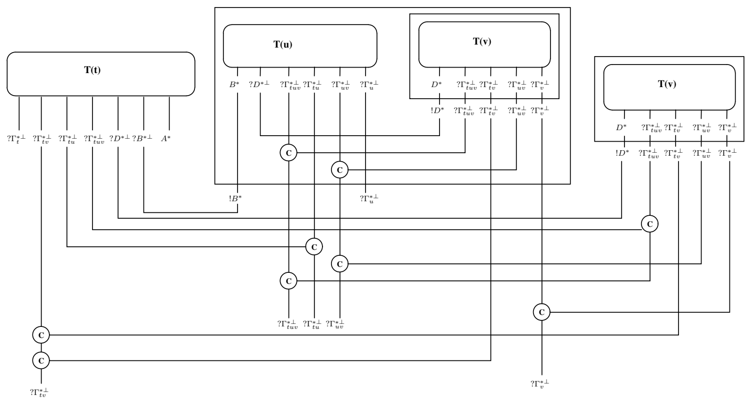
which reduces by $\to_{\mathtt{c\text{-}b}}$ to the proof-net



which reduces by $\to_{\mathtt{b\text{-}b}}$ to the proof-net

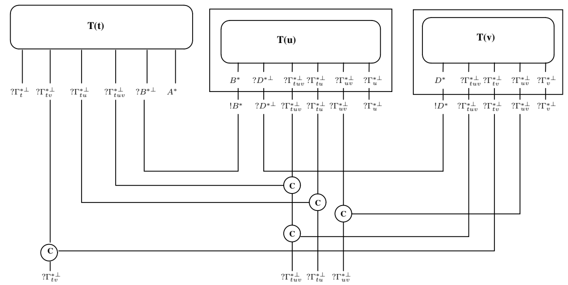


which is equivalent via $\sim_E$ to the proof-net $T(s')$

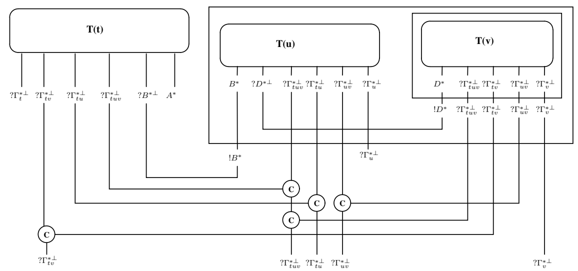


- $s = t[x/u][y/v] \to_{\texttt{Comp}_2} t[x/u[y/v]] = s'$, with $y \notin \texttt{fv}(t)$ & $y \in \texttt{fv}(u)$. We show here the case $x \in \texttt{fv}(t)$, the case $x \notin \texttt{fv}(t)$ being similar. Thus, $\Gamma \vdash s : A$ comes from $\Gamma_{tuv}, \Gamma_{tu}, \Gamma_{tv}, \Gamma_t, x : B \vdash t : A$ and $\Gamma_{tuv}, \Gamma_{tu}, \Gamma_{uv}, \Gamma_u, y : D \vdash u : B$ and $\Gamma_{tuv}, \Gamma_{tv}, \Gamma_{uv}, \Gamma_v \vdash v : D$, where $\Gamma_{tuv} := \texttt{fv}(t) \cap \texttt{fv}(u) \cap \texttt{fv}(u)$, $\Gamma_{tu} := \texttt{fv}(t) \cap \texttt{fv}(u) \setminus y \setminus \texttt{fv}(v)$, $\Gamma_{tv} := \texttt{fv}(t) \cap \texttt{fv}(v) \setminus \texttt{fv}(u)$, $\Gamma_{uv} := \texttt{fv}(u) \cap \texttt{fv}(v) \setminus \texttt{fv}(t)$, $\Gamma_t := \texttt{fv}(t) \setminus \texttt{fv}(u) \setminus \texttt{fv}(v)$, $\Gamma_u := \texttt{fv}(u) \setminus \texttt{fv}(t) \setminus \texttt{fv}(v)$ and $\Gamma_v := \texttt{fv}(v) \setminus \texttt{fv}(t) \setminus \texttt{fv}(u)$.

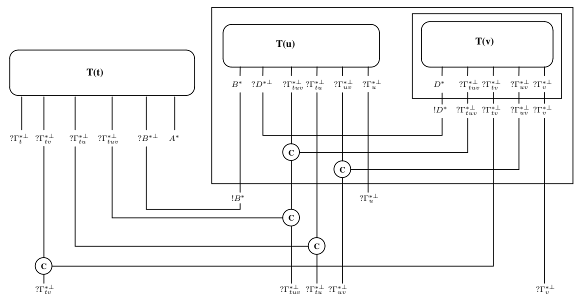
  This case is similar to $\texttt{App}_2$. The proof-net $T(s)$ is given by



which reduces by $\to_{\texttt{b-b}}$ to the proof-net



which is equivalent via $\sim_E$ to the proof-net $T(s')$



We now consider the cases where $s \to_{\lambda\mathtt{s}} s'$ is an internal reduction step.

- If $s =_{\mathtt{C}} s'$ or $s \to_{\mathtt{App_3},\mathtt{Lamb}} s'$ then the property trivially holds since $\sim_E$ is a congruence.
- If $s \to_{\mathtt{Bs}\setminus\{\mathtt{App_3},\mathtt{Lamb}\}} s'$ is $\lambda x.t \to \lambda x.t'$ or $t\ u \to t'\ u$ or $t[x/u] \to t'[x/u]$ coming from $t \to t'$, then we obtain $T(t) \to^+_{R/E} C[T(t')]$ by i.h. and the property holds by the fact that the context $C[\ ]$ of weakening wires surrounding $T(t')$ can also be considered as a context of weakening wires surrounding $T(s')$.
- If $s \to_{\mathtt{Bs}\setminus\{\mathtt{App_3},\mathtt{Lamb}\}} s'$ is $u\ t \to u\ t'$ or $u[x/t] \to u[x/t']$ coming from $t \to t'$, then we obtain $T(t) \to^+_{R/E} C[T(t')]$ by i.h. and the property holds by the fact that the context $C[\ ]$ of weakening wires surrounding $T(t')$ can be pushed outside the box containing $T(t')$ by using the rule $\to_{\mathtt{V}}$ in order to obtain a context of weakening wires surrounding $T(s')$.

∎

Remark that the only case where we get a non empty context in Lemma 0.1 is when simulating the rule `Gc`. This is because `Gc` is the only rule which looses free variables, all the other ones preserve the same set of free variables.

Corollary 0.2 (SN for $\lambda\mathtt{s}$-typed terms) *If $\Gamma \vdash_{\lambda\mathtt{s}} t : A$, then $t \in \mathcal{SN}_{\lambda\mathtt{s}}$.*

Proof. We can apply the abstract theorem 0.3 : $\mathcal{E}$ is $\mathtt{C}$, $\mathcal{R}_1$ is the relation $\to_{\mathtt{App_3},\mathtt{Lamb}}$ (for which we can trivially show that $\to_{\mathtt{App_3},\mathtt{Lamb}} / =_{\mathcal{E}}$ is well-founded), $\mathcal{R}_2$ is the relation $\to_{\mathtt{es}\setminus\{\mathtt{App_3},\mathtt{Lamb}\}}$, $\mathtt{K}$ is the relation given by the translation $T(\_)$, $\mathcal{S}$ is the reduction relation R/E on MELL proof-nets (which is well-founded Polonovski), and properties **(ES), (WS), (SS)** hold by Lemma 0.1. ∎

An abstract theorem

Theorem 0.3 *Let $\mathcal{O}$ and $\mathcal{P}$ be two sets. Let $\mathcal{R}_1, \mathcal{R}_2$ be two relations on $\mathcal{O} \times \mathcal{O}$, $\mathcal{S}$ be a relation on $\mathcal{P} \times \mathcal{P}$, $\mathtt{K}$ a relation $\subseteq \mathcal{O} \times \mathcal{P}$ and $\mathcal{E}$ an equivalence relation on $\mathcal{O}$ such that $\mathcal{R}_1/\mathcal{E}$ is well-founded. Suppose also*

(ES) *$t\ \mathcal{E}\ t'$ and $t\ \mathtt{K}\ T$ implies $t'\ \mathtt{K}\ T$*

(WS) *$t\ \mathcal{R}_1\ t'$ and $t\ \mathtt{K}\ T$ implies there is T' such that $t'\ \mathtt{K}\ T'$ and $T\ \mathcal{S}^*\ T'$*

(SS) *$t\ \mathcal{R}_2\ t'$ and $t\ \mathtt{K}\ T$ implies there is T' such that $t'\ \mathtt{K}\ T'$ and $T\ \mathcal{S}^+\ T'$*

Then, if $t\ \mathtt{K}\ T$ and $\mathcal{S}$ is a well-founded relation on T, then $(\mathcal{R}_1 \cup \mathcal{R}_2)/\mathcal{E}$ is well-founded on t.

Proof. Suppose $(\mathcal{R}_1 \cup \mathcal{R}_2)/\mathcal{E}$ is not well-founded on t. Since $\mathcal{R}_1/\mathcal{E}$ is well-founded by hypothesis, there is an infinite sequence on $\mathcal{O}$ where $\mathcal{R}_2/\mathcal{E}$ occurs infinitely many times so it is of the form

$$t\ \ldots (\mathcal{R}_2/\mathcal{E})\ t_1 \ldots (\mathcal{R}_2/\mathcal{E})\ t_2 \ldots (\mathcal{R}_2/\mathcal{E})\ t_i \ldots$$

that is,

$$t\ (\mathcal{R}_1/\mathcal{E})^*\ \mathcal{E}\ \mathcal{R}_2\ \mathcal{E}\ t_1 (\mathcal{R}_1/\mathcal{E})^*\ \mathcal{E}\ \mathcal{R}_2\ \mathcal{E}\ t_2 \ldots (\mathcal{R}_1/\mathcal{E})^*\ \mathcal{E}\ \mathcal{R}_2\ \mathcal{E}\ t_i \ldots$$

But $t_j\ \mathtt{K}\ T_j$ and $t_j (\mathcal{R}_1/\mathcal{E})^*\ \mathcal{E}\ \mathcal{R}_2\ \mathcal{E}\ t_{j+1}$ imply, by **(ES)**, **(WS)** and **(SS)** , that there is T_{j+1} s.t. $t_{j+1}\ \mathtt{K}\ T_{j+1}$ and $T_j\ \mathcal{S}^+\ T_{j+1}$. Thus, there are $T_1, T_2, \ldots, T_i, \ldots \in \mathcal{P}$ such that $t_1\ \mathtt{K}\ T_1, t_2\ \mathtt{K}\ T_2, \ldots, t_i\ \mathtt{K}\ T_i, \ldots$ and the following infinite $\mathcal{S}$-reduction sequence exists

$$T\ \mathcal{S}^+\ T_1\ \mathcal{S}^+\ T_2\ \mathcal{S}^+\ \ldots \mathcal{S}^+\ T_i\ \ldots$$

This leads to a contradiction with the fact that $\mathcal{S}$ is well-founded on T. ∎