Some properties of realizability models

Jean-Louis Krivine

PPS Group, University Paris-Diderot, CNRS
krivine@pps.univ-paris-diderot.fr

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Outline

We give some general properties of classical realizability and we look at some particular models:

- True arithmetical formulas, and even true $\Pi^1_1$ formulas are realized; thus, realizability models cannot give undecidability results in arithmetic.
- A model is given by forcing iff its Boolean algebra $\mathbb{B}$ is trivial.
- We build models in which $\mathbb{B}$ is non trivial and finite.
- Following T. Ehrhard and T. Streicher, the usual models of lambda-calculus have, in fact, a structure of realizability algebra. Therefore, they give rise to realizability models of ZF.

We study a simple case, in which $\mathbb{B}$ is non trivial and integers are preserved.
A game on first order formulas

We consider first order formulas written with: 
→, ∀, ⊤, ⊥, ≠, predicate constants, function symbols for recursive functions. 

A 1st order formula has the form ∀⃗x[Φ1,...,Φn → A] where Φ1,...,Φn are 1st order formulas and A is atomic (i.e. Rt1...tk or t0 ≠ t1 or ⊤ or ⊥).

In the following, we only consider closed 1st order formulas.

The atomic closed formula t0 ≠ t1 is interpreted as ⊤ (resp. ⊥) if it is true (resp. false) in ℤ.

We define a game with two players: ∃ (the client) and ∀ (the server).

At each step, the position is a sequent U ⊢ A with closed 1st order formulas; the formulas of A are atomic and ⊥ ∈ A; U and A increase at each step.

The game starts with a sequent U₀ ⊢ A₀.
A move in this game is as follows:
Player $\exists$ chooses $\Psi \in \mathcal{U}$, $\Psi = \forall \vec{y}[\Phi_1(\vec{y}), \ldots, \Phi_n(\vec{y}) \rightarrow B(\vec{y})]$ and $\vec{j} \in \mathbb{N}^l$ such that $B(\vec{j}) \in \mathcal{A}$ (if this is impossible, then $\exists$ has lost).
Player $\forall$ chooses a formula $\Phi \in \mathcal{V} = \{\Phi_1(\vec{j}), \ldots, \Phi_n(\vec{j})\}$,
$\Phi \equiv \forall \vec{x}[\Psi_1(\vec{x}), \ldots, \Psi_m(\vec{x}) \rightarrow A(\vec{x})]$; $\forall$ chooses also $\vec{i} \in \mathbb{N}^k$.
The atomic formula $A(\vec{i})$ must not be $\top$ (otherwise, $\forall$ has lost).
Then $\Psi_1(\vec{i}), \ldots, \Psi_m(\vec{i})$ are added to $\mathcal{U}$ and $A(\vec{i})$ is added to $\mathcal{A}$.
$\exists$ wins iff $\forall$ cannot play at some step
(every formula of $\mathcal{V}$ ends with $\top$, in particular if $\mathcal{V} = \emptyset$).
In fact, player $\forall$ tries to build a model over $\mathbb{N}$ in which
the formula $\mathcal{V}_0 = \bigwedge \mathcal{U}_0 \rightarrow \bigvee \mathcal{A}_0$ is false, and $\exists$ tries to avoid this:
**Theorem.** i) Any model $\mathcal{M}$ over $\mathbb{N}$ s.t. $\mathcal{M} \not\models \mathcal{V}_0$ gives a winning strategy for $\forall$.

ii) There exists a “trivial” strategy for the player $\exists$ such that each play $\exists$ loses using it, gives a model $\mathcal{M}$ over $\mathbb{N}$, $\mathcal{M} \not\models \mathcal{V}_0$.

i) We define a strategy for $\forall$ such that, at each step:

   every formula of $\mathcal{U}$ (resp. $\mathcal{A}$) is true (resp. false) in $\mathcal{M}$.

This is true at the beginning of the game.

Then $\exists$ chooses $\Psi \in \mathcal{U}$, $\Psi = \forall \vec{y}[\Phi_1(\vec{y}), \ldots, \Phi_n(\vec{y}) \rightarrow B(\vec{y})]$ and $\vec{j} \in \mathbb{N}^l$ such that $B(\vec{j}) \in \mathcal{A}$. Therefore, $\mathcal{M} \models \neg B(\vec{j})$ and $\mathcal{M} \models \Psi$.

Thus, $\forall$ can choose $\Phi \in \mathcal{V} = \{\Phi_1(\vec{j}), \ldots, \Phi_n(\vec{j})\}$ s.t. $\mathcal{M} \models \neg \Phi$.

Let $\Phi = \forall \vec{x}[\Psi_1(\vec{x}), \ldots, \Psi_m(\vec{x}) \rightarrow A(\vec{x})]$.

Then $\forall$ can choose $\vec{i} \in \mathbb{N}^k$ s.t. $\mathcal{M} \models \Psi_1(\vec{i}), \ldots, \Psi_m(\vec{i})$ and $\neg A(\vec{i})$.

Finally $\Psi_1(\vec{i}), \ldots, \Psi_m(\vec{i})$ are added to $\mathcal{U}$ and $A(\vec{i})$ to $\mathcal{A}$.

Thus $\mathcal{U}$ and the negation of formulas of $\mathcal{A}$ remain true in $\mathcal{M}$. 


ii) Here is the “trivial” strategy for $\exists$:

*fix an enumeration of all ordered pairs $<\Psi, \vec{j}>$ ($\Psi$ is a closed formula, $\vec{j} \in \mathbb{N}$).

*At each step, $\exists$ chooses the first allowed pair $<\Psi, \vec{j}>$, not chosen before.*

Suppose $\exists$ loses some play with this strategy. Let $\mathcal{M}$ be the model which satisfies exactly the closed atomic formulas never put in $\mathcal{A}$ during this play.

A pair $<\Psi, \vec{j}>$ is called *acceptable* if $\Psi$ is put in $\mathcal{U}$ and $B(\vec{j})$ in $\mathcal{A}$ at some step (not necessarily the same) where $B(\vec{y})$ is the final atom of $\Psi$.

Every acceptable pair is effectively played by $\exists$ at some step: namely when every acceptable pair strictly less than it has been played.

We prove, by induction, that $\mathcal{M}$ satisfies *every formula $\Psi$ which is put in $\mathcal{U}$* and the negation of *every formula $\Phi$ chosen by $\forall$* during the play.
Proof for $\Psi$. The result is clear if $\Psi$ is atomic because, if $\Psi$ is both in $\mathcal{U}$ and $\mathcal{A}$ then $<\Psi, \varnothing>$ is acceptable and thus will be chosen by $\exists$; then $\exists$ wins.
Otherwise, let $\Psi = \forall \vec{y}[\Phi_1(\vec{y}), \ldots, \Phi_n(\vec{y}) \rightarrow B(\vec{y})]$. We must show that $\mathcal{M} \models \Phi_1(\vec{j}), \ldots, \Phi_n(\vec{j}) \rightarrow B(\vec{j})$ for every $\vec{j} \in \mathbb{N}^k$.
This is clear if $B(\vec{j})$ is never put in $\mathcal{A}$, because $\mathcal{M} \models B(\vec{j})$.
Otherwise, $<\Psi, \vec{j}>$ is acceptable and is chosen by $\exists$ at some step.
Then $\mathcal{V} = \{\Phi_1(\vec{j}), \ldots, \Phi_n(\vec{j})\}$ and $\Phi_1(\vec{j})$, for instance, is chosen by $\forall$.
By induction hypothesis, we have $\mathcal{M} \models \neg \Phi_1(\vec{j})$, which gives the result.

Proof for $\Phi$. Let $\Phi = \forall \vec{x}[\Psi_1(\vec{x}), \ldots, \Psi_m(\vec{x}) \rightarrow A(\vec{x})]$; $\forall$ chooses $\vec{i}$
and puts $A(\vec{i})$ in $\mathcal{A}$ and $\Psi_1(\vec{i}), \ldots, \Psi_m(\vec{i})$ in $\mathcal{U}$. By induction hypothesis, $\mathcal{M} \models \Psi_1(\vec{i}), \ldots, \Psi_m(\vec{i})$; and, by definition, $\mathcal{M} \not\models A(\vec{i})$. Thus $\mathcal{M} \models \neg \Phi$.
It follows that $\mathcal{M} \not\models \mathcal{V}_0$ since $\mathcal{M} \models \mathcal{U}_0$ and $\mathcal{M} \models \neg A$ for $A \in \mathcal{A}_0$.  

QED
Well founded recursive relations

Let $f : \mathbb{N}^2 \to \{0,1\}$ be arbitrary. The predicate $f(x, y) = 1$ is well founded iff the formula $\forall X \forall z \{ \forall x [ \forall y (f(x,y) = 1 \rightarrow Xy) \rightarrow Xx] \rightarrow Xz \}$ is true in $\mathbb{N}$. We show that, in this case, this formula is even realized.

**Theorem.** If the predicate $f(x,y) = 1$ is well founded, then $Y \models \forall X \forall z \{ \forall x [ \forall y (f(x,y) = 1 \rightarrow Xy) \rightarrow Xx] \rightarrow Xz \}$.

Let $t \models \forall x [ \forall y (f(x,y) = 1 \rightarrow Xy) \rightarrow Xx]$ and $n \in \mathbb{N}$; we show by induction on $n$, following the well founded predicate “$f(n,y) = 1$”, that $Y t \models Xn$.

Since $Yt \ast \pi > t \ast Yt \cdot \pi$, it suffices to show that $Y t \mid \forall y (f(n,y) = 1 \rightarrow Xy)$ i.e. $Y t \models f(n,p) = 1 \rightarrow Xp$. This is trivial if $f(n,p) \neq 1$ and this follows from the induction hypothesis if $f(n,p) = 1$.

Thus, if $\pi \in \| Xn \|$, we have $t \ast Yt \cdot \pi \in \perp$ and therefore $Y \ast t \cdot \pi \in \perp$. **QED**

This shows that a *recursive* well founded predicate on integers is also well founded *in every realizability model.*
**True $\Pi^1_1$ formulas**

A $\Pi^1_1$ formula is of the form $F \equiv \forall \vec{X} \Phi[\vec{X}]$ where $\Phi$ is a 1st order formula written with the function symbols $0, 1, +, \times$ and the predicate symbols $\neq, \vec{X}$.

**Theorem.** If $F$ is a true $\Pi^1_1$ formula, then $F^{\text{int}}$ is realized.

This shows, in particular, that the integers of any realizability model are *elementary equivalent* to the integers of the ground model.

It is not possible to show the independence of some *arithmetical* (and even $\Pi^1_1$) formula by means of realizability models.

Open problems: What about $\Sigma^1_1$ (or higher) formulas?

Are the *constructible universes* of the ground model and the realizability model elementarily equivalent? This is (trivially) true in the case of forcing.
**Proof.** Fix a recursive enumeration of closed formulas and also of sequents $\mathcal{U} \vdash A$. Let $F \equiv \forall \bar{X} \neg \Phi[\bar{X}]$ be a true $\Pi^1_1$ formula.

The meaning of $F$ is that the 1st order formula $\Phi \rightarrow \bot$ has no model.

Thus, *the “trivial” strategy for $\exists$ is winning* in the game which starts with the sequent $\Phi \vdash \bot$.

Now, let $f(x, y) = 1$ be the recursive predicate which says that $x, y$ are (numbers of) successive positions chosen by $\forall$ such that, between them, $\exists$ has applied (once) the trivial strategy.

This strategy is winning for $\exists$ iff each play is finite, i.e. iff *the predicate $f(x, y) = 1$ is well founded*.

Now, by the above theorem, we obtain:

$\mathcal{Y} \parallel \forall X \{ \forall x [ \forall y (f(x, y) = 1 \leftrightarrow X y) \rightarrow X x] \rightarrow \forall x X x \}$. 

But we have just proved that: "$f(x, y) = 1$ is well founded" $\rightarrow F$.

Let $\theta$ be a proof-like term associated with this proof. Then $\theta \mathcal{Y} \parallel \vdash F$.  

QED
The case of arithmetical formulas

An arithmetical formula is of the form
\[ \forall x_1 \exists y_1 \ldots \forall x_n \exists y_n (f(x_1, y_1, \ldots, x_n, y_n) \neq 0) \]
where \( f : \mathbb{N}^{2n} \rightarrow \{0, 1\} \) is recursive.

**Theorem.** Let \( f : \mathbb{N}^{2n} \rightarrow \{0, 1\} \) be an arbitrary function, such that
\[ \forall x_1 \exists y_1 \ldots \forall x_n \exists y_n (f(x_1, y_1, \ldots, x_n, y_n) \neq 0) \] is true in \( \mathbb{N} \). Then
\[ \forall x_1 \exists y_1^{\text{int}} \ldots \forall x_n \exists y_n^{\text{int}} (f(x_1, y_1, \ldots, x_n, y_n) \neq 0) \]
is realized
by a proof-like term that depends only on \( n \).

This theorem shows once again that any true arithmetical formula is realized.
For $n = 1$, the proof is very simple:

**Theorem.** Let $\theta \in \text{QP}$ be such that $\theta \star \underline{n} \cdot \xi \cdot \pi > \xi \star \underline{n} \cdot \theta \underline{n^+} \xi \cdot \pi$ with \(n^+ = (s)n\).

Then $\theta_0 \parallel \forall x \left(\forall \int(f(x, y) \neq 0 \rightarrow \bot) \rightarrow \bot\right)$

for every $f : \mathbb{N}^2 \rightarrow 2$ such that $\mathbb{N} \models \forall x \exists y(f(x, y) = 1)$.

We simply need to prove $\theta_0 \parallel \forall \int(f(y) \neq 0 \rightarrow \bot) \rightarrow \bot$

for every $f : \mathbb{N} \rightarrow 2$ such that $\mathbb{N} \models \exists y(f(y) = 1)$.

**Lemma.** Let $\xi \models \forall \int(f(y) \neq 0 \rightarrow \bot)$; if $\theta \underline{n} \xi \not\models \bot$, then $f(n) = 0$ and $\theta \underline{n^+} \xi \not\models \bot$.

We have $\theta \star \underline{n} \cdot \xi \cdot \pi \not\in \bot$, thus $\xi \star \underline{n} \cdot \theta \underline{n^+} \xi \cdot \pi \not\in \bot$;

therefore $\theta \underline{n^+} \xi \not\models f(n) \neq 0$ hence the result. QED

Suppose $\theta \star \underline{0} \cdot \xi \cdot \pi \not\in \bot$; the lemma gives $f(n) = 0$ for all $n \in \mathbb{N}$, a contradiction. QED
We consider now the case \( n = 2 \), which is typical for the general case.

**Theorem.** Let \( \theta = \lambda x \lambda t \lambda \sigma \lambda m \lambda n (xm) \lambda y (H \sigma m y n)((t)(\Sigma) \sigma m y) m' n' \) where \( H, \Sigma \) are closed \( \lambda \)-terms defined below ; \( <m', n'> \) is the successor of \( <m, n> \) in \( \mathbb{N}^2 \).

Then, for every \( f : \mathbb{N}^3 \rightarrow \{0, 1\} \), there exists \( \phi : \mathbb{N} \rightarrow \mathbb{N} \) such that :

\[
\lambda x((Y)(\theta)x)000 \not\models \forall x \exists y \forall z (f[x, y, z] = 1) \rightarrow \forall x \forall z (f[x, \phi x, z] \neq 0).
\]

**Definition of** \( H, \Sigma \). The variables \( m, n \) represent integers ; \( \eta \) an arbitrary term ; the variable \( \sigma \) represents a finite sequence of ordered pairs \( <m, \eta> \).

If no pair \( <m, \cdot> \) is in \( \sigma \), set \( \Sigma m \eta = \sigma \rightarrow <m, \eta>, H \sigma m \eta = \eta. \)

Else, set \( \Sigma m \eta = \sigma ; H \sigma m \eta = \zeta \) for the first \( <m, \zeta> \) appearing in \( \sigma \).

**Proof by contradiction.** Suppose \( \xi \not\models \forall x \exists y \forall z (f[x, y, z] = 1) ; ((Y)(\theta)\xi)000 \not\models \bot \) and \( f[x_0, \phi x_0, z_0] = 0. \)

We show, by recurrence on \( <m, n> \leq <x_0, z_0> \), that \( ((Y)(\theta)\xi)\sigma m n m n \not\models \bot \), with \( \sigma m n, \eta m n, b m n \) defined by recurrence ; it’s true for \( \sigma_{00} = 0 \). If it’s true for \( <m, n> \) we have \( ((Y)(\theta)\xi)\sigma m n m n \triangleright \pi \not\models \bot \), i.e. \( \theta \xi \triangleright (Y)(\theta)\xi \cdot \sigma m n \cdot m \cdot n \cdot \pi \not\models \bot \), or else :
\( \xi \star m \cdot \lambda y (H \sigma_{mn} m y n) ((Y)(\theta)\xi)(\Sigma) \sigma_{mn} m y \cdot \pi \notin \bot \). Thus, there exists \( b_{mn} \) s.t.: \\
\( \lambda y (H \sigma_{mn} m y n) ((Y)(\theta)\xi)(\Sigma) \sigma_{mn} m y \cdot \pi \notin \bot \) and thus there exists \( \eta_{mn} \| - \forall z^{\text{int}} (f [m, b_{mn}, z] = 1) \) such that 

\((*)\) \( H \sigma_{mn} m \eta_{mn} \star n \cdot ((Y)(\theta)\xi)(\Sigma) \sigma_{mn} m n \eta_{mn} m' n' \cdot \pi \notin \bot \). 

**Definition of \( \phi m \):** i) if no pair \( <m, \_> \) appears in \( \sigma_{mn} \) then set \( \phi [m] = b_{mn} \); ii) else, let \( <m, \eta_{mq}> \) be the first (indeed only) pair \( <m, \_> \) appearing in \( \sigma_{mn} \); then, set \( \phi [m] = b_{mq} \). Now, \( H \sigma_{mn} m \eta_{mn} \| - \forall z^{\text{int}} (f (m, \phi m, z) = 1) \) because:

in case (i) \( H \sigma_{mn} m \eta_{mn} = \eta_{mn} \) and \( \phi m = b_{mn} \); in case (ii), by induction on \( <m, n> \) since \( H \sigma_{mn} m \eta_{mn} = \eta_{mq} \) with \( <m, q> \) strictly before \( <m, n> \).

Thus \( H \sigma_{mn} m \eta_{mn} n \| - f (m, \phi m, n) \neq 1 \rightarrow \bot \).

Now, we set \( \sigma_{m'n'} = (\Sigma) \sigma_{mn} m \eta_{mn} \).

Thus, by \((*)\), we have \( ((Y)(\theta)\xi) \sigma_{m'n'} m'n' \| - f (m, \phi m, n) \neq 1 \).

Therefore \( f [m, \phi m, n] = 1 \) and \( ((Y)(\theta)\xi) \sigma_{m'n'} m'n' \| - \bot \).

Since \( f [x_0, \phi x_0, z_0] = 0 \), we have a contradiction if \( <m, n> = <x_0, z_0> \).

Else, we have done the recurrence step. 

QED
Consider now a function $f : \mathbb{N}^4 \to \{0, 1\}$ s.t. $\mathbb{N} \models \forall u \exists x \forall y \exists z (f(u, x, y, z) = 0)$. This gives

$$\forall u \left( \forall x \exists y \forall z (f(u, x, y, z) \neq 0) \rightarrow \bot \right).$$

Thus, for every $u \in \mathbb{N}$ and $\phi : \mathbb{N} \to \mathbb{N}$ we get:

$$\| \forall x \forall z (f(u, x, \phi x, z) \neq 0) \| = \| \bot \| = \Pi.$$

It follows from the previous theorem that

$$\lambda x ((Y)(\theta)x)000 \models \forall u \neg \forall x^{\text{int}} \exists y \forall z^{\text{int}} (f(u, x, y, z) = 1)$$

which is the case $n = 2$ for arithmetical formulas.
Let \( \delta \) be a proof-like term s.t. \( \delta \vdash \forall x \exists y \, (x \neq 0, x \neq 1 \rightarrow \bot) \) (i.e. \( \exists \) is trivial).

We have \( \delta \in |\top, \bot \rightarrow \bot| \cap |\bot, \top \rightarrow \bot| \). Let \( \delta' = \lambda x \lambda y cc \lambda k(\delta)((k)x)(k)y \); then

\[
\xi \star \pi \in \bot \text{ or } \eta \star \pi \in \bot \Rightarrow \delta' \star \xi \cdot \eta \cdot \pi \in \bot
\]

Thus, \( \delta' \vdash X, Y \rightarrow X \) and \( \delta' \vdash X, Y \rightarrow Y \) for every truth values \( X, Y \).

**Theorem.** \( (\exists \Phi \in \text{QP})(\forall \theta \in \text{QP})(\forall X \subset \Pi)(\theta \vdash X \Rightarrow \Phi \vdash X) \).

Define \( e \) (read eval) by the following program:

\[
e_0 = B, \ e_1 = C, \ e_2 = E, \ e_3 = I, \ e_4 = K, \ e_5 = W, \ e_6 = cc, \ e_7 = \delta;
\]

\[
e_{n+8} = ((e)(p_0)n)(e)(p_1)n;
\]

where \( p_0, p_1 \) define a recursive bijection from \( \mathbb{N} \) onto \( \mathbb{N}^2 \).

For every \( \theta \in \text{QP} \), there is an integer \( n \) s.t. \( e_n > \theta \).

Now define \( \phi \) by: \( \phi \star n \cdot \pi \geq \delta' \star e_n \cdot (\phi)(s)n \cdot \pi \). Finally \( \Phi \) is \( \phi_0 \).

Let \( \theta \in \text{QP} \) s.t. \( \theta \vdash X \); thus, we have \( \phi_n \vdash X \) for some \( n \),

then \( \phi_{n-1} \vdash X, \ldots \); eventually \( \phi_0 \vdash X \).
Let $B = \mathcal{P}(\Pi)$ be the Boolean algebra of truth values. The order is defined by $A \leq B \iff (\exists \theta \in \text{QP})(\theta \models A \rightarrow B)$.

Thus, the order on $B$ is defined by $A \leq B \iff \Phi \models A \rightarrow B$.

**Theorem.** $B$ is a complete Boolean algebra:

If $B_i (i \in I)$ is a family of truth values, then $\inf_{i \in I} B_i = \bigcup_{i \in I} B_i$.

Let $A \leq B_i$ for $i \in I$. Then $\Phi \models A \rightarrow B_i$, thus $\Phi \models A \rightarrow \bigcup_{i \in I} B_i$.

Conversely $I \models \bigcup_{i \in I} B_i \rightarrow B_{i_0}$.

Thus, the realizability model is, in fact, a **forcing model**.

The converse is also true: in the case of forcing, the realizability algebra is a commutative idempotent monoid with a unity 1; then $\text{QP} = \{1\}$.

We have $1 \models X, Y \rightarrow X$ and $X, Y \rightarrow Y$; thus $\exists 2$ is trivial.
Theorem. Let d be a term such that:
If two out of three processes $\xi \star \pi, \eta \star \pi, \zeta \star \pi$ are in $\perp$, then $d \star \xi \cdot \eta \cdot \zeta \cdot \pi \in \perp$.
Then $d \models " \exists 2 \ has \ at \ most \ 4 \ elements."

We have $d \in |T, \top, \top \rightarrow \top \cap \top, T, \top \rightarrow \top \cap \top, \top \rightarrow \top|$. Thus $d \models \forall x \exists 2 \forall y \in \exists 2 (x \neq 0, y \neq 1, x \neq y \rightarrow xy \neq x)$ QED

We now build a model in which $\exists 2$ has exactly 4 elements.

The only term constants are the elementary combinators, $cc$ and a new constant d.
There are two stack constants $\pi^0, \pi^1$. Let $\omega = (WI)(W)I = (\lambda x xx)xx$.
For $i \in \{0, 1\}$, let $\Lambda^i$ (resp. $\Pi^i$) be the set of terms (resp. stacks)
which contain the only stack constant $\pi^i$. 
2 with 4 elements

For $i, j \in \{0, 1\}$, define $\perp^i_j$ as the least set $P \subset \Lambda^i \star \Pi^i$ of processes such that:

1. $\omega \star j \pi \in P$ for every $\pi \in \Pi^i$;
2. $\xi \star \pi \in \Lambda^i \star \Pi^i, \xi \star \pi > \xi' \star \pi' \in P \Rightarrow \xi \star \pi \in P$ ($P$ is saturated in $\Lambda^i \star \Pi^i$);
3. if 2 out of 3 processes $\xi \star \pi, \eta \star \pi, \zeta \star \pi$ are in $P$, then $d \star \xi \cdot \eta \cdot \zeta \cdot \pi \in P$.

We define $\perp$ by:

$$\Lambda \star \Pi \setminus \perp = \bigcup_{i \in \{0, 1\}} (\Lambda^i \star \Pi^i \setminus \perp^i_i)$$

In other words, a process is in $\perp$ iff either it is in $\perp^0_0 \cup \perp^1_1$ or it contains both stack constants $\pi^0, \pi^1$.

**Lemma.** If $\xi \star \pi \in \perp^i_j$ and $\xi \star \pi > \xi' \star \pi'$ then $\xi' \star \pi' \in \perp^i_j$ (closure by reduction).

Suppose $\xi_0 \star \pi_0 > \xi'_0 \star \pi'_0 ; \xi_0 \star \pi_0 \in \perp^i_j ; \xi'_0 \star \pi'_0 \notin \perp^i_j$;

We may suppose that $\xi_0 \star \pi_0 > \xi'_0 \star \pi'_0$ is exactly one step of execution.

Then $\perp^i_j \setminus \{\xi_0 \star \pi_0\}$ has properties 1,2,3 defining $\perp^i_j$; contradiction. QED
Lemma. $\bot_i \cap \bot_i = \emptyset$.

We prove that $\Lambda^i \ast \Pi^i \setminus \bot_i \supset \bot_0$ by showing properties 1, 2, 3.

1. $\omega \ast \pi^i \notin \bot_i$ because $\bot_i \setminus \{\omega \ast \pi^i\}$ has properties 1, 2, 3 defining $\bot_i$.

2. Follows from previous lemma.

3. Suppose $\xi \ast \pi, \eta \ast \pi \notin \bot_i$; then $d \ast \xi \ast \eta \ast \pi \notin \bot_i$ because $\bot_i \setminus \{d \ast \xi \ast \eta \ast \pi\}$ has properties 1, 2, 3 defining $\bot_i$.

QED

Theorem. This realizability model is coherent.

Let $\theta \in \text{QP}$ s.t. $\theta \ast \pi^0 \in \bot_0$ and $\theta \ast \pi^1 \in \bot_1$. Then $\theta \ast \pi^0 \in \bot_0 \cap \bot_1$.

QED

Remark. If $\pi \in \Pi \setminus (\Pi_0 \cup \Pi_1)$, then $\xi \ast \pi \in \bot$ for every term $\xi$.

Thus, we can remove these stacks and consider only $\Pi^0 \cup \Pi^1$. 
\(\mathcal{J}_2\) with 4 elements

We define two individuals in this realizability model:
\[
\gamma_0 = (\{0\} \times \Pi^0) \cup (\{1\} \times \Pi^1); \quad \gamma_1 = (\{1\} \times \Pi^0) \cup (\{0\} \times \Pi^1).
\]
Obviously, \(\gamma_0, \gamma_1 \subseteq \mathcal{J}_2 = \{0, 1\} \times \Pi\). Now we have:

\[
\|\forall x(x \notin \gamma_0)\| = \Pi^0 \cup \Pi^1 = \|\bot\|; \quad \omega_0 \vdash 0 \notin \gamma_0 \text{ et } \omega_1 \vdash 1 \notin \gamma_0.
\]

It follows that \(\gamma_0\) is not \(\varepsilon\)-empty and that every \(\varepsilon\)-element of \(\gamma_0\) is \(\neq 0,1\).

Thus the Boolean algebra \(\mathcal{J}_2\) is not trivial and has exactly 4 \(\varepsilon\)-elements.

We have \(\xi \vdash \forall x \mathcal{J}_2(x \varepsilon \gamma_0, x \varepsilon \gamma_1 \rightarrow \bot)\) for every term \(\xi\):

Indeed, \(|i \varepsilon \gamma_0| = \{k_{\pi} ; \pi \in \Pi^i\}\) for \(i = 0,1\) and \(\xi \star k_{\rho_0} \star k_{\rho_1} \cdot \pi \in \bot\) if \(\rho_i \in \Pi^i\).

It follows that \(\gamma_0, \gamma_1\) are the singletons of the \(\varepsilon\)-elements \(\neq 0,1\) of \(\mathcal{J}_2\).

**Remark.** *We can easily modify this construction in order to obtain for \(\mathcal{J}_2\) any finite Boolean algebra.*
Denotational semantics

T. Ehrhard has found a method which converts usual models of $\lambda$-calculus into realizability algebras, by defining stacks, $cc$ and $k_\pi$ in such models. The construction of stacks was also given by T. Streicher. We need to avoid parallel or, because we don’t want to get forcing models. Thus, our example will be the simplest coherent model of $\lambda$-calculus. Let us recall (one of) its construction.

Let $o$ be a fixed set which is not an ordered pair.

The set $V$ of formulas is the smallest set such that:
- $o \in V$;
- if $\alpha \in V$, $a \in \mathcal{P}_f(V)$ and $<a, \alpha> \neq <\emptyset, o>$ then $<a, \alpha> \in V$

($\mathcal{P}_f(V)$ is the set of finite subsets of $V$).

If $a \in \mathcal{P}_f(V)$ and $\alpha \in V$, we set $a \rightarrow \alpha = <a, \alpha>$ except that $(\emptyset \rightarrow o) = o$.

Every element of $V$ except $o$ is an ordered pair.

If $\alpha \in V$, its rank $r(\alpha)$ is the total number of $\rightarrow$ in $\alpha$. 


Each $\alpha \in V$ has a unique normal form $\alpha = (a_1, \ldots, a_k \rightarrow o)$ with $k \in \mathbb{N}, a_1, \ldots, a_k \in \mathcal{P}_f(V)$ and $a_k \neq \emptyset$. Then $\alpha = (a_1, \ldots, a_k, \emptyset, \ldots, \emptyset \rightarrow o)$. The truth value $|\alpha| \in \{0, 1\}$ of a formula $\alpha$ is defined by induction:

$|o| = 0$; $|a_1, \ldots, a_k \rightarrow o| = 1$ iff $(\exists \beta \in a_1 \cup \ldots \cup a_k)(|\beta| = 0)$.

If $\alpha = (a_1, \ldots, a_k \rightarrow o), \beta = (b_1, \ldots, b_k \rightarrow o)$ we define

$\alpha \cap \beta = (a_1 \cup b_1, \ldots, a_k \cup b_k \rightarrow o)$.

This operation is associative, commutative and idempotent; $o$ is neutral; it defines an order relation: $\alpha \leq \beta \iff b_1 \subset a_1, \ldots, b_k \subset a_k$.

Define a subset $D$ of $V$ (the web) by induction on the rank:

$(a_1, \ldots, a_k \rightarrow o) \in D$ iff, for $1 \leq i \leq k,$ $a_i \subset D$ and $(\forall \beta, \gamma \in a_i)(\beta \neq \gamma \Rightarrow \beta \cap \gamma \notin D)$ ($a_i$ is an antichain of $D$).

$D$ is a final segment of $V$: let $\alpha = (a_1, \ldots, a_k \rightarrow o), \beta = (b_1, \ldots, b_k \rightarrow o), \alpha \in D, \alpha \leq \beta$. Then $b_i \subset a_i$ and $a_i$ is an antichain of $D$, thus so is $b_i$.

$\alpha, \beta \in D$ are called compatible if $\alpha \cap \beta \in D$; in symbols $\alpha \simeq \beta$.

If $\alpha_1, \ldots, \alpha_n$ are pairwise compatible, then $\alpha_1 \cap \ldots \cap \alpha_n \in D$. 

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The realizability algebra

\( \Lambda_D \) is the set \( \mathcal{A}(D) \) of antichains of \( D \), i.e. \( t \subset D \) is a term iff
\[
(\forall \alpha, \beta \in t)(\alpha \cap \beta \in D \rightarrow \alpha = \beta).
\]
\( \Pi_D \) is the set \( \mathcal{S}(D) \) of filters of \( D \), i.e. \( \pi \subset D \) is a stack iff
\[
(\forall \alpha, \beta \in \pi) \alpha \cap \beta \in \pi ; \forall \alpha \forall \beta (\alpha \in \pi, \alpha \leq \beta \rightarrow \beta \in \pi) ; \ 0 \in \pi.
\]

**Remark.** \( \Pi_D \) can be identified with \( \Lambda_D^\mathbb{N} \) : a sequence of terms \( t_n(n \in \mathbb{N}) \) corresponds with the filter \( \{(a_0, \ldots, a_k \rightarrow 0) ; k \in \mathbb{N}, a_0 \subset t_0, \ldots, a_k \subset t_k\} \).

\( \Lambda_D \star \Pi_D \) is \( \{0, 1\} \) and \( \bot \) is \( \{1\} \).

If \( t \in \Lambda_D, \pi \in \Pi_D \) then \( t \star \pi \in \bot \) iff \( t \cap \pi \neq \emptyset \) (i.e. \( t \cap \pi \) is a singleton).

\( t \cdot \pi = \{a \rightarrow \alpha ; a \subset t, \alpha \in \pi\} \);

\( tu = \{\alpha \in D ; (\exists a \subset u)(a \rightarrow \alpha) \in t\} \);

\( K \) is the set of all formulas : \( \{\alpha\}, \emptyset \rightarrow \alpha \) for \( \alpha \in D \).

\( S \) is the set of all formulas :
\[
\{a_0, \{\alpha_1, \ldots, \alpha_k \} \rightarrow \beta\}, \{a_1 \rightarrow \alpha_1, \ldots, a_k \rightarrow \alpha_k\}, a_0 \cup a_1 \cup \ldots \cup a_k \rightarrow \beta
\]
with \( \{\alpha_1, \ldots, \alpha_k\} \in \mathcal{A}(D) \) and \( a_0 \cup a_1 \cup \ldots \cup a_k \in \mathcal{A}(D) \).
\( k_\pi \) is the set of formulas: \( (\{\alpha\} \rightarrow o) \) for \( \alpha \in \pi \);
\( cc \) is the set of all formulas:
\[
\{a \rightarrow \alpha\} \rightarrow \alpha \cap \alpha_1 \cap ... \cap \alpha_k \quad \text{with} \quad a = \{\{\alpha_1\} \rightarrow o, ..., \{\alpha_k\} \rightarrow o\} \quad \text{and} \quad \alpha \cap \alpha_1 \cap ... \cap \alpha_k \in D.
\]

QP is defined as the set of \( t \in \Lambda_D \) s.t. \(|t| = 1\) i.e. \((\forall \alpha \in t)(|\alpha| = 1)\).

We have \( K, S, cc \in QP \); \( t, u \in QP \Rightarrow tu \in QP \).

The model is \textit{coherent} because \(|t| = 1 \Rightarrow o \notin t\) i.e. \( t \star \{o\} \notin \bot \).

**Lemma 1.** \( t \parallel \nexists T, ..., T \rightarrow \bot \iff t = \{o\} \).

Indeed, \( t \star \emptyset \cdots \emptyset \{o\} \in \bot \Rightarrow t = \{o\} \). \( \text{QED} \)

**Lemma 2.** If \( t \in |T, \bot \rightarrow \bot| \cap |\bot, T \rightarrow \bot| \) then \( t = \{o\} \).

We have \( t \cap \emptyset \cdot \{o\} \cdot \{o\} \neq \emptyset \) and \( t \cap \{o\} \cdot \emptyset \cdot \{o\} \neq \emptyset \); thus
\( (\emptyset, a \rightarrow o) \in t \) and \( (b, \emptyset \rightarrow o) \in t \) with \( a, b \subset \{o\} \).

These two formulas are compatible and therefore equal; thus \( a = b = \emptyset \). \( \text{QED} \)

It follows that \( I \parallel \nexists T, \bot \rightarrow \bot | \cap | \bot, T \rightarrow \bot | \rightarrow \bot \) i.e.
\( I \parallel \forall x \exists \exists (x \neq 0, x \neq 1 \rightarrow \bot) \rightarrow \bot \). Therefore:

\textit{The Boolean algebra} \( \exists \exists \) \textit{is non trivial.}
Lemma 3. If $u \vdash \bot, \bot \rightarrow \bot$ then $u$ contains one of the formulas:

$$\circ; \{\circ\} \rightarrow \circ; \emptyset, \{\circ\} \rightarrow \circ; \{\circ\}, \{\circ\} \rightarrow \circ.$$ 

We have $u \cap \{\circ\} \cdot \{\circ\} \cdot \{\circ\} \neq \emptyset$, thus there exist $a, b \subset \{\circ\}$ s.t. $(a, b \rightarrow \circ) \in u$. QED

Lemma 4. Let $t \in \Lambda_D$ contain the 4 incompatible formulas:

$$\{\circ\} \rightarrow \circ; \{\{\circ\} \rightarrow \circ\}, \{\circ\} \rightarrow \circ; \{\emptyset, \{\circ\} \rightarrow \circ\}, \{\circ\} \rightarrow \circ; \{\{\circ\}, \{\circ\} \rightarrow \circ\}, \{\circ\} \rightarrow \circ.$$ 

Then $t \vdash \top, \bot \rightarrow \bot \cap \bot, \top \rightarrow \bot \cap \bot$, and $t \vdash (\bot, \bot \rightarrow \bot), \bot \rightarrow \bot$.

By lemma 2, the first conclusion is $t \vdash \bot \rightarrow \bot$; it is satisfied because $(\{\circ\} \rightarrow \circ) \in t$.

Now, let $u \vdash \bot, \bot \rightarrow \bot$; we must show $t \cap u \cdot \{\circ\} \cdot \{\circ\} \neq \emptyset$

which follows immediately from lemma 3. QED

Theorem. The Boolean algebra $\mathbb{J}_2$ is atomless.

We have $t \vdash \forall x \mathbb{J}_2 \left( \forall y \mathbb{J}_2 (x y \neq 0, x y \neq x \rightarrow \bot), x \neq 0 \rightarrow \bot \right)$ iff

$$t \vdash \top, \bot \rightarrow \bot \cap \top, \top \rightarrow \bot, \top \rightarrow \bot \text{ and } t \vdash (\bot, \bot \rightarrow \bot), \bot \rightarrow \bot.$$ 

Hence the result by lemma 4. QED
Integers

In the sequel, we use truth values defined by subsets $|U|$ of $\Lambda$. They may be used in formulas only before a $\rightarrow$.

If $|U| \subset \Lambda$, $\|A\| \subset \Pi$, we define $\|U \rightarrow A\| = \{t \cdot \pi ; t \in |U|, \pi \in \|A\|\}$.

In particular $\|\neg U\| = \{t \cdot \pi ; t \in |U|, \pi \in \Pi\}$.

**Lemma 5.** If $(\forall t \in \Lambda)(t \in |U| \Rightarrow \theta t \in |U'|)$ then $\lambda x x \cdot \theta \models \neg U' \rightarrow \neg U$.

We shall sometimes write $\theta \models U \rightarrow U'$ in such a case.

Now, define the formulas :

$v_0 = (\varnothing \rightarrow o) ; v_1 = (\varnothing, \{o\} \rightarrow o) ; \ldots ; v_n = (\varnothing, \ldots, \varnothing, \{o\} \rightarrow o) ; \ldots ;$

and the terms $\overline{n} = \{v_n\} ; \text{suc} = \{(v_0) \rightarrow v_1), \ldots, (v_i) \rightarrow v_{i+1}\), \ldots\}.$

Define the unary predicate $N$ by :

$|Nn| = \{\overline{n}\}$ if $n \in \mathbb{N} ; |Nn| = \varnothing$ if $n \notin \mathbb{N}$.

Then we have easily $\lambda x (x)\overline{0} \models \neg \neg N0 ; \text{suc} \models Nn \rightarrow N(n + 1)$ for every $n$ ; i.e. $\lambda x x \cdot \text{suc} \models \forall x (\neg N(x + 1) \rightarrow \neg Nx)$.

We have shown : $\models \forall x^{\text{int}} \neg \neg Nx$. 
**Theorem 6.** Let \( u_n(n \in \mathbb{N}) \) be any sequence of terms and define:
\[
\theta = \{\{v_n\} \to \alpha) ; n \in \mathbb{N}, \alpha \in u_n\}.\]

Then \( \theta n = u_n \) for all \( n \in \mathbb{N} \).

If every \( u_n \) is in QP, then \( \theta \in QP \).

We show that \( \theta \in \Lambda_D : \) if \( \{v_m\} \to \alpha = \{v_n\} \to \beta \) then \( \{v_m, v_n\} \) is an antichain and therefore \( m = n \); thus \( \alpha, \beta \in u_m \); but \( \alpha \simeq \beta \) and therefore \( \alpha = \beta \).

\( \theta \{v_n\} = u_n \) is obvious.

Define the unary predicate \( \text{ent}(x) \) by:
\[
|\text{ent}(n)| = \{n\} \quad \text{(Church integer)} \quad \text{for} \quad n \in \mathbb{N} ; \quad |\text{ent}(n)| = \emptyset \quad \text{if} \quad n \notin \mathbb{N}.
\]

We already know (general theory) that \( \text{ent}(x) \) is equivalent to \( \text{int}(x) \).

Apply lemma 5 and theorem 6 above with \( u_n = \{n\} \).

This gives \( \theta \vdash \mathcal{N}n \to \text{ent}(n) \) and therefore \( \lambda x \ x \circ \theta \vdash \forall x (\neg \text{ent}(x) \to \neg \mathcal{N}x) \).

Finally we have shown that the predicates \( \mathcal{N}x, \text{int}(x), \text{ent}(x) \) are equivalent.

In the following, we use \( \mathcal{N}x \) which is the simplest.
Corollary. If $\theta_n \models F(n)$, with $\theta_n \in QP$ for all $n \in \mathbb{N}$, then there exists $\theta \in QP$ s.t. $\theta \models \forall n \text{int } F(n)$.

Applying theorem 6, we get $\theta_n \models F(n)$ for all $n \in \mathbb{N}$, thus $\theta \models \forall n \text{int } F(n)$. QED

By the above corollary, the set of formulas which are realized by a proof-like term is closed by the $\omega$-rule.

Thus there exists a realizability model which is an $\omega$-model.

Let $\mathcal{B} = \mathcal{P}(\Pi)$ be the Boolean algebra of truth values.

The order is defined by $\|A\| \leq \|B\| \iff (\exists \theta \in QP)(\theta \models A \to B)$.

Theorem. $\mathcal{B}$ is a countably complete Boolean algebra:

If $\|B(n)\|_{n \in \mathbb{N}}$ is a sequence of truth values, then $\inf_{n \in \mathbb{N}} \|B(n)\| = \|\forall x \text{int } B(x)\|$.

Let $\|A\| \leq \|B(n)\|$ for every $n \in \mathbb{N}$. Then $\theta_n \models A \to B(n)$ for some sequence $\theta_n \in QP$.

By the previous corollary, we get $\theta \models \|A \to \forall x \text{int } B(x)\|$ i.e. $\|A\| \leq \|\forall x \text{int } B(x)\|$.

Conversely, $\|\forall x \text{int } B(x)\| \leq \|B(n)\|$ because $\lambda x(x) \models \forall x \text{int } B(x) \to B(n)$. QED