A general storage theorem for integers in call-by-name λ-calculus

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Abstract. The notion of *storage operator* introduced in [5, 6] appears to be an important tool in the study of data types in second order λ-calculus. These operators are λ-terms which simulate call-by-value in the call-by-name strategy, and they can be used in order to modelize assignment instructions. The main result about storage operators is that there is a very simple second order type for them, using Gödel's "not-not translation" of classical into intuitionistic logic.
We give here a new and simpler proof of a strengthened version of this theorem, which contains all previous results in intuitionistic and in classical logic ([6, 7]), and gives rise to new "storage theorems". Moreover, this result has a simple and intuitive meaning, in terms of realizability.

Introduction

This paper deals with the "call-by-name" λ-calculus (cf. [14]), i.e. λ-calculus in which the strategy of reduction is the "weak head reduction", which consists in reducing only the head redex, until the λ-term begins by a λ. Thus, the rule for "call-by-name" reduction is:

$$(\lambda x\, u)tt_1 \dots t_k \rightsquigarrow u[t/x]t_1 \dots t_k.$$

We consider a second order type assignment system for this λ-calculus, in which types are formulas of second order predicate logic. Such a type system has already been used in [9, 4, 8]. It allows to get λ-terms from proofs in second order *intuitionistic* logic, by means of the well known Curry-Howard isomorphism.

The notion of *storage operator* defined in [5, 6] appears as an important tool in the study of this second order call-by-name λ-calculus. They are closed λ-terms which allow, for a given data type (the type of integers, for example), to simulate call-by-value in the call-by-name strategy. In [6] was proved a "storage theorem" for intuitionistic logic; it asserts that the formula $\forall x[\neg Int(x) \to \neg Int^*(x)]$ (where $Int(x)$ is the definition of integers in second order logic, and * stands for Gödel's translation; cf. Definitions

& Notations below) is a specification for storage operators on integers in second order intuitionistic logic. This theorem can be generalized to other data types (cf. [10, 11]).

In [7], this storage theorem was extended to the case of second order classical logic (the idea of using storage operators in classical logic is due to M. Parigot [13] who found that they can decode integers obtained from classical proofs).

Our method in order to obtain λ-terms from proofs in second order *classical* logic is very simple: we only add a new λ-constant, denoted by C, with the declaration $\mathrm{C} : \forall X(\neg\neg X \to X)$. In this way, classical proofs, considered as intuitionistic proofs with the axiom $\forall X(\neg\neg X \to X)$, give λ-terms with the constant C inside. Now, for such λ-terms, we extend the weak head reduction strategy with the following rule of reduction:

$$\mathrm{C}tt_1 \dots t_n \rightsquigarrow (t)\lambda x\, xt_1 \dots t_n.$$

This is a particular case of a rule given by Felleisen [1] for control operators. Indeed, this new instruction C allows to introduce in λ-terms, mechanisms of "escape" which are much used in real programming languages, particularly in order to handle errors. Examples of such instructions are `Call/cc` in SCHEME, and `setjmp, longjmp` in the C language.

The idea of using classical logic in order to give types to "escape" instructions is due to T. Griffin[3].

In the present paper, we prove a general semantic property (theorem 2) for any λ-term T with type $\forall x[\neg Int(x) \to \neg Int^*(x)]$. This "storage property" can be expressed intuitively as follows:

T turns any program ϕ which can only accept as input an integer which is already computed, into a program $T\phi$ which accepts an integer in any form (and gives the same result as ϕ).

This property appears to be a good mathematical modelization of the simulation of call-by-value inside the call-by-name strategy. Indeed, the term T can only do this job by first computing the integer, before giving it, as an input, to the program ϕ.

As corollaries of this result, we get new and simpler proofs of the storage theorems for intuitionistic and classical logic, and also stronger forms of these theorems, which do not seem to be obtainable by the methods of [6, 7].

Definitions and notations

Formulas

We consider a second order language $\mathcal{L}$, the logical symbols of which are $\perp$, $\to$, $\forall$. There are individual (or first order) variables $x, y, \dots$, and predicate (or second order) variables $X, Y, \dots$ of each arity. There may be function symbols of any arity; in particular, a constant symbol 0, and a unary function symbol s (for successor). For simplicity, we shall assume there is no predicate symbol (i.e. predicate constant) except $\perp$.

The terms of $\mathcal{L}$ (built, in the usual way, with first order variables and function symbols) will be called *$\mathcal{L}$-terms* in order not to confuse them with λ-terms.

The set of *closed* $\mathcal{L}$-terms (i.e. without variables) will be denoted by $\mathcal{T}$.
Equality is defined: $x = y$ is the formula $\forall X(Xx \to Xy)$ where X is a unary predicate variable.
The formula $\neg F$ is, by definition, $F \to \bot$. The formula $\exists x F$ is $\neg\forall x \neg F$. We shall use the notation $A, B \to C$ for $A \to (B \to C)$.
The formula $Int(x)$ is $\forall X[\forall y(Xy \to Xsy), X0 \to Xx]$, and reads "$x$ is an integer".
The *Gödel translation* of a formula F of $\mathcal{L}$ is, by definition, the formula F^* obtained by replacing, in F, each atomic formula, except $\bot$, with its negation. For example, the formula $Int^*(x)$ is $\forall X[\forall y(\neg Xy \to \neg Xsy), \neg X0 \to \neg Xx]$.

Let $\mathcal{E}$ be a set of equations, i.e. formulas of the form $t = u$ where t, u are $\mathcal{L}$-terms. $\mathcal{E}$ is called a *system of equations for integers* if $\mathcal{L}$ contains the symbol of constant 0 and the symbol of unary function s, and, for any $\mathcal{L}$-term t, and any $k \in \mathbf{N}$:
$st = 0$ is *not* an equational consequence of $\mathcal{E}$;
if $st = s^{k+1}0$ is an equational consequence of $\mathcal{E}$, so is $t = s^k 0$.
Notice that $\emptyset$ is clearly a system of equations for integers.
Let $\mathcal{E}$ be a system of equations of $\mathcal{L}$. On the set $\mathcal{T}$ of closed terms of $\mathcal{L}$, we define an equivalence relation $\simeq_\mathcal{E}$ as follows: $t \simeq_\mathcal{E} u$ if and only if $t = u$ is an equational consequence of $\mathcal{E}$.
The quotient set $\mathcal{T}/\simeq_\mathcal{E}$ will be denoted by $\mathcal{T}_\mathcal{E}$. It is easy to check that $\mathcal{T}_\mathcal{E}$ is a model of $\mathcal{E}$, when function symbols are interpreted in the canonical way.
If $n \in \mathbf{N}$, the *integer* n of $\mathcal{T}_\mathcal{E}$ is, by definition, the equivalence class of the term $s^n 0$, for the equivalence relation $\simeq_\mathcal{E}$.
If $\mathcal{E}$ is a set of equations for integers, then the integers of $\mathcal{T}_\mathcal{E}$ are distinct. Furthermore, if $t \in \mathcal{T}_\mathcal{E}$ is such that st is an integer of $\mathcal{T}_\mathcal{E}$, then t itself is an integer of $\mathcal{T}_\mathcal{E}$.

λ-terms

We denote by Λ the set of λ-terms, modulo α-equivalence, and by $\mathcal{V}$ the set of λ-variables. If $t, u \in \Lambda$, we denote by $(t)u$, or tu, or $t.u$, the application of t to u; and by $(t)u_1 \ldots u_k$, or $tu_1 \ldots u_{k-1}u_k$, or $t.u_1 \ldots u_{k-1}.u_k$, the λ-term $(\ldots((t)u_1) \ldots u_{k-1})u_k$.
A *substitution* is a map $\sigma : \mathcal{V}' \to \Lambda$, where $\mathcal{V}'$ is any subset of $\mathcal{V}$. It has a canonical extension (which we shall denote also by σ) into a map $\sigma : \Lambda \to \Lambda$, defined in the usual way. When $\mathcal{V}'$ is a finite set $\{x_1, \ldots, x_k\}$ of variables, and $\sigma x_i = t_i$, the substitution σ is also denoted by $[t_1/x_1, \ldots, t_k/x_k]$; and, for any $t \in \Lambda$, σt is denoted then by $t[t_1/x_1, \ldots, t_k/x_k]$.

The *weak head reduction* (called *call-by-name* in [14]), is a binary relation on Λ, which will be denoted by $\succ$; it is defined as the least reflexive and transitive relation on Λ such that
$(\lambda x\, u)tt_1 \ldots t_k \succ u[t/x]t_1 \ldots t_k$ for any $k \in \mathbf{N}, u, t, t_1, \ldots, t_k \in \Lambda$.
A subset $\mathcal{X}$ of Λ is called *saturated* if $t \in \mathcal{X}, t' \succ t \Rightarrow t' \in \mathcal{X}$. In other words, $\mathcal{X}$ is saturated if and only if:
$u[t/x]t_1 \ldots t_k \in \mathcal{X} \Rightarrow (\lambda x\, u)tt_1 \ldots t_k \in \mathcal{X}$ for any $k \in \mathbf{N}, u, t, t_1, \ldots, t_k \in \Lambda$.

If $\mathcal{X}, \mathcal{Y} \subseteq \Lambda$, then we define $(\mathcal{X} \to \mathcal{Y}) \subseteq \Lambda$ as $\{t \in \Lambda; tu \in \mathcal{Y}$ for every $u \in \mathcal{X}\}$. Clearly,

$(\mathcal{X} \to \mathcal{Y})$ is saturated for every saturated $\mathcal{Y}$.

Models

A Λ-*model* $\mathcal{M}$ is composed of the following data:

- A set $\Re_{\mathcal{M}}$ of saturated subsets of Λ, called the *truth values set* of $\mathcal{M}$, with the following properties:
  i) if $\mathcal{X} \subset \Lambda$ and $\mathcal{Y} \in \Re_{\mathcal{M}}$ then $(\mathcal{X} \to \mathcal{Y}) \in \Re_{\mathcal{M}}$;
  ii) any intersection of elements of $\Re_{\mathcal{M}}$ is in $\Re_{\mathcal{M}}$; (it follows that $\Lambda \in \Re_{\mathcal{M}}$: take the void intersection).
  Notice that condition (i) may be replaced by:
  i') if $t \in \Lambda$ and $\mathcal{Y} \in \Re_{\mathcal{M}}$ then $(\{t\} \to \mathcal{Y}) \in \Re_{\mathcal{M}}$.

- A fixed element of $\Re_{\mathcal{M}}$, denoted by $|\bot|_{\mathcal{M}}$.

A Λ-model $\mathcal{M}$ is called *standard* or *intuitionistic* if $\Re_{\mathcal{M}}$ is the set of *all* saturated subsets of Λ.

A *formula of* $\mathcal{L}$ *with parameters in* $\mathcal{M}$ is an expression of the form $F[\Phi_1/X_1, \ldots, \Phi_k/X_k]$, where F is a formula of $\mathcal{L}$, X_i a predicate variable of arity n_i, and $\Phi_i : \mathcal{T}_{\mathcal{E}}^{n_i} \to \Re_{\mathcal{M}}$ for $1 \leq i \leq k$ (the functions Φ_i are the parameters of the formula).
For each closed formula F of $\mathcal{L}$, possibly with parameters, in $\mathcal{M}$, we define its *value in the model* $\mathcal{M}$, which is an element of $\Re_{\mathcal{M}}$, denoted by $|F|_{\mathcal{M}}$. This is done, by induction on F, in the following way:
If F is atomic, then $F \equiv \Phi(t_1, \ldots, t_n)$ with $t_1, \ldots, t_n \in \mathcal{T}$, or $F \equiv \bot$. In both cases, $|F|_{\mathcal{M}}$ is defined in an evident way.
If $F \equiv G \to H$, then $|F|_{\mathcal{M}} = |G|_{\mathcal{M}} \to |H|_{\mathcal{M}}$.
If $F \equiv \forall x G$, then $|F|_{\mathcal{M}} = \bigcap\{|G[t/x]|_{\mathcal{M}}; t \in \mathcal{T}\}$.
If $F \equiv \forall X G$, X being a predicate variable of arity k, then $|F|_{\mathcal{M}} = \bigcap\{|G[\Phi/X]|_{\mathcal{M}}; \Phi : \mathcal{T}_{\mathcal{E}}^k \to \Re_{\mathcal{M}}\}$.
When $t \in |F|_{\mathcal{M}}$, we say that t *realizes the formula* F *in* $\mathcal{M}$ (notation $t \Vdash F$ in $\mathcal{M}$).

Typed terms

We shall consider *typed terms* i.e. expressions of the following form:
$x_1 : A_1, \ldots, x_k : A_k \vdash_{\mathcal{E}} \tau : A$, where $A_1, \ldots, A_k, A$ are formulas of $\mathcal{L}$, $x_1, \ldots, x_k$ are λ-variables, τ is a λ-term, and $\mathcal{E}$ a system of equations for integers.
The rules of construction of typed terms are as follows ([4, 8, 9]):

1. $x_1 : A_1, \ldots, x_k : A_k \vdash_{\mathcal{E}} x_i : A_i$.
2. If $x_1 : A_1, \ldots, x_k : A_k, x : A \vdash_{\mathcal{E}} \tau : B$, then $x_1 : A_1, \ldots, x_k : A_k \vdash_{\mathcal{E}} \lambda x\, \tau : A \to B$.
3. If $x_1 : A_1, \ldots, x_k : A_k \vdash_{\mathcal{E}} \tau : A \to B$, $\tau' : A$ then $x_1 : A_1, \ldots, x_k : A_k \vdash_{\mathcal{E}} \tau\tau' : B$.
4. If $x_1 : A_1, \ldots, x_k : A_k \vdash_{\mathcal{E}} \tau : \forall x\, A$, then $x_1 : A_1, \ldots, x_k : A_k \vdash_{\mathcal{E}} \tau : A[t/x]$ for every $\mathcal{L}$-term t.
5. If $x_1 : A_1, \ldots, x_k : A_k \vdash_{\mathcal{E}} \tau : A$, then $x_1 : A_1, \ldots, x_k : A_k \vdash_{\mathcal{E}} \tau : \forall x\, A$, if x is a first order variable which does not appear in $A_1, \ldots, A_k$.
6. If $x_1 : A_1, \ldots, x_k : A_k \vdash_{\mathcal{E}} \tau : \forall X\, A$, then $x_1 : A_1, \ldots, x_k : A_k \vdash_{\mathcal{E}} \tau : A[F/Xy_1 \ldots y_n]$

(X is a predicate variable of arity n, F is any formula; $A[F/Xy_1 \dots y_n]$ is the formula obtained by replacing, in A, each atomic formula $Xt_1 \dots t_n$ by $F[t_1/y_1, \dots, t_n/y_n]$).
7. If $x_1 : A_1, \dots, x_k : A_k \vdash_{\mathcal{E}} \tau : A$, then $x_1 : A_1, \dots, x_k : A_k \vdash_{\mathcal{E}} \tau : \forall X\, A$, if X is a predicate variable which does not appear in $A_1, \dots, A_k$.
8. If $x_1 : A_1, \dots, x_k : A_k \vdash_{\mathcal{E}} \tau : A\,[t/x]$, then $x_1 : A_1, \dots, x_k : A_k \vdash_{\mathcal{E}} \tau :\ A\,[u/x]$, if $t = u$ is an equational consequence of $\mathcal{E}$.

These are the rules for second order intuitionistic logic.

Lemma 1 (Adequacy lemma) *If $x_1 : A_1, \dots, x_k : A_k \vdash_{\mathcal{E}} t : A$ is a typed term, where $A_1, \dots, A_k, A$ are closed formulas of $\mathcal{L}$, and if $\mathcal{M}$ is a Λ-model, then, for every $u_1 \in |A_1|_{\mathcal{M}}, \dots, u_k \in |A_k|_{\mathcal{M}}$, we have $t[u_1/x_1, \dots, u_k/x_k] \in |A|_{\mathcal{M}}$.*

The proof is postponed in an appendix. It is not really necessary for a first reading, because of the following remark.

Remark. In fact, we shall never use precisely the rules of construction of typed terms, but only the property expressed by the adequacy lemma. Therefore, in all what follows, we could indeed as well define a typed term as an expression $x_1 : A_1, \dots, x_k : A_k \vdash_{\mathcal{E}} t : A$, where $A, A_1, \dots, A_k$ are closed formulas of $\mathcal{L}$, such that, for every Λ-model $\mathcal{M}$, if $u_1 \in |A_1|_{\mathcal{M}}, \dots, u_k \in |A_k|_{\mathcal{M}}$, then $t[u_1/x_1, \dots, u_k/x_k] \in |A|_{\mathcal{M}}$.

The R_n-reduction

In the following, we shall consider a system $\mathcal{E}$ of equations for integers, a Λ-model $\mathcal{N}$ and some fixed $O \in \Re_{\mathcal{N}}$.

If $t, u \in \Lambda$, and $i \in \mathbf{N}$, we define $X(t, u, i) \subseteq \Lambda$ by induction on i:
$X(t, u, 0) = \{\eta \in \Lambda; u\eta \in O\} \to O$;
$X(t, u, i+1) = \{\eta \in \Lambda; t\zeta\eta \in O$ for each $\zeta \in X(t, u, i)\} \to O$.
We set $X(t, u, a) = \Lambda$ for every $a \in \mathcal{T}_{\mathcal{E}}$ which is not an integer of $\mathcal{T}_{\mathcal{E}}$. Clearly, $X(t, u, a) \in \Re_{\mathcal{N}}$ for every $t, u \in \Lambda$, $a \in \mathcal{T}_{\mathcal{E}}$.

Lemma 2 *For every $t, u \in \Lambda$ we have $u \in X(t, u, 0)$ and $t \in |\forall y[X(t, u, y) \to X(t, u, sy)]|_{\mathcal{N}}$.*

Let $Y(u) = \{\eta \in \Lambda; u\eta \in O\}$. Then, clearly, $u \in (Y(u) \to O) = X(t, u, 0)$.
Let $i \in \mathcal{T}_{\mathcal{E}}$; we have to show that $t \in (X(t, u, i) \to X(t, u, si))$. This is clear if i is not an integer of $\mathcal{T}_{\mathcal{E}}$, because neither is si (by the properties of $\mathcal{E}$), and, therefore, $X(t, u, si) = \Lambda$.
Now, if i is an integer of $\mathcal{T}_{\mathcal{E}}$, let $Z(t, u, i) = \{\eta; t\zeta\eta \in O$ for each $\zeta \in X(t, u, i)\}$. Let $\zeta \in X(t, u, i)$; we have to show that $t\zeta \in X(t, u, i+1)$, that is $\eta \in Z(t, u, i) \Rightarrow t\zeta\eta \in O$. But this is clear by definition of $Z(t, u, i)$.
Q.E.D.

It follows from this lemma that $(t)^i u \in X(t, u, i)$, for every $t, u \in \Lambda$ and $i \in \mathbf{N}$;

therefore, $X(t,u,a) \neq \emptyset$ for any $t,u \in \Lambda, a \in \mathcal{T}_{\mathcal{E}}$.

Let $\mathcal{V}$ be the set of λ-variables, with a given enumeration. Let $\mathcal{W}$ be an infinite subset of $\mathcal{V}$ such that $\mathcal{V}-\mathcal{W}$ is also infinite. Let $(t_k,u_k,v_k,n_k)_{k\geq 1}$ be a one-to-one enumeration of $\Lambda^3 \times \mathbf{N}$.
We define, by induction, a sequence $(b_k)_{k\in\mathbf{N}}$ of distinct elements of $\mathcal{W}$: b_0 is the first element of $\mathcal{W}$; b_{k+1} is the first element of $\mathcal{W}$, which is different from $b_0,\dots,b_k$, and which does not appear in t_i,u_i,v_i $(1\leq i\leq k+1)$.
Let $\mathcal{B}=\{b_k;\ k\in\mathbf{N}\}$. For $k\geq 1$, we set:

$$\phi(b_k)=t_k,\ \psi(b_k)=u_k,\ \chi(b_k)=v_k,\ \pi(b_k)=n_k,\ ht(b_k)=k.$$

In this way, we have defined five fonctions of domain $\mathcal{B}_0=\mathcal{B}-\{b_0\}$; $ht(b)$ will be called the *height* of the variable b. For every $u\in\Lambda$, we define $ht(u)$ to be the maximum of $ht(b)$ for all free variables b of u, $b\in\mathcal{B}_0$ ($ht(u)=0$ if u contains no variables in $\mathcal{B}_0$).
Let us notice that the map $b_k\mapsto(t_k,u_k,v_k,n_k)$ is one-to-one from $\mathcal{B}_0$ onto $\Lambda^3\times\mathbf{N}$; and that the only variables of $\mathcal{B}$ which appear in t_k,u_k,v_k are amongst $b_0,\dots,b_{k-1}$, by definition of $\mathcal{B}$.

Now, let n be a fixed integer.
A substitution $\sigma:\{b_0,\dots,b_k\}\to\Lambda$ will be called *n-suitable* if:
$\sigma b_0\in|Int(n)|_{\mathcal{N}}$ and $\sigma b_i\in X(\sigma\phi b_i,\sigma\psi b_i,\pi b_i)$ for $1\leq i\leq k$.
A substitution $\sigma:\mathcal{B}\to\Lambda$ will be called *n-suitable* if:
$\sigma b_0\in|Int(n)|_{\mathcal{N}}$, and $\sigma b\in X(\sigma\phi b,\sigma\psi b,\pi b)$ for every $b\in\mathcal{B}_0$.

Lemma 3 *Every n-suitable substitution $\sigma:\{b_0,\dots,b_k\}\to\Lambda$ can be extended into an n-suitable substitution $\bar\sigma:\mathcal{B}\to\Lambda$.*

We already know that $X(t,u,i)\neq\emptyset$, and $\sigma b_0\in|Int(n)|_{\mathcal{N}}$. We define $\bar\sigma b_j=\sigma b_j$ for $0\leq j\leq k$; for $j>k$, the definition of $\bar\sigma b_j$ is done by induction on j, by choosing $\bar\sigma b_j\in X(\bar\sigma\phi b_j,\bar\sigma\psi b_j,\pi b_j)$. Notice that the only variables of $\mathcal{B}$ which appear in ϕb_j, ψb_j are taken among $b_0,\dots,b_{j-1}$; therefore $\bar\sigma\phi b_j$, $\bar\sigma\psi b_j$ are already defined, and so is $X(\bar\sigma\phi b_j,\bar\sigma\psi b_j,\pi b_j)$.
Q.E.D.

Lemma 4 *Let σ be an n-suitable substitution on $\mathcal{B}$, $b\in\mathcal{B}_0$, and $\zeta\in X(\sigma\phi b,\sigma\psi b,\pi b)$. Then, there is an n-suitable substitution σ' on $\mathcal{B}$, such that $\sigma' b=\zeta$ and $\sigma'\phi b=\sigma\phi b$, $\sigma'\psi b=\sigma\psi b$, $\sigma'\chi b=\sigma\chi b$.*

Let $k\geq 1$ be such that $b=b_k$; we first define σ' on $\{b_0,\dots,b_k\}$ by setting $\sigma' b_i=\sigma b_i$ for $0\leq i<k$, and $\sigma' b_k=\zeta$. Then σ' is clearly an n-suitable substitution on $\{b_0,\dots,b_k\}$. By lemma 3, we extend it into an n-suitable substitution, also denoted by σ', defined on the whole of $\mathcal{B}$. We have $\sigma' b=\zeta$, by construction of σ'. And also $\sigma'\phi b=\sigma\phi b$ because the only variables of $\mathcal{B}$ which appear in ϕb are among $b_0,\dots,b_{k-1}$. The same apply to $\psi b,\chi b$.
Q.E.D.

We now define a binary relation on Λ, which we call *R_n-reduction*, and denote by $\succ_{R_n}$. It is the least reflexive and transitive relation, such that

- $(\lambda x\ u)tt_1 \ldots t_k \succ_{R_n} u[t/x]t_1 \ldots t_k$, for every $k \in \mathbf{N}$ and $u, t, t_1, \ldots, t_k \in \Lambda$;
- $bw \succ_{R_n} (\psi b)w$, if $w \in \Lambda$, and $b \in \mathcal{B}_0$ is such that $\pi(b) = 0$;
- $bw \succ_{R_n} (\phi b)b'w$ if $w \in \Lambda$, and $b \in \mathcal{B}_0$ is such that $1 \leq \pi(b) \leq n-1$; b' is the single variable in $\mathcal{B}_0$ such that $\phi(b') = \phi(b)$, $\psi(b') = \psi(b)$, $\chi(b') = w$, $\pi(b') = \pi(b) - 1$;
- $b_0tuv \succ_{R_n} tbv$, where b is the single variable in $\mathcal{B}_0$ such that $\phi(b) = t, \psi(b) = u, \chi(b) = v, \pi(b) = n-1$; t, u, v are arbitrary λ-terms.

Theorem 1 *Let $e \in \Lambda$ be such that $\sigma e \in O$ for every substitution σ on $\mathcal{B}$. If $E \in \Lambda$ is such that $E \succ_{R_n} e$, then $\sigma E \in O$ for every n-suitable substitution σ on $\mathcal{B}$.*

The proof is by induction on the length of the R_n-reduction $E \succ_{R_n} e$. The result is clear if $E = e$. Let us consider now the first rule of R_n-reduction which is applied in the R_n-reduction of E:

1. $E = (\lambda x\ u)tt_1 \ldots t_k$; by the induction hypothesis, we have $\sigma\{u[t/x]t_1 \ldots t_k\} \in O$, for every n-suitable substitution σ on $\mathcal{B}$. Therefore, $\sigma E = \sigma\{(\lambda x\ u)tt_1 \ldots t_k\} \in O$, since O is saturated.

2. $E = bw$ with $b \in \mathcal{B}_0, \pi(b) = 0$. By the induction hypothesis, we have $\sigma((\psi b)w) \in O$, that is $\sigma\psi b.\sigma w \in O$, for every n-suitable substitution σ on $\mathcal{B}$. But $\sigma b \in X(\sigma\phi b, \sigma\psi b, 0)$, by definition of an n-suitable substitution, since $\pi b = 0$. Therefore, $(\sigma b)\eta \in O$, for every η such that $(\sigma\psi b)\eta \in O$. If we take $\eta = \sigma w$, we get $\sigma b.\sigma w \in O$, that is $\sigma E \in O$.

3. $E = bw$, with $b \in \mathcal{B}_0, \pi(b) = i$, $1 \leq i \leq n-1$. By the induction hypothesis, we have $\sigma'((\phi b)b'w) \in O$ for *every* n-suitable substitution σ' on $\mathcal{B}$. Therefore, $\sigma'\phi b.\sigma' b'.\sigma' w \in O$. Moreover, we have $\phi(b') = \phi(b)$, $\psi(b') = \psi(b)$, $\chi(b') = w$ and $\pi(b') = i-1$.
Suppose now that σ is an n-suitable substitution on $\mathcal{B}$. Let ζ be an arbitrary element of $X(\sigma\phi b', \sigma\psi b', i-1)$. By lemma 4, there exists an n-suitable substitution σ' on $\mathcal{B}$ such that $\sigma' b' = \zeta$, $\sigma'\phi b' = \sigma\phi b'$, $\sigma'\psi b' = \sigma\psi b'$, $\sigma'\chi b' = \sigma\chi b'$. Since $\phi(b') = \phi(b)$, we have $\sigma'\phi b = \sigma\phi b$. From $\chi(b') = w$, we obtain $\sigma' w = \sigma w$. Therefore $\sigma'\phi b.\sigma' b'.\sigma' w = \sigma\phi b.\zeta.\sigma w$, and it follows that $\sigma\phi b.\zeta.\sigma w \in O$. And this is true for every $\zeta \in X(\sigma\phi b', \sigma\psi b', i-1) = X(\sigma\phi b, \sigma\psi b, i-1)$ (because $\phi b' = \phi b$, $\psi b' = \psi b$). On the other hand, since σ is an n-suitable substitution, and $\pi(b) = i$, we have $\sigma b \in X(\sigma\phi b, \sigma\psi b, i)$. From the definition of $X(t, u, i)$ for $i \geq 1$, it follows that $(\sigma b)\eta \in O$ for every $\eta \in \Lambda$ such that $(\sigma\phi b)\zeta\eta \in O$ for every $\zeta \in X(\sigma\phi b, \sigma\psi b, i-1)$. If we take $\eta = \sigma w$, we obtain $\sigma b.\sigma w \in O$, in other words $\sigma E = \sigma(bw) \in O$.

4. $E = b_0tuv$; by the induction hypothesis, we have $\sigma'(tbv) \in O$ for each n-suitable substitution σ' on $\mathcal{B}$. Here, $b \in \mathcal{B}_0$, $\pi(b) = n-1$, $\phi(b) = t$, $\psi(b) = u$, $\chi(b) = v$.
Now, let σ be an n-suitable substitution on $\mathcal{B}$, and ζ an arbitrary element of $X(\sigma t, \sigma u, n-1) = X(\sigma\phi b, \sigma\psi b, \pi(b))$. By lemma 4, there exists an n-suitable substitution σ' on $\mathcal{B}$ such that $\sigma' b = \zeta$, $\sigma' t = \sigma t$, $\sigma' u = \sigma u$, $\sigma' v = \sigma v$. Thus, $\sigma'(tbv) = \sigma' t.\sigma' b.\sigma' v = \sigma t.\zeta.\sigma v$ and it follows that $\sigma t.\zeta.\sigma v \in O$; and this is true for every $\zeta \in X(\sigma t, \sigma u, n-$

1). But, by the fact that σ is n-suitable, we have $\sigma b_0 \in |Int(n)|_{\mathcal{N}}$, and, therefore, $\sigma b_0 \in |\forall y[X(\sigma t, \sigma u, y) \to X(\sigma t, \sigma u, sy)], X(\sigma t, \sigma u, 0) \to X(\sigma t, \sigma u, n)|_{\mathcal{N}}$. On the other hand, by lemma 2, we know that $\sigma u \in X(\sigma t, \sigma u, 0)$ and $\sigma t \in |\forall y[X(\sigma t, \sigma u, y) \to X(\sigma t, \sigma u, sy)]|_{\mathcal{N}}$. It follows that $\sigma b_0.\sigma t.\sigma u \in X(\sigma t, \sigma u, n)$. By definition of $X(\sigma t, \sigma u, n)$, we have $\sigma b_0.\sigma t.\sigma u.\eta \in O$, for every $\eta \in \Lambda$ such that $(\sigma t)\zeta\eta \in O$ for each $\zeta \in X(\sigma t, \sigma u, n-1)$. Therefore, by taking $\eta = \sigma v$, it follows that $\sigma b_0.\sigma t.\sigma u.\sigma v \in O$, that is $\sigma E = \sigma(b_0 tuv) \in O$.
Q.E.D.

A subset $\mathcal{X} \subseteq \Lambda$ will be said *R_n-saturated* if $t \in \mathcal{X}, t' \succ_{R_n} t \Rightarrow t' \in \mathcal{X}$.

Lemma 5 *For every model $\mathcal{M}$ such that $|\bot|_{\mathcal{M}}$ is R_n-saturated, we have $b_0 \in |Int^*(n)|_{\mathcal{M}}$.*

Assume that $t \in |\forall y(\neg Xy \to \neg Xsy)|_{\mathcal{M}}$, $u \in |\neg X0|_{\mathcal{M}}$, $v \in |Xn|_{\mathcal{M}}$. We have to show that $b_0 tuv \in |\bot|_{\mathcal{M}}$. Let us first show that, for every $b \in \mathcal{B}_0$ such that $\phi b = t, \psi b = u$ and $\pi(b) = i$ $(0 \leq i \leq n-1)$, we have $b \in |\neg Xi|_{\mathcal{M}}$; this will be done by induction on i.
If $\pi(b) = 0$, we have $bw \succ_{R_n} uw$. But, if $w \in |X0|_{\mathcal{M}}$, then $uw \in |\bot|_{\mathcal{M}}$, and therefore (since $|\bot|_{\mathcal{M}}$ is R_n-saturated) $bw \in |\bot|_{\mathcal{M}}$. Thus $b \in |\neg X0|_{\mathcal{M}}$.
If $\pi(b) = i$ $(1 \leq i \leq n-1)$, we have $bw \succ_{R_n} tb'w$, with $b' \in \mathcal{B}_0$, $\phi(b') = \phi(b) = t$, $\psi(b') = \psi(b) = u$ and $\pi(b') = i-1$. By the induction hypothesis, we get $b' \in |\neg X(i-1)|_{\mathcal{M}}$, and therefore $tb' \in |\neg Xi|_{\mathcal{M}}$. Thus, if $w \in |Xi|_{\mathcal{M}}$, we have $tb'w \in |\bot|_{\mathcal{M}}$; since $|\bot|_{\mathcal{M}}$ is R_n-saturated, it follows that $bw \in |\bot|_{\mathcal{M}}$, and finally, $b \in |\neg Xi|_{\mathcal{M}}$.
Now $b_0 tuv \succ_{R_n} tbv$ with $\phi(b) = t$, $\psi(b) = u$ and $\pi(b) = n-1$. By what has just been proved, we have $b \in |\neg X(n-1)|_{\mathcal{M}}$. Therefore $tb \in |\neg Xn|_{\mathcal{M}}$. Since $v \in |Xn|_{\mathcal{M}}$, we obtain $tbv \in |\bot|_{\mathcal{M}}$. Therefore $b_0 tuv \in |\bot|_{\mathcal{M}}$ because $|\bot|_{\mathcal{M}}$ is R_n-saturated.
Q.E.D.

Lemma 6 *Let $\mathcal{E}$ be a system of equations for integers, $n \in \mathbf{N}$, T a λ-term such that $\vdash_{\mathcal{E}} T : \neg Int(n) \to \neg Int^*(n)$, and f a variable $\in \mathcal{W} - \mathcal{B}$. Then there exists $\alpha_n \in \Lambda$, $\alpha_n \simeq_\beta \lambda f \lambda x\, f^n x$ (or, possibly, $\alpha_n \simeq_\beta \lambda x\, x$ if $n = 1$) such that $Tfb_0 \succ_{R_n} f\alpha_n$.*

Let $\mathcal{M}$ be a standard model of $\mathcal{E}$ such that $|\bot|_{\mathcal{M}} = \{t \in \Lambda;\ t \succ_{R_n} f\alpha$ for some $\alpha \simeq_\beta n$ or $\alpha \simeq_\beta \lambda x\, x$ if $n = 1\}$. Since $\mathcal{M}$ is a standard model, we have, by lemma 8 below, $|Int(n)|_{\mathcal{M}} \subseteq \{\alpha \in \Lambda;\ \alpha \simeq_\beta n\}$ if $n \neq 1$, and $|Int(1)|_{\mathcal{M}} \subseteq \{\alpha \in \Lambda;\ \alpha \simeq_\beta 1$ or $\alpha \simeq_\beta \lambda x\, x\}$. In either case, it follows that $f \in |\neg Int(n)|_{\mathcal{M}}$. Since, by the adequacy lemma 1, $T \in |\neg Int(n) \to \neg Int^*(n)|_{\mathcal{M}}$, we have $Tf \in |\neg Int^*(n)|_{\mathcal{M}}$. But, by lemma 5, $b_0 \in |Int^*(n)|_{\mathcal{M}}$, because $|\bot|_{\mathcal{M}}$ is R_n-saturated. Therefore $Tfb_0 \in |\bot|_{\mathcal{M}}$, and this is the desired result.
Q.E.D.

Definition. If $\vdash_{\mathcal{E}} T : \neg Int(n) \to \neg Int^*(n)$ with $n \in \mathbf{N}$, the λ-term α_n defined in lemma 6 will be called the *T-value* of the integer n. We define $Val(T, n) \subseteq \Lambda$ as $Val(T, n) = \{\sigma\alpha_n; \sigma$ substitution on $\mathcal{B} \cup \{f\}\}$. If $n \in \mathcal{T}$ is not an integer of $\mathcal{T}_{\mathcal{E}}$, we define $Val(T, n) = \emptyset$.

Lemma 7 *Let σ be a substitution on $\mathcal{V} - \mathcal{B}$. Then, there exists an extension of σ into a substitution σ', defined on the whole of $\mathcal{V}$, such that $\sigma' b_0 = b_0$, and, for every $T, U \in \Lambda$, $T \succ_{R_n} U \Rightarrow \sigma' T \succ_{R_n} \sigma' U$.*

We define $\sigma' b_k$ by induction on k: $\sigma' b_0 = b_0$; $\sigma' b_k$ for $k \geq 1$ is the single variable $b \in \mathcal{B}_0$ such that $\phi b = \sigma' \phi b_k$, $\psi b = \sigma' \psi b_k$, $\chi b = \sigma' \chi b_k$, $\pi b = \pi b_k$. Notice that the heights of $\phi b_k, \psi b_k, \chi b_k$ are $< k$, so that $\sigma' \phi b_k, \sigma' \psi b_k, \sigma' \chi b_k$ are already defined.
Thus, we have $\phi \sigma' b = \sigma' \phi b$, $\psi \sigma' b = \sigma' \psi b$, $\chi \sigma' b = \sigma' \chi b$ and $\pi \sigma' b = \pi b$ for every $b \in \mathcal{B}_0$.
We prove that $T \succ_{R_n} U \Rightarrow \sigma' T \succ_{R_n} \sigma' U$ by induction on the length of the R_n-reduction $T \succ_{R_n} U$.
If $T = (\lambda x\, u) t t_1 \ldots t_k$, one step of R_n-reduction gives the λ-term $T_1 = u[t/x] t_1 \ldots t_k$. By the induction hypothesis, we have $\sigma' T_1 \succ_{R_n} \sigma' U$, and, clearly, $\sigma' T \succ \sigma' T_1$. Thus $\sigma' T \succ_{R_n} \sigma' U$.
If $T = bw$ with $b \in \mathcal{B}_0, \pi b = 0$, then one step of R_n-reduction gives the λ-term $T_1 = \psi b.w$. The induction hypothesis gives $\sigma' T_1 \succ_{R_n} \sigma' U$. We have $\sigma' T = \sigma' b.\sigma' w \succ_{R_n} \psi \sigma' b.\sigma' w = \sigma' \psi b.\sigma' w = \sigma' T_1$. Therefore $\sigma' T \succ_{R_n} \sigma' U$.
If $T = bw$ with $b \in \mathcal{B}_0$ and $1 \leq \pi b \leq n-1$, then we set $T_1 = (\phi b) b' w$, where $b' \in \mathcal{B}_0$ is determined by the conditions $\phi b' = \phi b$, $\psi b' = \psi b$, $\chi b' = w$ and $\pi b' = \pi b - 1$. The induction hypothesis gives $\sigma' T_1 \succ_{R_n} \sigma' U$. But $\sigma' T = \sigma' b.\sigma' w \succ_{R_n} \phi \sigma' b.b''.\sigma' w$, with $b'' \in \mathcal{B}_0$ such that $\phi b'' = \phi \sigma' b$, $\psi b'' = \psi \sigma' b$, $\chi b'' = \sigma' w$ and $\pi b'' = \pi \sigma' b - 1$. It follows that $b'' = \sigma' b'$, since $\phi \sigma' b' = \sigma' \phi b' = \sigma' \phi b = \phi \sigma' b$; $\psi \sigma' b' = \sigma' \psi b' = \sigma' \psi b = \psi \sigma' b$; $\chi \sigma' b' = \sigma' \chi b' = \sigma' w$; and $\pi \sigma' b' = \pi b' = \pi b - 1 = \pi \sigma' b - 1$. Thus, $\sigma' T \succ_{R_n} \phi \sigma' b.\sigma' b'.\sigma' w = \sigma' \phi b.\sigma' b'.\sigma' w = \sigma' T_1$, and, eventually $\sigma' T \succ_{R_n} \sigma' U$.
If $T = b_0 t u v$, one step of R_n-reduction gives the λ-term $T_1 = t b v$. We have $\sigma' T_1 = \sigma' t.\sigma' b.\sigma' v$. But b is the variable of $\mathcal{B}_0$ defined by $\phi b = t$, $\psi b = u$, $\chi b = v$, $\pi b = n-1$; by definition of σ', we have $\sigma' b \in \mathcal{B}_0, \phi \sigma' b = \sigma' t$, $\psi \sigma' b = \sigma' u$, $\chi \sigma' b = \sigma' v$, $\pi \sigma' b = n-1$. It follows that $\sigma' T = b_0 \sigma' t.\sigma' u.\sigma' v \succ_{R_n} \sigma' T_1$. By the induction hypothesis, we get $\sigma' T_1 \succ_{R_n} \sigma' U$, and it follows that $\sigma' T \succ_{R_n} \sigma' U$.
Q.E.D.

Storage theorems

We can now prove a general result about the behaviour of λ-terms of type $\neg Int(x) \to \neg Int^*(x)$.

Theorem 2 (General storage theorem)
Suppose that $\mathcal{E}$ is a set of equations for integers, n is a $\mathcal{L}$-term, $\mathcal{N}$ is a Λ-model and $O \in \Re_{\mathcal{N}}$. Then
i) If $\vdash_{\mathcal{E}} T : \neg Int(n) \to \neg Int^*(n)$, then $T \in |(Val(T,n) \to O) \to (Int(n) \to O)|_{\mathcal{N}}$.
ii) If $\vdash_{\mathcal{E}} T : \forall x[\neg Int(x) \to \neg Int^*(x)]$, then $T \in |\forall x[(Val(T,x) \to O) \to (Int(x) \to O)]|_{\mathcal{N}}$.

Remark. The intuitive meaning of this theorem, when n is an integer of $\mathcal{T}_{\mathcal{E}}$, is as follows: let T be a λ-term of type $\forall x[\neg Int(x) \to \neg Int^*(x)]$; for example, $T = \lambda f \lambda x(x) \lambda h \lambda g(h) g \circ suc.\lambda g\, g0.f$, which is given by the simplest proof of $\forall x[\neg Int(x) \to \neg Int^*(x)]$; suc is a λ-term for the successor, and $\circ$ is the operator of composition $(f \circ g = \lambda x(f)(g)x)$.

Then, by lemma 6, to each $n \in \mathbf{N}$, is associated a λ-term $\alpha_n \simeq_\beta n$, which is a kind of computed value which we called the T-value of n (for the above example, α_n is $suc^n 0$). Notice that, from the point of view of computation, all λ-terms $\sigma\alpha_n$, for any substitution σ, are equivalent to α_n; indeed, free variables in α_n are dummy variables since α_n is β-equivalent to a closed term. It follows that the λ-terms we substitute for them are never computed in the call-by-name strategy.
Now, let ϕ be a λ-term which can only handle integers in the reduced form α_n (or $\sigma\alpha_n$, for any substitution σ). This means that we know the behavior of ϕ only on these integers; this is expressed by the fact that $\phi \in |Val(T,n) \to O|$.
Then $T\phi$ will be able to handle any "general" integer n, i.e. any λ-term which realizes $Int(n)$, and it will have the same behavior as ϕ. This is expressed by the fact that $T\phi$ will realize $Int(n) \to O$.
Clearly, T can only accomplish this by first computing the T-value α_n of the general integer n, *before* giving it, as an argument, to the program ϕ. This is exactly the simulation of call-by-value in the head reduction (call-by-name) strategy.

Proof of theorem 2.
i) We first suppose that n is an integer. Let $\phi \in |Val(T,n) \to O|_{\mathcal{N}}$, $\theta \in |Int(n)|_{\mathcal{N}}$. We have to show that $T\phi\theta \in O$. We first remark that the set $\mathcal{W}$ of variables can be chosen arbitrarily. Therefore, we can assume that no variable of $\mathcal{W}$, and thus of $\mathcal{B} \cup \{f\}$, has a free occurrence in ϕ. By lemma 6, we have $Tfb_0 \succ_{R_n} f\alpha_n$. Let σ_0 be the substitution $[\phi/f]$ defined on $\mathcal{V} - \mathcal{B}$, and σ'_0 its extension defined in lemma 7. Then, by lemma 7, $\sigma'_0(Tfb_0) \succ_{R_n} \sigma'_0(f\alpha_n)$, that is $T\phi b_0 \succ_{R_n} \phi.\sigma'_0\alpha_n$.
Now, for any substitution σ on $\mathcal{B}$, we have $\sigma(\phi.\sigma'_0\alpha_n) = \phi.\sigma'\alpha_n$, where $\sigma' = \sigma \circ \sigma'_0$ is a substitution. Therefore $\sigma(\phi.\sigma'_0\alpha_n) \in O$, because $\phi \in |Val(T,n) \to O|_{\mathcal{N}}$. By theorem 1, we deduce that $\sigma(T\phi b_0) \in O$ for every n-suitable substitution σ on $\mathcal{B}$. We choose σ such that $\sigma(b_0) = \theta$ (by means of lemma 3, in which we set $k = 0$), and eventually, we get $T\phi\theta \in O$.
Suppose now that n is not an integer. Then $Val(T,n) = \emptyset$, so that $|Val(T,n) \to O|_{\mathcal{N}} = \Lambda$. Thus, if ϕ is any λ-term, and $\theta \in |Int(n)|_{\mathcal{N}}$, we have to show that $T\phi\theta \in O$. Let $\mathcal{M}$ be the standard model such that $|\bot|_{\mathcal{M}} = O$. Since n is not an integer of $\mathcal{T}_{\mathcal{E}}$, and $\mathcal{M}$ is standard, it follows, by lemma 8 below, that $|Int(n)|_{\mathcal{M}} = \emptyset$. Therefore $\phi \in |\neg Int(n)|_{\mathcal{M}}$. By the adequacy lemma, we have $T \in |\neg Int(n) \to \neg Int^*(n)|_{\mathcal{M}}$, and it follows that $T\phi \in |\neg Int^*(n)|_{\mathcal{M}}$.
We now show that $\theta \in |Int^*(n)|_{\mathcal{M}}$; the result will follow, since we get $T\phi\theta \in |\bot|_{\mathcal{M}} = O$. Let Ξ be an arbitrary function $\mathcal{T}_{\mathcal{E}} \to \Re_{\mathcal{M}}$. We have to prove that $\theta \in |\forall y(\Xi' y \to \Xi' sy), \Xi' 0 \to \Xi' n|_{\mathcal{M}}$, with $\Xi'\eta = (\Xi\eta \to O)$ for every $\eta \in \mathcal{T}_{\mathcal{E}}$. But, since $O \in \Re_{\mathcal{N}}$, we have $\Xi'\eta \in \Re_{\mathcal{N}}$ for every $\eta \in \mathcal{T}_{\mathcal{E}}$, and the desired result follows from the fact that $\theta \in |Int(n)|_{\mathcal{N}} = |\forall X[\forall y(Xy \to Xsy), X0 \to Xn]|_{\mathcal{N}}$.
ii) Immediate from (i), by definition of $|\forall x[(Val(T,x) \to O) \to (Int(x) \to O)]|_{\mathcal{N}}$.
Q.E.D.

We obtain, as corollaries of theorem 2, the storage theorems for intuitionistic logic [6] and for classical logic [7], and some strengthening of them (theorems 3 to 5).

Theorem 3 (storage theorem for intuitionistic logic)
If $\vdash_{\mathcal{E}} T : \neg Int(n) \to \neg Int^*(n)$, $n \in \mathbf{N}$, *f is a variable, and* $\theta \simeq_\beta n$, *then* $Tf\theta \succ f.\sigma\alpha_n$ *for some substitution* σ, α_n *being the* T*-value of* n.

We apply theorem 2(i), taking for $\mathcal{N}$ a standard model. We set $O = \{t \in \Lambda;\ t \succ f.\sigma\alpha_n$ for some substitution $\sigma\}$. From this definition of O, it follows that $f \in |Val(T,n) \to O|_{\mathcal{N}}$. From theorem 2, we deduce that $Tf \in |Int(n) \to O|_{\mathcal{N}}$. But, since $\theta \simeq_\beta n$, the following lemma shows that $\theta \in |Int(n)|_{\mathcal{N}}$, and therefore $Tf\theta \in O$.
Q.E.D.

Lemma 8 *If* $\theta \simeq_\beta \lambda f \lambda x\, f^n x$, $n \in \mathbf{N}$, *then* $\theta \in |Int(n)|_{\mathcal{N}}$, *for every model* $\mathcal{N}$. *Conversely, if* $\mathcal{M}$ *is a standard model and* $\theta \in |Int(u)|_{\mathcal{M}}$ ($u \in \mathcal{T}$), *then* $u \simeq_{\mathcal{E}} s^n 0$ *for some* $n \in \mathbf{N}$, *and* $\theta \simeq_\beta \lambda f \lambda x\, f^n x$ *(or, possibly, if* $n = 1$, $\theta \simeq_\beta \lambda x\, x$*).*

Let $\phi \in |\forall y(\Xi y \to \Xi sy)|_{\mathcal{N}}$ and $\alpha \in \Xi 0$, with $\Xi : \mathcal{T}_{\mathcal{E}} \to \Re_{\mathcal{N}}$. We have to show that $\theta\phi\alpha \in \Xi n$. Let f, a be λ-variables not in θ. Since $\theta \simeq_\beta \lambda f \lambda x\, f^n x$, we have $\theta f a \succ f t_n;\ t_n \succ f t_{n-1}; \ldots;\ t_2 \succ f t_1;\ t_1 \succ a$. Since the weak head reduction is compatible with substitution, we deduce that $\theta\phi\alpha \succ \phi t'_n;\ t'_n \succ \phi t'_{n-1}; \ldots;\ t'_2 \succ \phi t'_1;\ t'_1 \succ \alpha$, where $t'_i = t_i[\phi/f, \alpha/a]$. By induction on i, we show that $t'_{i+1} \in \Xi i$, for $0 \le i < n$. This is clear for $i = 0$, since $\Xi 0$ is saturated. If $t'_i \in \Xi(i-1)$, then $\phi t'_i \in \Xi i$ and therefore, $t'_{i+1} \in \Xi i$, because Ξi is saturated ($1 \le i < n$). Thus $t'_n \in \Xi(n-1)$, so that $\phi t'_n \in \Xi n$, and $\theta\phi\alpha \in \Xi n$, by saturation of Ξn.
Conversely, let $\theta \in |Int(u)|_{\mathcal{M}}$. We define $\Xi : \mathcal{T}_{\mathcal{E}} \to \Re_{\mathcal{M}}$ by $\Xi(v) = \emptyset$ if $v \not\simeq_{\mathcal{E}} s^k 0$ for any $k \in \mathbf{N}$; $\Xi(s^k 0) = \{\tau \in \Lambda;\ \tau \simeq_\beta f^k a\}$, f and a being variables not in θ. Then $a \in \Xi(0)$, and $f \in |\forall y(\Xi(y) \to \Xi(sy))|_{\mathcal{M}}$ (indeed, if $v \in \mathcal{T}$ is not an integer, then $\Xi(v) = \emptyset$, and thus $(\Xi(v) \to \Xi(sv)) = \Lambda$).
It follows that $\theta f a \in \Xi(u)$; thus, $\Xi(u) \ne \emptyset$, and $u \simeq_{\mathcal{E}} s^n 0$ for some $n \in \mathbf{N}$. Therefore, $\theta f a \simeq_\beta f^n a$, and it follows easily that $\theta \simeq_\beta \lambda f \lambda a\, f^n a$ (or, possibly, if $n = 1$, $\theta \simeq_\beta \lambda f\, f$).
Q.E.D.

In order to state the storage theorem for classical logic, we introduce a constant C in λ-calculus. We define the *head* C*-reduction* on λ-terms (notation $\succ_c$) as the least reflexive and transitive binary relation on Λ such that:
$(\lambda x\, t)u u_1 \ldots u_k \succ_c t[u/x]u_1 \ldots u_k$;
$\mathrm{C} u u_1 \ldots u_k \succ_c (u)\lambda x(x)u_1 \ldots u_k$, x being a variable not in $u_1, \ldots, u_k$.
A subset $\mathcal{X}$ of Λ will be called C*-saturated* if $t \in \mathcal{X}, t' \succ_c t \Rightarrow t' \in \mathcal{X}$.
The λ-constant C is declared to be of type $\forall X(\neg\neg X \to X)$. In this way, without changing the rules of construction of typed terms, we get typed terms in classical logic. We write: $x_1 : A_1, \ldots, x_k : A_k \vdash^c_{\mathcal{E}} t : A$ instead of $x_1 : A_1, \ldots, x_k : A_k, \mathrm{C} : \forall X(\neg\neg X \to X) \vdash_{\mathcal{E}} t : A$.
A Λ-model $\mathcal{M}$ will be called $\perp$*-standard* or *classical* if $|\perp|_{\mathcal{M}}$ is C-saturated, and $\Re_{\mathcal{M}}$ is the set of all intersections of subsets of Λ of the form $|\mathcal{X}_1, \ldots, \mathcal{X}_k \to \perp|_{\mathcal{M}}$, with $k \in \mathbf{N}, \mathcal{X}_1, \ldots, \mathcal{X}_k \subseteq \Lambda$ (or, what amounts to the same thing, the set of all intersections of subsets of Λ of the form $|\{t_1\}, \ldots, \{t_k\} \to \perp|_{\mathcal{M}}$, with $k \in \mathbf{N}, t_1, \ldots, t_k \in \Lambda$).

Lemma 9 *If $\mathcal{M}$ is a Λ-model which is $\perp$-standard, then* $\mathsf{C}\in |\forall X(\neg\neg X \to X)|_{\mathcal{M}}$.

We have to prove that $\mathsf{C}\in |\neg\neg\mathcal{X} \to \mathcal{X}|_{\mathcal{M}}$ for every $\mathcal{X} \in \Re_{\mathcal{M}}$. Let us suppose first that $\mathcal{X} = |\{t_1\}, \ldots, \{t_k\} \to \perp|_{\mathcal{M}}$, with $t_1, \ldots, t_k \in \Lambda$. Then, we remark that $\lambda x(x)t_1 \ldots t_k \in |\neg\mathcal{X}|_{\mathcal{M}}$ (x being not free in $t_1, \ldots, t_k$): indeed, for every $t \in \mathcal{X}$, we have $tt_1 \ldots t_k \in |\perp|_{\mathcal{M}}$. But $|\perp|_{\mathcal{M}}$ is C-saturated, and, therefore, $(\lambda x(x)t_1 \ldots t_k)t \in |\perp|_{\mathcal{M}}$.
Now let $t \in |\neg\neg\mathcal{X}|_{\mathcal{M}}$; from what has just been proved, it follows that $(t)\lambda x(x)t_1 \ldots t_k \in |\perp|_{\mathcal{M}}$. Since $|\perp|_{\mathcal{M}}$ is C-saturated, we get $\mathsf{C}tt_1 \ldots t_k \in |\perp|_{\mathcal{M}}$, that is $\mathsf{C}t \in \mathcal{X}$. It follows that $\mathsf{C}\in |\neg\neg\mathcal{X} \to \mathcal{X}|_{\mathcal{M}}$.
Let now $\mathcal{X}$ be any element of $\Re_{\mathcal{M}}$. Thus we have $\mathcal{X} = \bigcap_{i\in I} \mathcal{X}_i$, each $\mathcal{X}_i$ being of the form $|\{t_1\}, \ldots, \{t_k\} \to \perp|_{\mathcal{M}}$. Now, we know that $\mathsf{C} \in |\neg\neg\mathcal{X}_i \to \mathcal{X}_i|_{\mathcal{M}}$. If $t \in |\neg\neg\mathcal{X}|_{\mathcal{M}}$, then $t \in |\neg\neg\mathcal{X}_i|_{\mathcal{M}}$ for all $i \in I$, since $\mathcal{X} \subseteq \mathcal{X}_i$; therefore $\mathsf{C}t \in \mathcal{X}_i$ for each $i \in I$, and it follows that $\mathsf{C}t \in \mathcal{X}$, which is the desired result.
Q.E.D.

Theorem 4 (1st storage theorem for classical logic)
If $\vdash_{\mathcal{E}} T : \neg Int(n) \to \neg Int^*(n)$, *and* $\vdash^c_{\mathcal{E}} \tau : \neg\neg Int(n)$ *(n being an integer), then* $(\tau)(T)f \succ_c f.\sigma\alpha_n$ *for some substitution σ, α_n being the T-value of n. Furthermore, for any λ-terms $t_1, \ldots, t_k$, we have* $(\tau)\lambda x\, Tfxt_1 \ldots t_k \succ_c f.\sigma\alpha_n.t_1 \ldots t_k$ *for some substitution σ.*

Corollary 1 *If* $\vdash^c_{\mathcal{E}} \theta : Int(n)$, *then, for any* $k \in \mathbf{N}$ *and* $t_1, \ldots, t_k \in \Lambda$, *we have* $Tf\theta t_1 \ldots t_k \succ_c f.\sigma\alpha_n.t_1 \ldots t_k$ *for some substitution σ.*

Indeed, if $\vdash^c_{\mathcal{E}} \theta : Int(n)$, then $\vdash^c_{\mathcal{E}} \tau : \neg\neg Int(n)$, with $\tau = \lambda g\, g\theta$ (g being a variable which does not appear in θ), and we only have to apply theorem 4.
For $k = 0$, this gives theorem III.3 of [7].

Proof of theorem 4:
We apply theorem 2(i), taking for $\mathcal{N}$ a $\perp$-standard model of $\mathcal{E}$. We first set $O = |\perp|_{\mathcal{N}} = \{t \in \Lambda;\ t \succ_c f.\sigma\alpha_n$ for some substitution $\sigma\}$. From this definition of O, it follows that $f \in |Val(T, n) \to O|_{\mathcal{N}}$. From theorem 2, we deduce that $Tf \in |Int(n) \to O|_{\mathcal{N}}$ i.e. $Tf \in |\neg Int(n)|_{\mathcal{N}}$. But, by hypothesis, we have C: $\forall X(\neg\neg X \to X) \vdash_{\mathcal{E}} \tau : \neg\neg Int(n)$, and lemma 9 gives $\mathsf{C}\in |\forall X(\neg\neg X \to X)|_{\mathcal{N}}$. From adequacy lemma, it follows that $\tau \in |\neg\neg Int(n)|_{\mathcal{N}}$, and therefore $(\tau)(T)f \in |\perp|_{\mathcal{N}} = O$, which is the first desired result.
Now, we set $|\perp|_{\mathcal{N}} = \{t \in \Lambda;\ t \succ_c f.\sigma\alpha_n.t_1 \ldots t_k$ for some substitution $\sigma\}$, and $O = \{t_1\}, \ldots, \{t_k\} \to |\perp|_{\mathcal{N}}$. These definitions ensure that $f \in |Val(T, n) \to O|_{\mathcal{N}}$. By theorem 2, we obtain $Tf \in |Int(n) \to O|_{\mathcal{N}}$. By definition of O, it follows that $\lambda x\, Tfxt_1 \ldots t_k \in |\neg Int(n)|_{\mathcal{N}}$. Again from lemma 9 and adequacy lemma, we have $(\tau)\lambda x\, Tfxt_1 \ldots t_k \in |\perp|_{\mathcal{N}}$, which gives the second part of the theorem.
Q.E.D.

In the following theorem, $\phi(x)$ is a term of the language $\mathcal{L}$, the only free variable of which is x. $Rg(\phi)$ is its range, i.e. the set of all $\mathcal{L}$-terms which are $\simeq_{\mathcal{E}} \phi(t)$, for some closed $\mathcal{L}$-term t.

Theorem 5 (2nd storage theorem for classical logic)
If $\vdash_{\mathcal{E}} T : \forall x[\neg Int(x) \to \neg Int^*(x)]$*, and* $\vdash^c_{\mathcal{E}} \tau : \exists x Int[\phi(x)]$*, (*$\phi(x)$ *being a* $\mathcal{L}$*-term) then* $(\tau)(T)f \succ_c f.\sigma\alpha_n$ *for some integer* $n \in Rg(\phi)$ *and some substitution* σ*,* α_n *being the* T*-value of* n*. Furthermore, for any* λ*-terms* $t_1, \ldots, t_k$*, we have* $(\tau)\lambda x\, T f x t_1 \ldots t_k \succ_c f.\sigma\alpha_n.t_1 \ldots t_k$ *for some integer* $n \in Rg(\phi)$ *and some substitution* σ*.*

The proof is almost the same, and we only prove the first part. We apply theorem 2(ii), taking for $\mathcal{N}$ a $\perp$-standard model. We set $O = |\perp|_{\mathcal{N}} = \{t \in \Lambda;\ t \succ_c f.\sigma\alpha_n$ for some integer $n \in Rg(\phi)$ and some substitution $\sigma\}$. From this definition of O, it follows that $f \in |\forall x(Val[T, \phi(x)] \to O)|_{\mathcal{N}}$. From theorem 2, we deduce that $Tf \in |\forall x(Int[\phi(x)] \to O)|_{\mathcal{N}}$ i.e. $Tf \in |\forall x \neg Int[\phi(x)]|_{\mathcal{N}}$. Now, by hypothesis, we have $\vdash^c_{\mathcal{E}} \tau : \exists x Int[\phi(x)]$, that is $\vdash^c_{\mathcal{E}} \tau : \neg\forall x \neg Int[\phi(x)]$. Again from lemma 9 and adequacy lemma, it follows that $(\tau)(T)f \in |\perp|_{\mathcal{N}} = O$.
Q.E.D.

Remark. If we take $\phi(x) = x$, this theorem suggests to represent the type of integers by the formula $\exists x Int(x)$. And also, by taking $\phi(x) = 2x$, for example, it suggests that we could define the type of even integers, by the formula $\exists x\, Int(2x)$.

Appendix: proof of the adequacy lemma

Let $\mathcal{M}$ be a Λ-model. A *valuation* δ *in* $\mathcal{M}$ is a map defined on the set of variables of $\mathcal{L}$, and such that: $\delta x \in \mathcal{T}$ if x is a first order variable; $\delta X : \mathcal{T}^n_{\mathcal{E}} \to \Re_{\mathcal{M}}$ if X is a second order variable of arity n.
If t is a $\mathcal{L}$-term, with variables $x_1, \ldots, x_m$, we define $\delta t \in \mathcal{T}$ by $\delta t = t[\delta x_1/x_1, \ldots, \delta x_m/x_m]$.
If A is a formula, possibly with parameters in $\mathcal{M}$, with free variables $x_1, \ldots, x_m, X_1, \ldots, X_n$, and δ is a valuation in $\mathcal{M}$, then δA is defined as the closed formula $A[\delta x_1/x_1, \ldots, \delta x_m/x_m, \delta X_1/X_1, \ldots, \delta X_n/X_n]$ (with parameters in $\mathcal{M}$).
We now prove the

Theorem 6 *If* $x_1 : A_1, \ldots, x_k : A_k \vdash_{\mathcal{E}} t : A$ *and* δ *is a valuation in* $\mathcal{M}$*, then, for every* $u_1 \in |\delta A_1|_{\mathcal{M}}, \ldots, u_k \in |\delta A_k|_{\mathcal{M}}$*, we have* $t[u_1/x_1, \ldots, u_k/x_k] \in |\delta A|_{\mathcal{M}}$*.*

The adequacy lemma is the particular case of this theorem when $A_1, \ldots, A_k, A$ are closed formulas of $\mathcal{L}$.

Let $x_1 : A_1, \ldots, x_k : A_k \vdash_{\mathcal{E}} t : A$ be a typed term, δ a valuation in $\mathcal{M}$, and $u_1 \in |\delta A_1|_{\mathcal{M}}, \ldots, u_k \in |\delta A_k|_{\mathcal{M}}$. We prove that $t[u_1/x_1, \ldots, u_k/x_k] \in |\delta A|_{\mathcal{M}}$ by induction on the length of the derivation of $x_1 : A_1, \ldots, x_k : A_k \vdash_{\mathcal{E}} t : A$. Let us look at the rule used in the last step:
If it is rule 1, we have $t \equiv x_i$ and $A \equiv A_i (1 \le i \le k)$. Then the result is trivial.
If it is rule 2, we have $t \equiv \lambda x\, \tau$ and $A \equiv B \to C$; the previous step gave $x_1 : A_1, \ldots, x_k : A_k, x : B \vdash_{\mathcal{E}} \tau : C$. By the induction hypothesis, if $u \in |\delta B|_{\mathcal{M}}$, we have:
$\tau[u_1/x_1, \ldots, u_k/x_k, u/x] \in |\delta C|_{\mathcal{M}}$. Since $|\delta C|_{\mathcal{M}}$ is a saturated set, we get:
$(\lambda x\, \tau[u_1/x_1, \ldots, u_k/x_k])u \in |\delta C|_{\mathcal{M}}$, i.e. $t[u_1/x_1, \ldots, u_k/x_k]u \in |\delta C|_{\mathcal{M}}$. Since this is

true for every $u \in |\delta B|_{\mathcal{M}}$, we get $t[u_1/x_1, \ldots, u_k/x_k] \in |\delta B \to \delta C|_{\mathcal{M}} = |\delta A|_{\mathcal{M}}$.
If it is rule 3, we have $t = vw$ and previous steps gave $x_1 : A_1, \ldots, x_k : A_k \vdash_{\mathcal{E}} v : B \to A$ and $x_1 : A_1, \ldots, x_k : A_k \vdash_{\mathcal{E}} w : B$. By the induction hypothesis, we have $v[u_1/x_1, \ldots, u_k/x_k] \in |\delta B \to \delta A|_{\mathcal{M}}$ and $w[u_1/x_1, \ldots, u_k/x_k] \in |\delta B|_{\mathcal{M}}$. Therefore, $(vw)[u_1/x_1, \ldots, u_k/x_k] \in |\delta A|_{\mathcal{M}}$, i.e. $t[u_1/x_1, \ldots, u_k/x_k] \in |\delta A|_{\mathcal{M}}$.
If it is rule 4, we have $A \equiv B[v/x]$, v being a $\mathcal{L}$-term. The previous step gave $x_1 : A_1, \ldots, x_k : A_k \vdash_{\mathcal{E}} t : \forall x B$. By the induction hypothesis, we have $t[u_1/x_1, \ldots, u_k/x_k] \in |\delta(\forall x B)|_{\mathcal{M}}$. But $\delta(B[v/x]) = \delta B[\delta v/x]$, and therefore $|\delta(\forall x B)|_{\mathcal{M}} \subset |\delta(B[v/x])|_{\mathcal{M}}$. It follows that $t[u_1/x_1, \ldots, u_k/x_k] \in |\delta(B[v/x])|_{\mathcal{M}} = |\delta A|_{\mathcal{M}}$.
If it is rule 5, we have $A \equiv \forall x B$, and the previous step gave $x_1 : A_1, \ldots, x_k : A_k \vdash_{\mathcal{E}} t : B$, x not being free in $A_1, \ldots, A_k$. Let $v \in \mathcal{T}$, and δ' be the valuation identical with δ, except on x, and such that $\delta' x = v$. Since x is not free in $A_1, \ldots, A_k$, we have $\delta A_i = \delta' A_i$, and therefore $u_i \in |\delta' A_i|_{\mathcal{M}}$. By the induction hypothesis applied with δ', we get $t[u_1/x_1, \ldots, u_k/x_k] \in |\delta' B|_{\mathcal{M}} = |\delta(B[v/x])|_{\mathcal{M}}$. Since this is true for each $v \in \mathcal{T}$, we obtain $t[u_1/x_1, \ldots, u_k/x_k] \in \bigcap_{v \in \mathcal{T}} |\delta(B[v/x])|_{\mathcal{M}} = |\delta(\forall x B)|_{\mathcal{M}} = |\delta A|_{\mathcal{M}}$.
If it is rule 6, we have $A \equiv B[F/Xy_1 \ldots y_n]$, and the previous derivation step gave $x_1 : A_1, \ldots, x_k : A_k \vdash_{\mathcal{E}} t : \forall X B$. By the induction hypothesis, we have $t[u_1/x_1, \ldots, u_k/x_k] \in |(\forall X B)|_{\mathcal{M}} = \bigcap\{\delta(B[\Phi/X]); \Phi : \mathcal{T}_{\mathcal{E}}^n \to \Re_{\mathcal{M}}\}$.
By the following lemma, we have $|\delta(B[F/Xy_1 \ldots y_n])|_{\mathcal{M}} = |\delta(B[\Phi_0/X])|_{\mathcal{M}}$ for some $\Phi_0 : \mathcal{T}_{\mathcal{E}}^n \to \Re_{\mathcal{M}}$. It follows that $|\delta(\forall X B)|_{\mathcal{M}} \subset |\delta(B[F/Xy_1 \ldots y_n])|_{\mathcal{M}} = |\delta A|_{\mathcal{M}}$. Thus $t[u_1/x_1, \ldots, u_k/x_k] \in |\delta A|_{\mathcal{M}}$.
If it is rule 7, we have $A \equiv \forall X B$, and the previous derivation step gave $x_1 : A_1, \ldots, x_k : A_k \vdash_{\mathcal{E}} t : B$, X not being free in $A_1, \ldots, A_k$. Let $\Phi : \mathcal{T}_{\mathcal{E}}^n \to \Re_{\mathcal{M}}$, n being the arity of X, and let δ' be the valuation identical with δ, except on X, and such that $\delta' X = \Phi$. Since X is not free in $A_1, \ldots, A_k$, we have $|\delta A_i|_{\mathcal{M}} = |\delta' A_i|_{\mathcal{M}}$, and therefore $u_i \in |\delta' A_i|_{\mathcal{M}}$. By the induction hypothesis, we get $t[u_1/x_1, \ldots, u_k/x_k] \in |\delta' B|_{\mathcal{M}} = |\delta(B[\Phi/X])|_{\mathcal{M}}$. Since this is true for every Φ, we get $t[u_1/x_1, \ldots, u_k/x_k] \in \bigcap\{|\delta(B[\Phi/X])|_{\mathcal{M}}; \Phi : \mathcal{T}_{\mathcal{E}}^n \to \Re_{\mathcal{M}}\} = |\delta(\forall X B)| = |\delta A|_{\mathcal{M}}$.
If it is rule 8, we have $A \equiv B[v/x]$, and the previous derivation step gave $x_1 : A_1, \ldots, x_k : A_k \vdash_{\mathcal{E}} t : B[u/x]$, u and v being $\mathcal{L}$-terms such that $\mathcal{E} \vdash u = v$.
By the induction hypothesis, we have $t[u_1/x_1, \ldots, u_k/x_k] \in |B[u/x]|_{\mathcal{M}}$. But $u \simeq_{\mathcal{E}} v$, and therefore $|B[u/x]|_{\mathcal{M}} = |B[v/x]|_{\mathcal{M}} = |A|_{\mathcal{M}}$. Hence the result.
Q.E.D.

Lemma 10 *Let B and F be formulas of $\mathcal{L}$, possibly with parameters in $\mathcal{M}$, and δ a valuation in $\mathcal{M}$. Then $|\delta(B[F/Xy_1 \ldots y_n])|_{\mathcal{M}} = |\delta(B[\Phi_0/X])|_{\mathcal{M}}$, where $\Phi_0 : \mathcal{T}_{\mathcal{E}}^n \to \Re_{\mathcal{M}}$ is defined by $\Phi_0(t_1, \ldots, t_n) = |\delta(F[t_1/y_1, \ldots, t_n/y_n])|_{\mathcal{M}}$ for every $t_1, \ldots, t_n \in \mathcal{T}_{\mathcal{E}}$.*

The proof is by induction on the formula B. If B is atomic, the only non-trivial case is $B \equiv Xv_1 \ldots v_n$, $v_1, \ldots, v_n$ being $\mathcal{L}$-terms. Then $|\delta(B[F/Xy_1 \ldots y_n])|_{\mathcal{M}} = |\delta(F[v_1/y_1, \ldots, v_n/y_n])|_{\mathcal{M}} = |\delta(F[\delta v_1/x_1, \ldots, \delta v_n/y_n])|_{\mathcal{M}} = \Phi_0(\delta v_1, \ldots, \delta v_n) = \delta(\Phi_0(v_1, \ldots, v_n))$.
If $B \equiv B' \to B''$, the result follows immediately from the induction hypothesis.
If $B \equiv \forall x B'$, we may assume that the first order variable x does not appear in F. Then $|\delta(B[F/Xy_1 \ldots y_n])|_{\mathcal{M}} = \bigcap_{t \in \mathcal{T}} |\delta(B'[F/Xy_1 \ldots y_n][t/x])|_{\mathcal{M}} =$

$\bigcap_{t\in\mathcal{T}} |\delta(B'[t/x][F/Xy_1 \ldots y_n])|_{\mathcal{M}} = \bigcap_{t\in\mathcal{T}} |\delta(B'[t/x][\Phi_0/X])|_{\mathcal{M}}$ (by the induction hypothesis) $= \bigcap_{t\in\mathcal{T}} |\delta(B'[\Phi_0/X][t/x])|_{\mathcal{M}} = |\delta(\forall x\, B'[\Phi_0/X])|_{\mathcal{M}} = |\delta(B[\Phi_0/X])|_{\mathcal{M}}$.
If $B \equiv \forall Y B'$, Y being a second order variable of arity p, (we may assume it does not appear in F) then:
$|\delta(B[F/Xy_1 \ldots y_n])|_{\mathcal{M}} = \bigcap\{|\delta(B'[F/Xy_1 \ldots y_n][\Psi/Y])|_{\mathcal{M}}; \Psi : \mathcal{T}_{\mathcal{E}}^p \to \Re_{\mathcal{M}}\}$
$= \bigcap\{|\delta(B'[\Psi/Y][F/Xy_1 \ldots y_n])|_{\mathcal{M}}; \Psi : \mathcal{T}_{\mathcal{E}}^p \to \Re_{\mathcal{M}}\}$
$= \bigcap\{|\delta(B'[\Psi/Y][\Phi_0/X])|_{\mathcal{M}}; \Psi : \mathcal{T}_{\mathcal{E}}^p \to \Re_{\mathcal{M}}\}$ (by the induction hypothesis)
$= \bigcap\{|\delta(B'[\Phi_0/X][\Psi/Y])|_{\mathcal{M}}; \Psi : \mathcal{T}_{\mathcal{E}}^p \to \Re_{\mathcal{M}}\} = |\delta(\forall X\, B'[\Phi_0/X])|_{\mathcal{M}} = |\delta(B[\Phi_0/X])|_{\mathcal{M}}$.
Q.E.D.

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