Applications of Kolmogorov complexity to classical and quantum computational complexity

Habilitation à diriger des recherches

Sophie Laplante

LRI, Université Paris-Sud XI

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Summary

- Context and overview of contributions
- Part I Foundations
  - Time bounded Kolmogorov complexity
  - Quantum Kolmogorov complexity
- Part II Applications
  - Quantum query complexity lower bounds
  - Formula size lower bounds
- Research projects

Computational complexity

Computational complexity

- Shannon (1949)
  - defines circuits as a model of computation
  - proposes circuit size as a measure of complexity
  - poses the problem of finding an explicit function for which exponential size circuits are required.

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! Despite much effort, still no separation in sight

Lower bound techniques

- Significant separations have been achieved by diagonalization

“So many problems,
so few machines…!”

- Many known techniques seem to be fundamentally information theoretic

“So much information,
so little time…!”

Kolmogorov complexity

Introduced by Solomonoff, Kolmogorov, and Chaitin (*algorithmic information*), in the 60s

$K(x)$ is the length of the shortest program that prints x.

- $K(\text{“0101010101010101 ....”}) \approx log(n)$

n

- $K($“  ....”$) \approx n$

$K(x|y)$ is the length of the shortest program that prints x when given string y as auxiliary input.

Incompressibility

Fundamental tool for proving lower bounds:

- For any finite set A, $\exists x \in A, K(x) \geq log(\#A)$

  (there are not enough short programs to describe all x in A)

Corresponding upper bound:

- For any finite set A, $\forall x \in A, K(x) \leq log(\#A)$

  (suffices to give an index into the set A)

Classical decision tree model

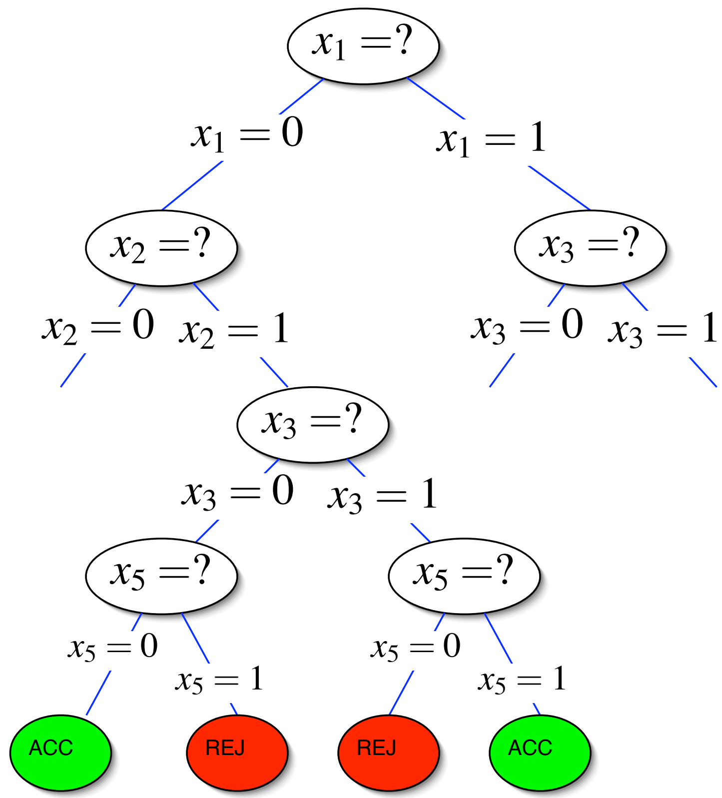
To compute a boolean function $f : \{0,1\}^n \rightarrow \{0,1\}$,

Model : decision tree

Cost : Number of queries to input

Query complexity of f:

DT(f) is depth of shallowest decision tree for f



Simple decision tree lower bound

<u>Proposition</u> [L] If $f(x) \neq f(y)$,
then there exists $i, x_i \neq y_i$:

$$K(i|x) \leq log(\text{depth}(T))$$

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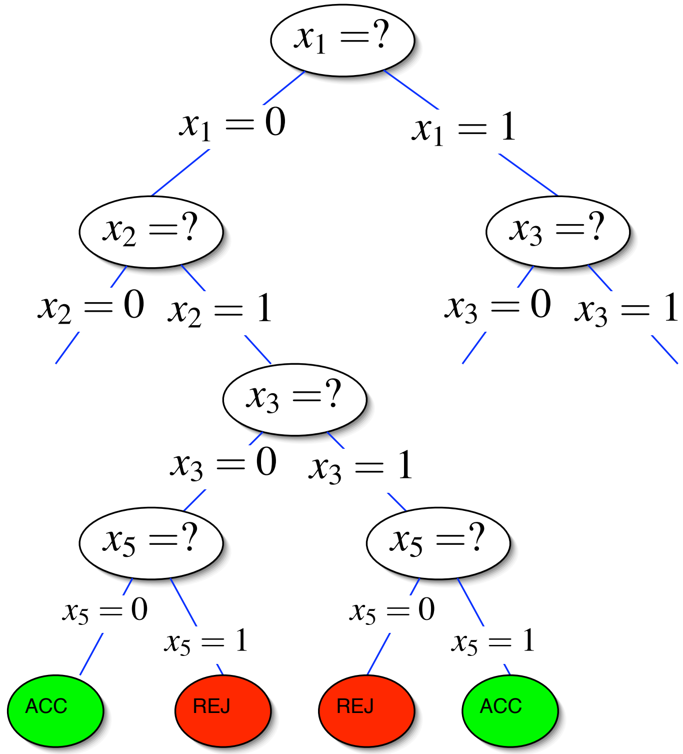
$$DT(f) \geq min_i\{max\{2^{K(i|x)}, 2^{K(i|y)}\}\}$$



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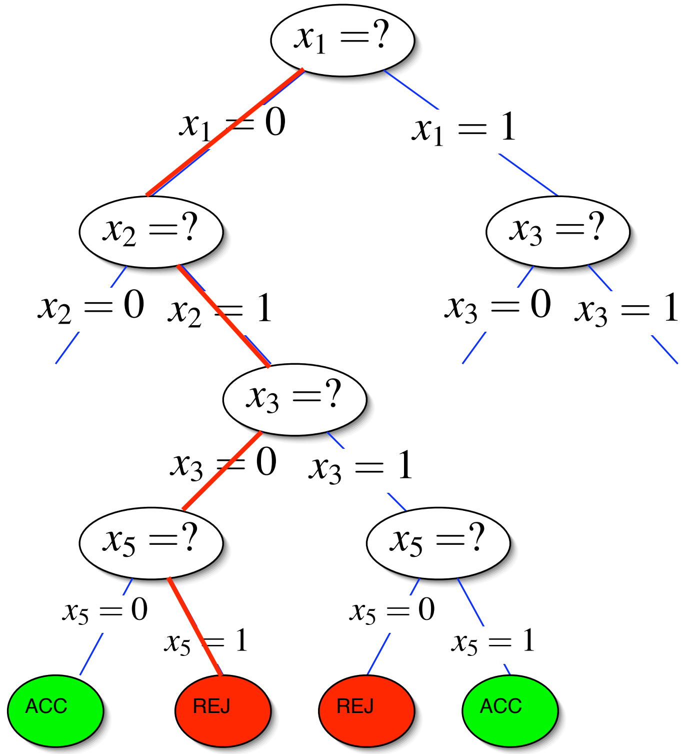

$f(x) = 0$

Simple decision tree lower bound

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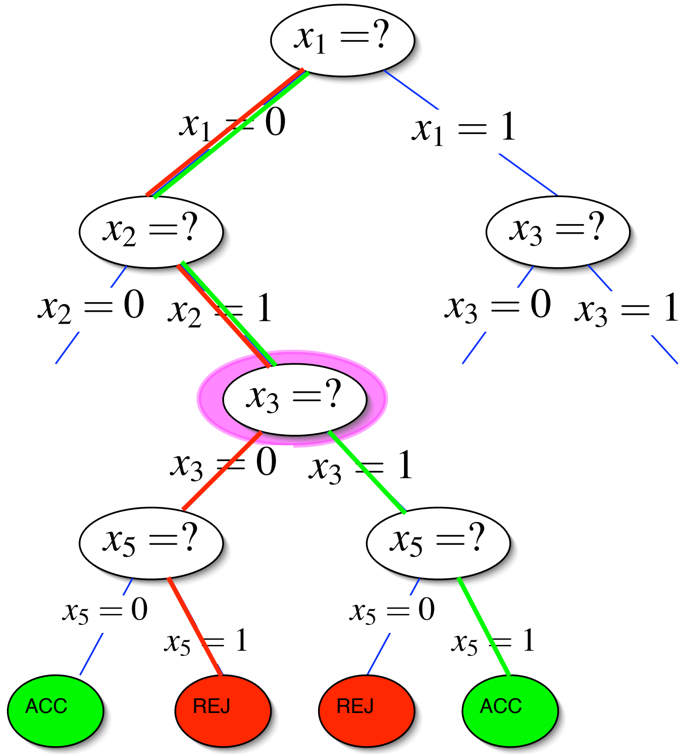
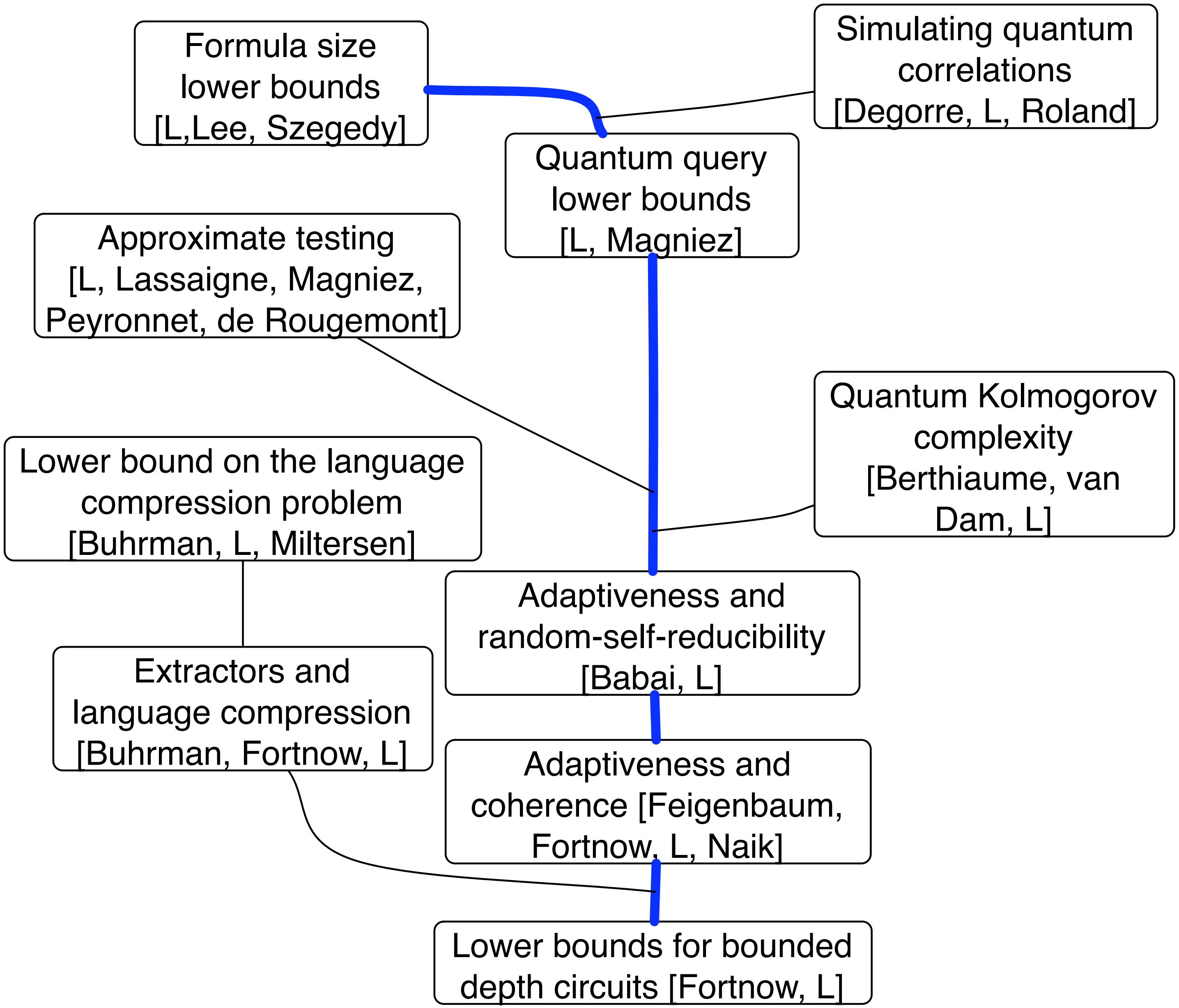


$f(x)=0$ $f(y)=1$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Simulating quantum correlations [Degorre, L, Roland]
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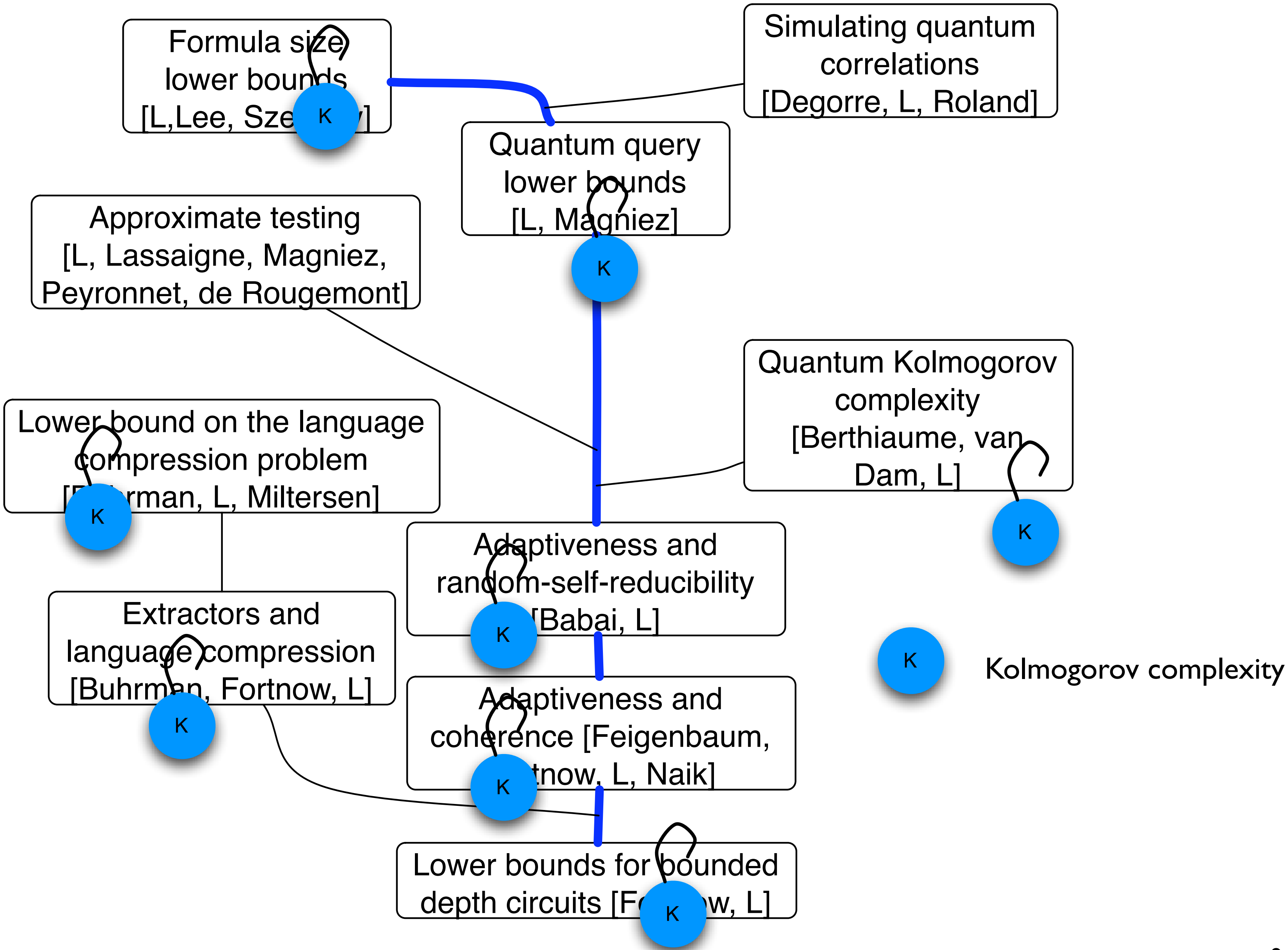

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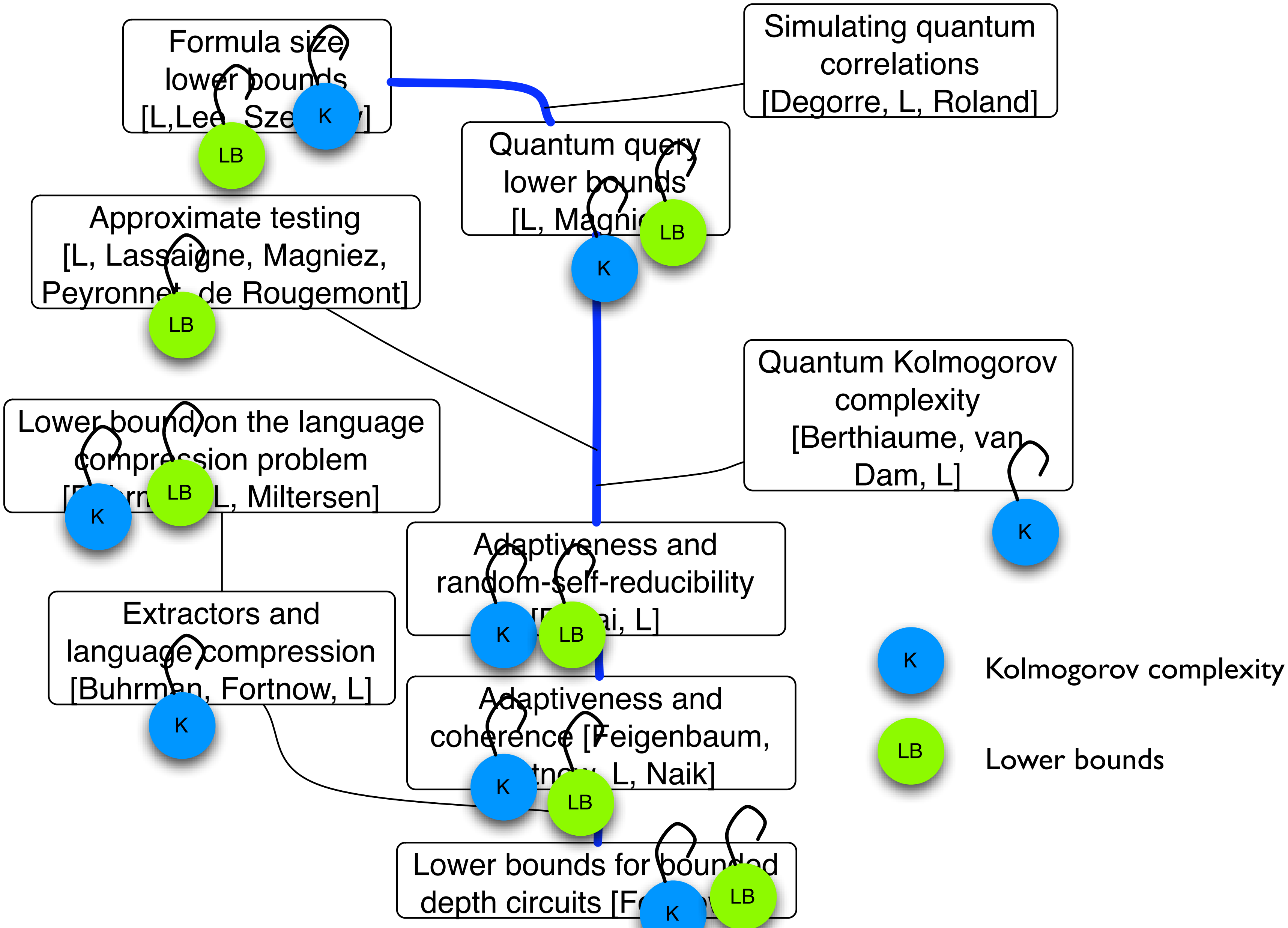

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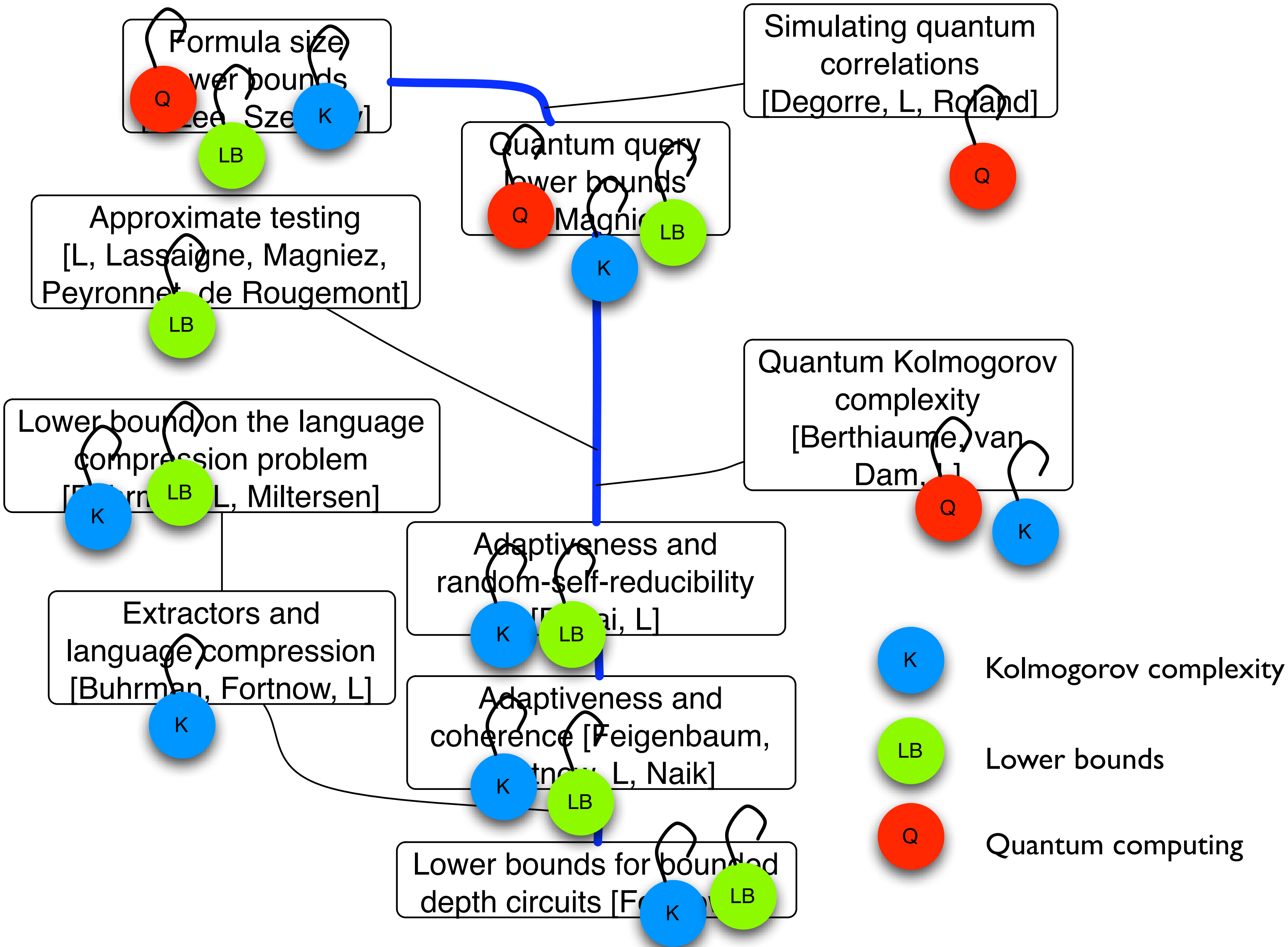
Lower bounds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Quantum computing

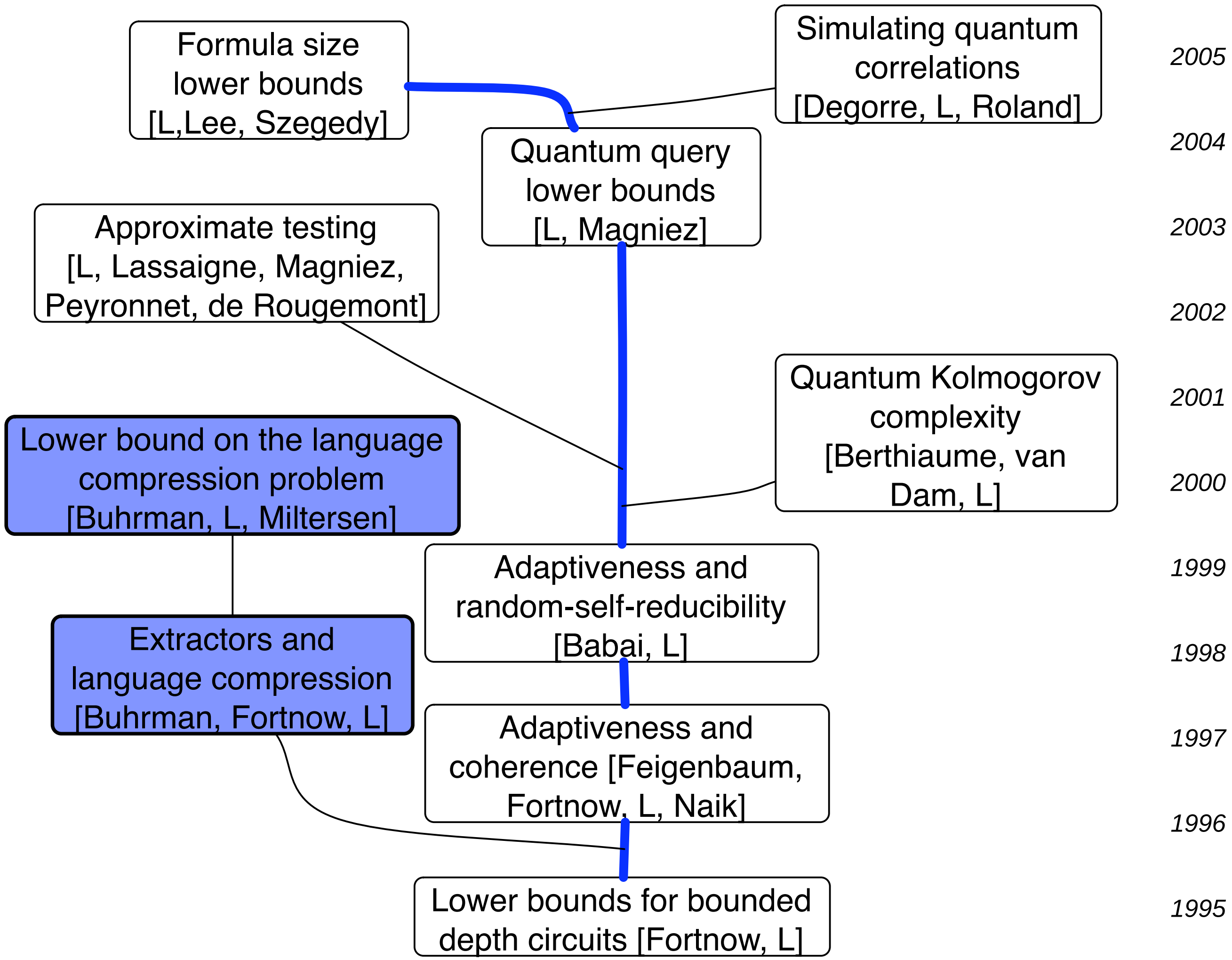
Part I

Foundations

Time bounded Kolmogorov complexity

Quantum Kolmogorov complexity

Foundations: Time bounded complexity



Time-bounded Kolmogorov complexity

$C^p(x)$ is the length of the shortest program that prints x in time $p(|x|)$.

$CD^p(x)$ is the length of the shortest program that runs in time $p(|z|)$ and accepts z if and only if $z = x$.

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$CD^p(x)$ is the length of the shortest program that runs in time $p(|z|)$ and accepts z if and only if $z = x$.

- In unbounded time, $CD^\infty = C^\infty$.
- For any finite set A, and $x \in A$ $CD^\infty(x) \leq log(\#A)$
- The language compression problem [S83]:

For any A, $x \in A$ $CD^p(x) \leq ??$ for polynomial p ?

Language compression problem

- For most r, $CD^p(x|r) \leq log(\#A)$ [S83]
- $CD^p(x) \leq 2\ log(\#A)$ [BFL02]

Chinese remainder theorem

- For all but ϵ fraction of $x \in A$,

$$CD^p(x|r) \leq log(\#A) + polylog(|x|/\epsilon)$$ [BFL02]

Extractors

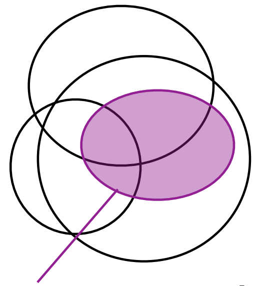
- Exists A, $x \in A$, $CD^p(x) \geq 2\ log(\#A)$ [BLM00]

Cover-free families

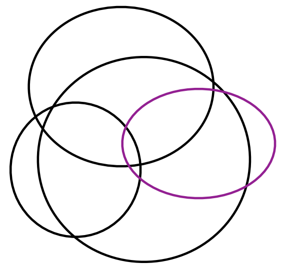
Cover-free families of sets

- Definition $\mathcal{F}$ is k-cover free if for any $F_0,\ldots F_k$ in $\mathcal{F}$, $F_0 \nsubseteq \cup_i F_i$
- Theorem [DR82] Let $\mathcal{F}$ be a family of N sets over a universe of M elements. If $\mathcal{F}$ is k-cover free and $N > k^3$, then

$$M \geq \frac{N^2 \log(N)}{2\log(k) + O(1)}$$



F_0 is covered by the other sets



$\mathcal{F}$ is 3-cover free

Lower bound on language compression

Theorem [BLM00] $\exists A, x \in A, CD^{p,A}(x) \geq 2\, log\, (\#A)$

Programs

Lower bound on language compression

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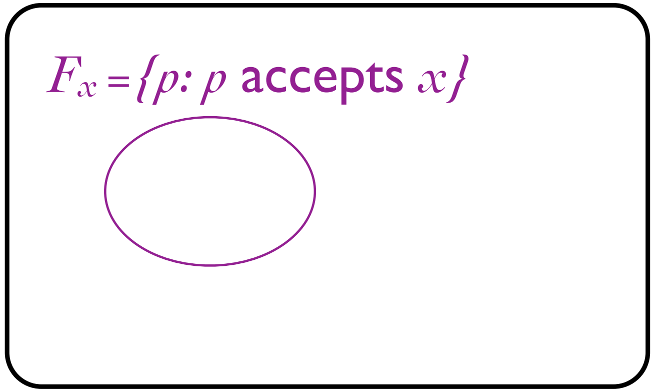
Programs



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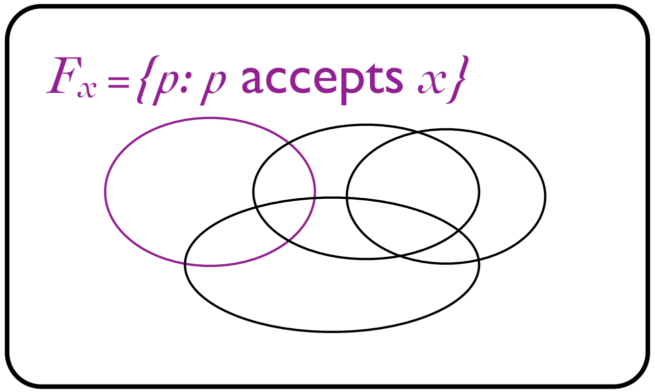

$\mathcal{F} = \{F_x \mid x \in A\}$ is k-cover free

Lower bound on language compression

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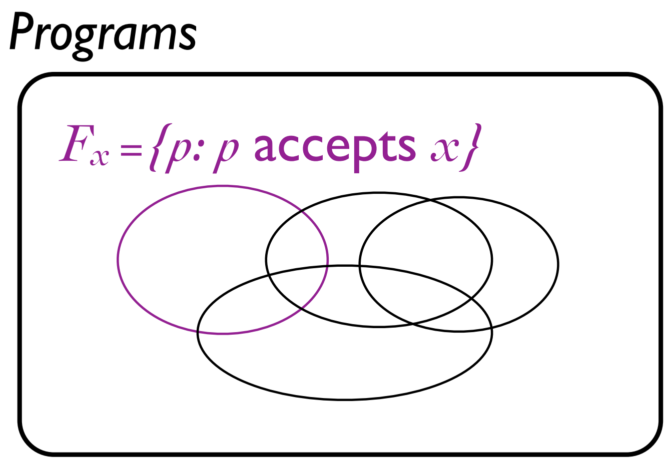
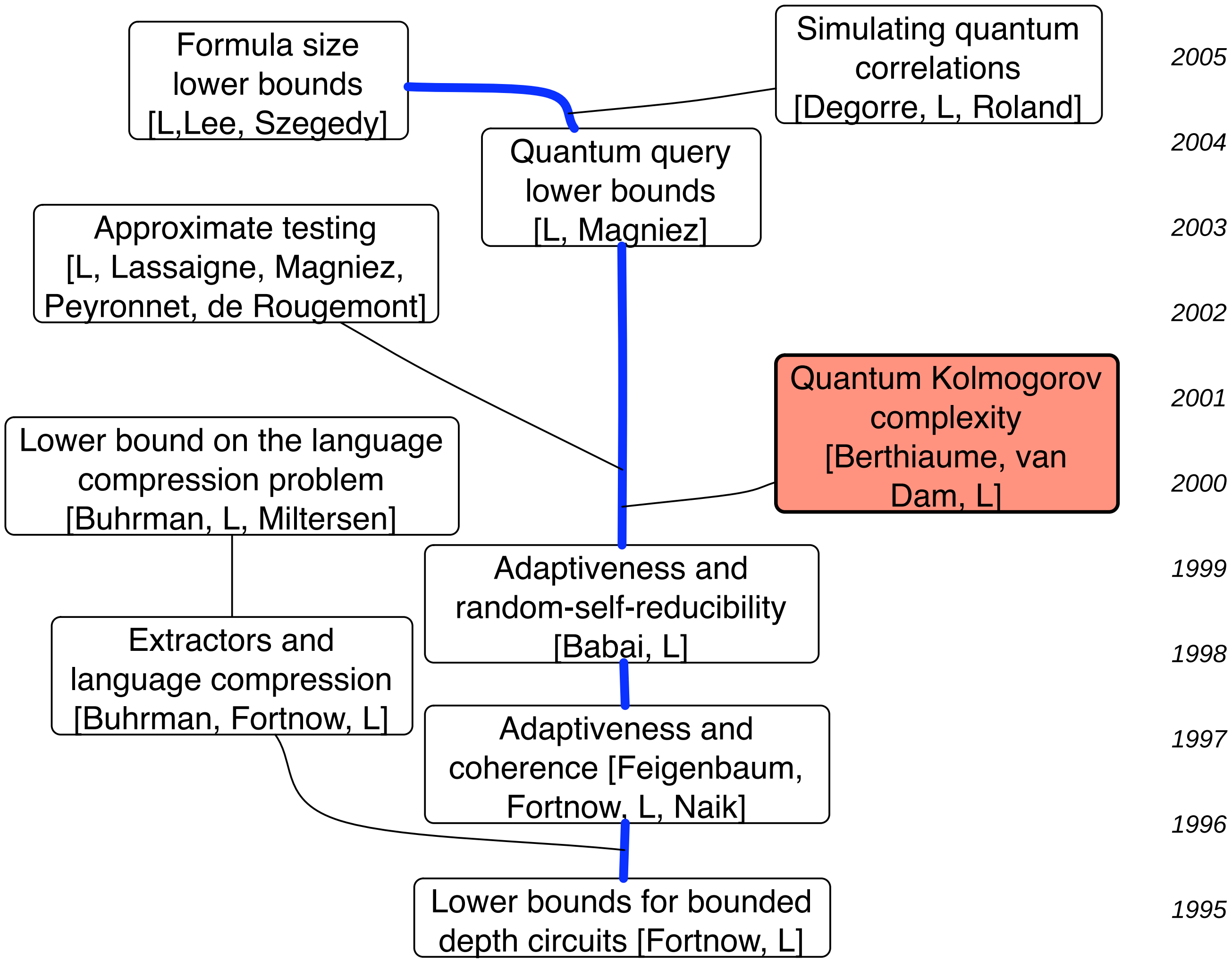


$\mathcal{F} = \{F_x \mid x \in A\}$ is k-cover free

N inputs ($\#\mathcal{F} = r^{1/3}$)
M programs
$k \sim N^{1/3} \sim r^{1/9}$

$$M \geq \frac{N^2 \log(N)}{2\log(k) + O(1)}$$

Foundations: Quantum Kolmogorov complexity



Quantum computation

- Computation acts on *qubits*
  - n-bit strings are vectors forming an orthonormal basis of 2^n-dimensional Hilbert space, $\{|i\rangle = e_i\}_{1 \leq i \leq 2^n}$
  - Qubits are unit, complex combinations of basis states
- Quantum gates are unitary operations
  - $U^\dagger U = I$
  - Linear, invertible, norm-preserving

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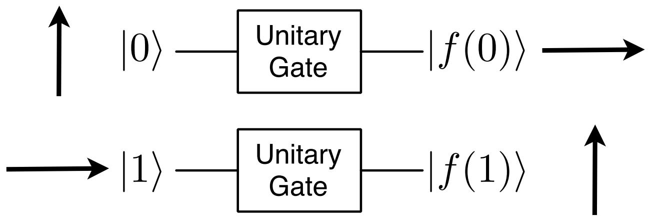

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$$\alpha|0\rangle + \beta|1\rangle \;—\; \boxed{\text{Unitary Gate}} \;—\; \alpha|f(0)\rangle + \beta|f(1)\rangle$$

Quantum Kolmogorov complexity

- Three definitions have been proposed
  - Classical description [V00]
  - Quantum description [BDL00]
  - Semi-density matrices [G01]
- We give a quantum description by means of universal quantum Turing machine U [BV97]
  - $QC(|\phi\rangle) = \min\{\underbrace{dim(|\psi\rangle)}_{\text{number of qubits}} : U|\psi\rangle \approx |\phi\rangle\}$

Properties of quantum Kolmogorov complexity

- Properties of [BDL00] definition
  - Existence of incompressible quantum states
  - Strong connection to quantum information theory (von Neumann entropy)
  - Quantification of no-cloning of quantum states:

$$QC(|\phi\rangle^{\otimes k} \mid |\phi\rangle)$$

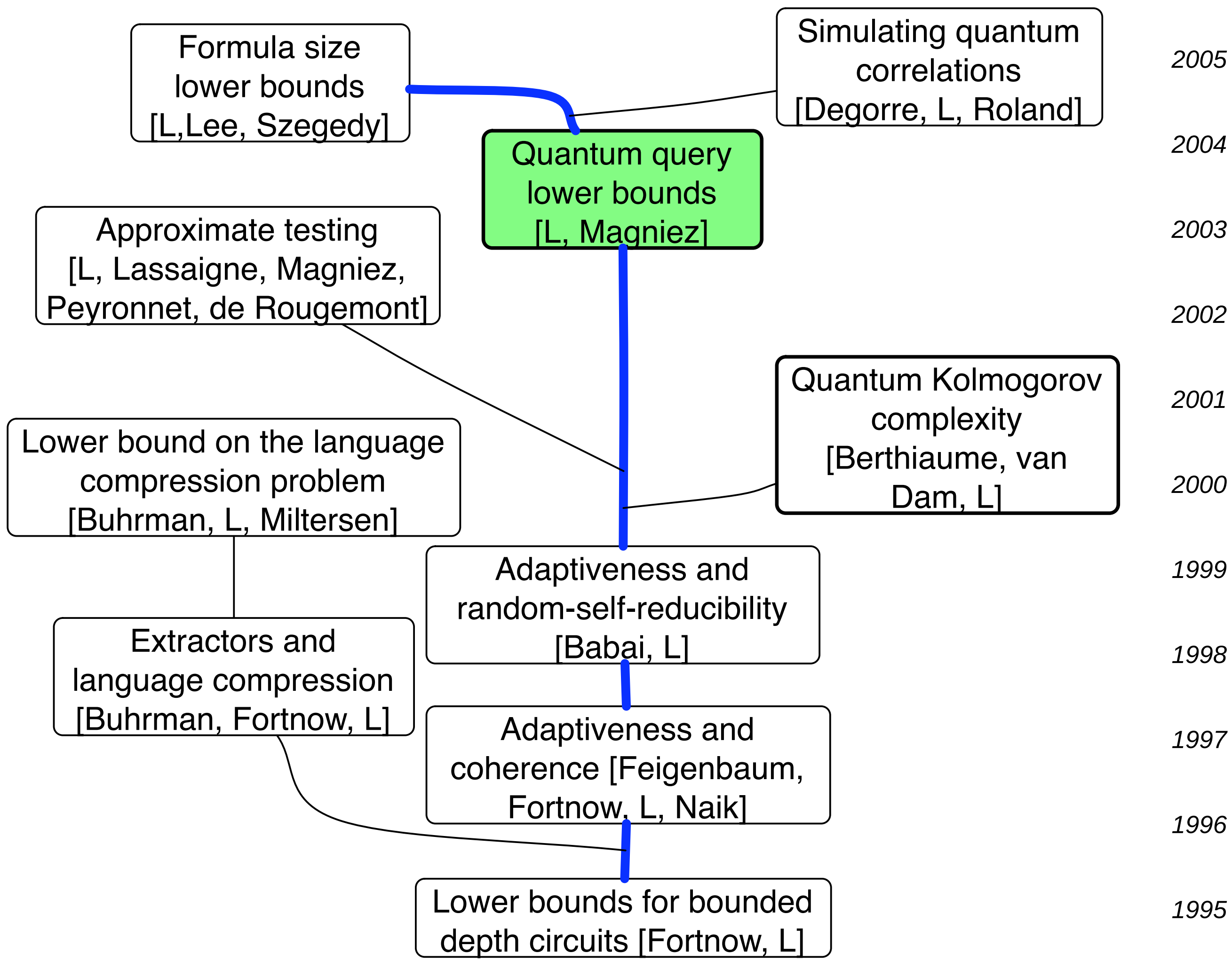
Part II

Applications

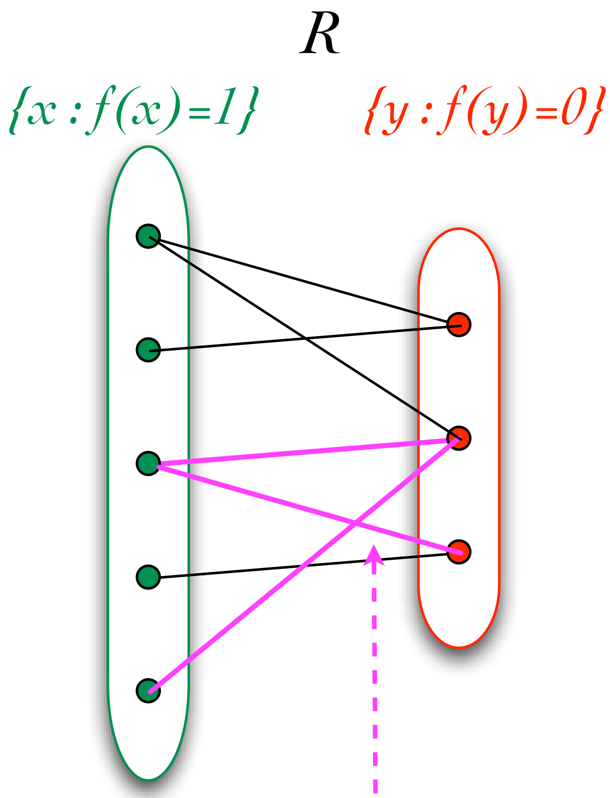
Quantum query complexity lower bounds

Formula size lower bounds

Applications: Quantum lower bounds

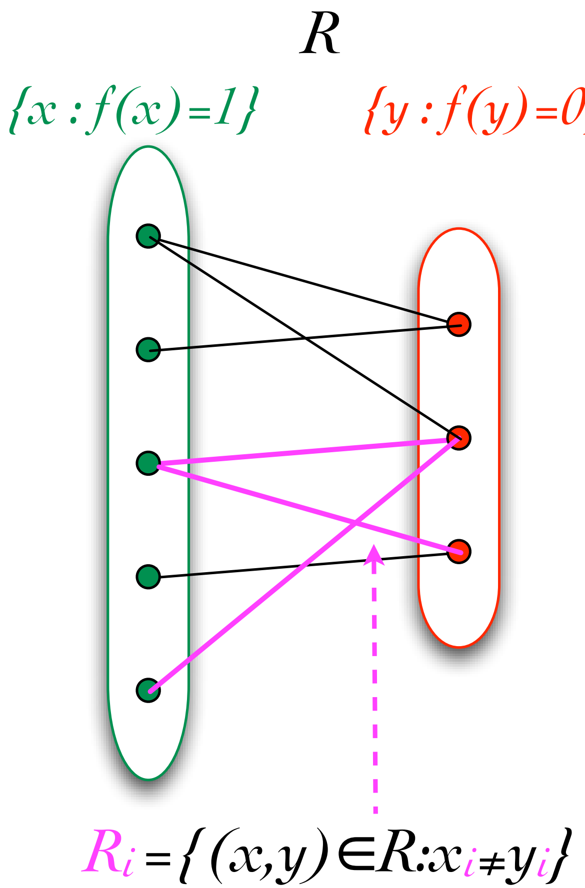


Adversary method



$R_i = \{ (x,y) \in R : x_i \neq y_i \}$

Adversary method



<u>Proposition</u> [L] For any relation R,

$$\forall i \; DT(f) \geq \deg_{min}(R)/(\deg(R_i))$$

Recall that

$$\forall i \; DT(f) \geq 2^{K(i|x)}$$

- $K(x,y) \geq \log(\#f^{-1}(1)\deg_{min}(R))$
- $K(x,y) \leq K(x) + K(i|x) + K(y|x,i)$
- $\#f^{-1}(1)\deg_{min}(R) \leq \#f^{-1}(1) \cdot 2^{K(i|x)} \cdot \deg(R_i)$
- $2^{K(i|x)} \geq \deg_{min}(R)/(\deg(R_i))$

Quantum adversary lower bounds

Theorem [LM04]

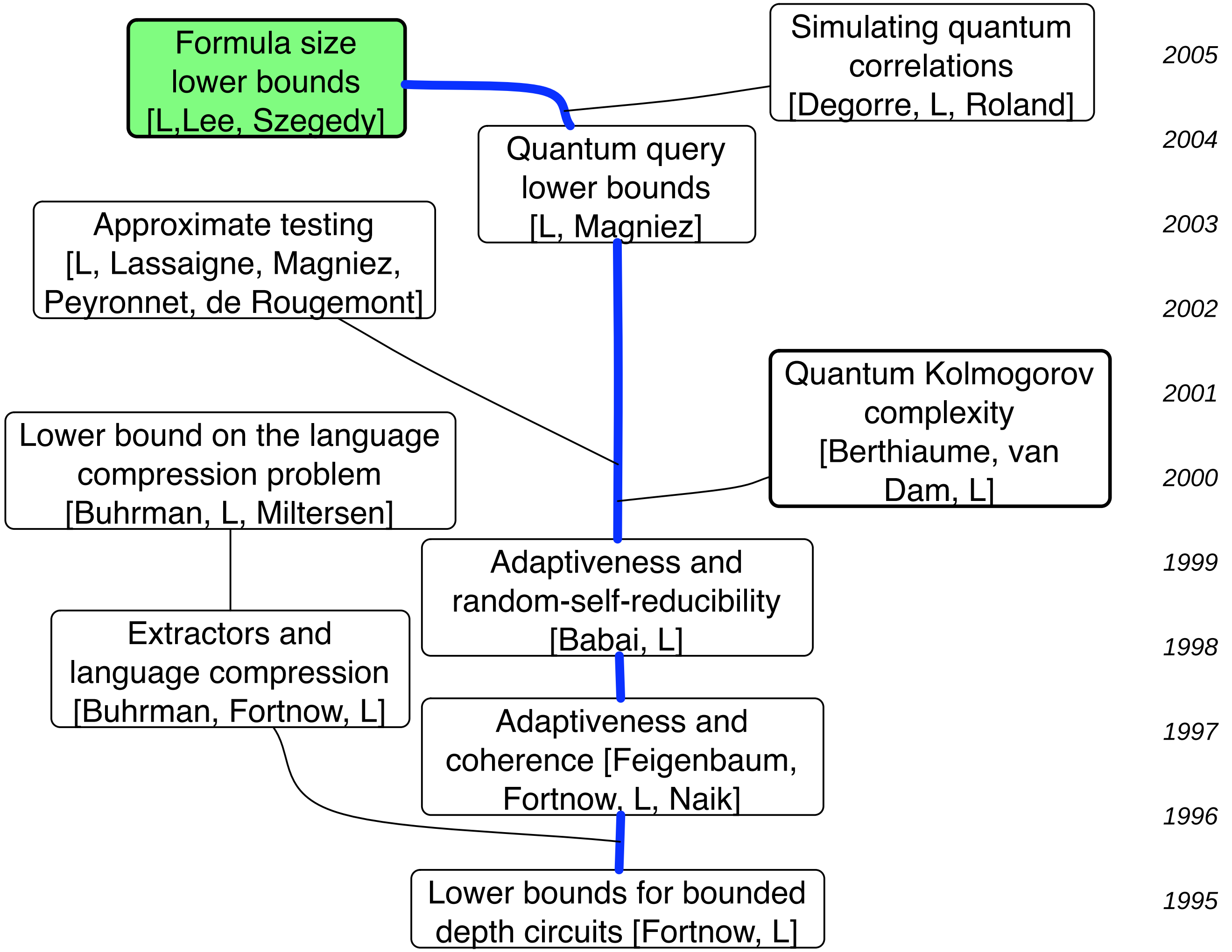
$$Q_\varepsilon(f) \geq \frac{c_\epsilon}{2} \frac{1}{\sum_{i:x_i \neq y_i} \sqrt{2^{-K(i|x)-K(i|y)}}}$$

Implies all previously known quantum adversary lower bounds

- Unweighted adversary [A02]
- Weighted adversary [A03]
- Spectral method [BSS03]

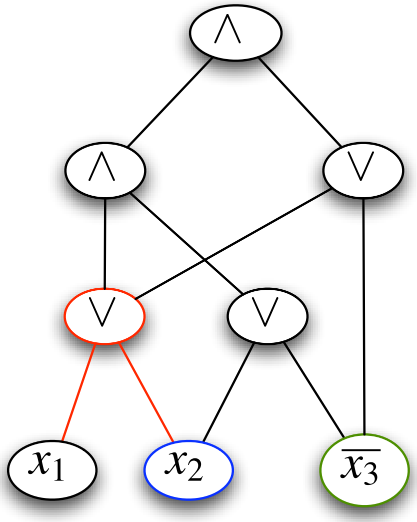
All these methods are equivalent [ŠS05]

Applications: Formula size lower bounds



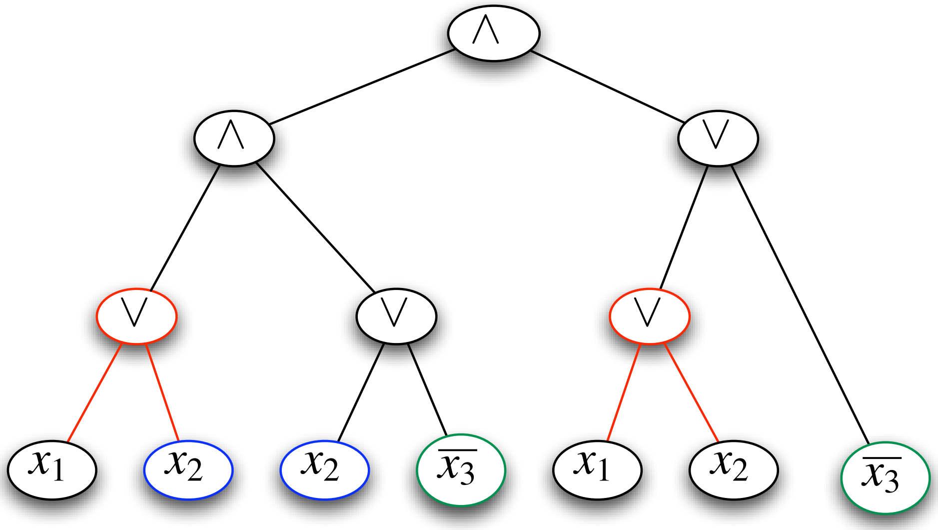
Boolean circuit and formula size

Boolean circuit



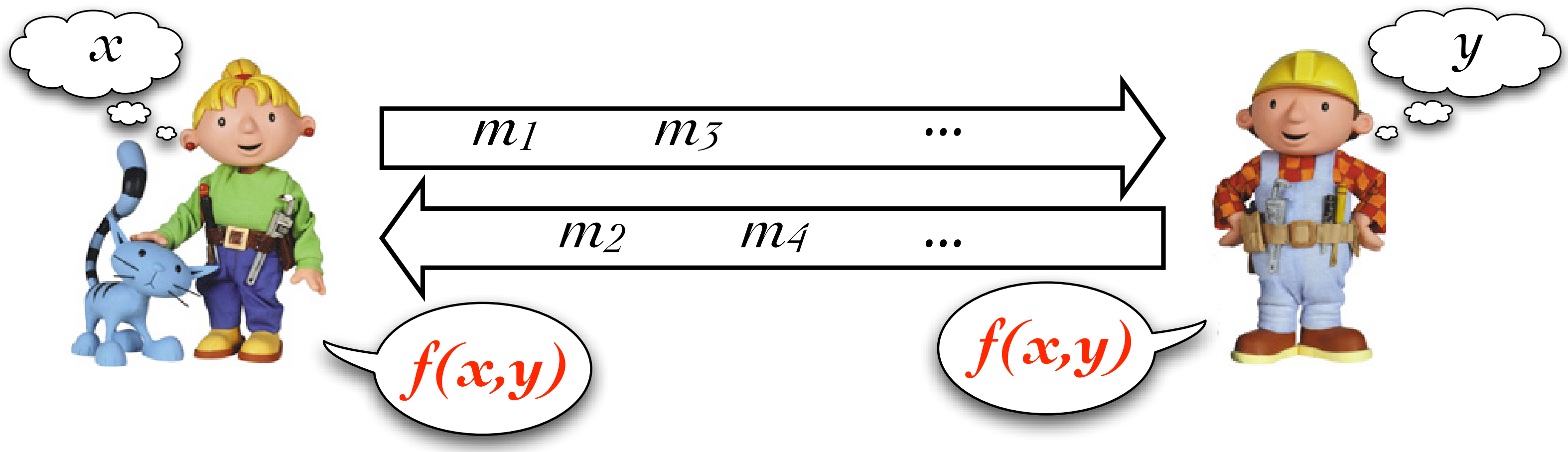
Best lower bound: 5n
[Lachish Raz 01,
Iwama Morizumi 02]

Boolean formula



Best lower bound: n^3
[Håstad 98]

Communication complexity

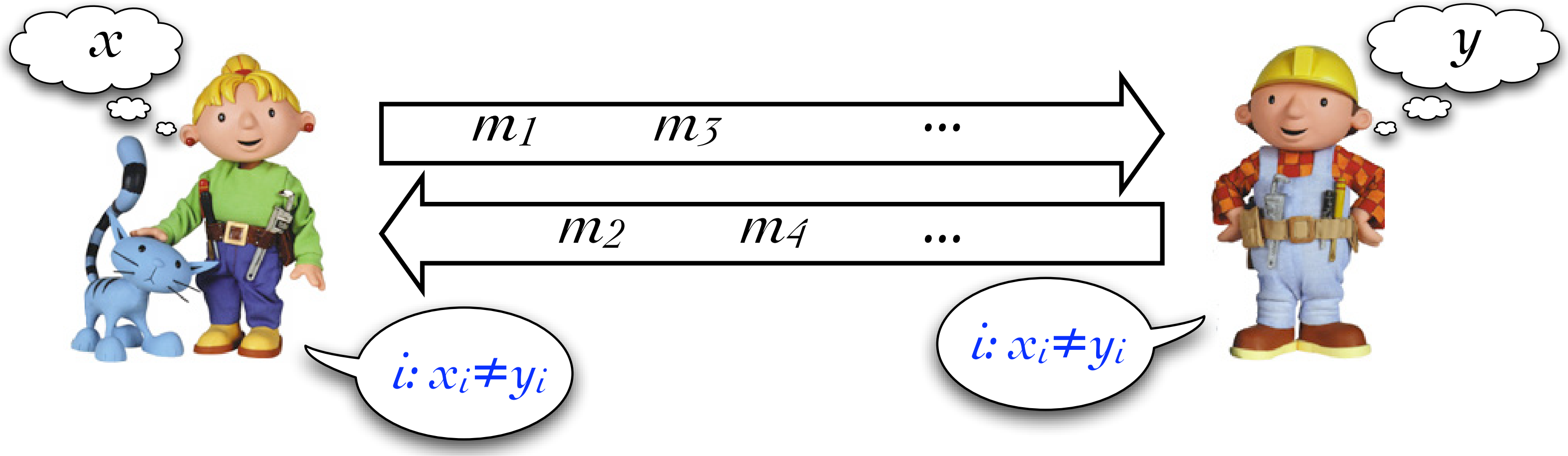


- $D(f)$ = amount of communication in the worst case, for the best protocol for f
- d(f) = $D(R_f)$ [KW88]

Circuit depth

Given x, y for which $f(x) \neq f(y)$, find i s.t. $x_i \neq y_i$

Circuit depth lower bound



Proposition [LLS05]

$$K(i|x)+K(i|y) \leq D(R_f)=\mathrm{d}(f)$$

Proof

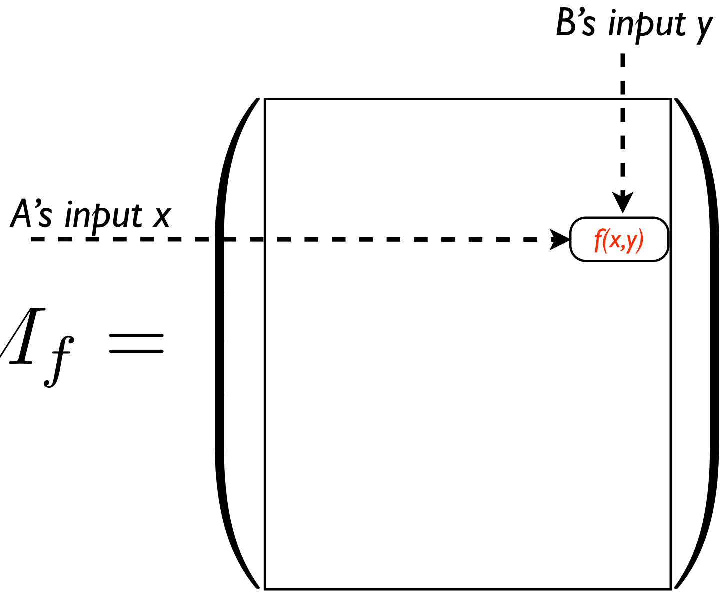
$$K(i|x) \leq |m_2| + |m_4| + \ldots$$
$$K(i|y) \leq |m_1| + |m_3| + \ldots$$

Background on communication complexity

$$M_f = \begin{pmatrix} \end{pmatrix}$$

Background on communication complexity

$$M_f =$$



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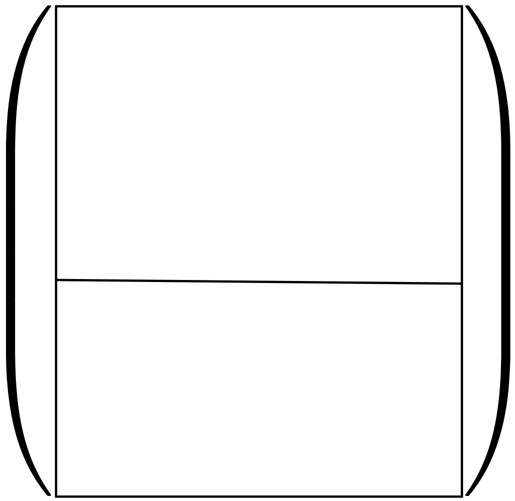

A sends 0

A sends 1

Background on communication complexity

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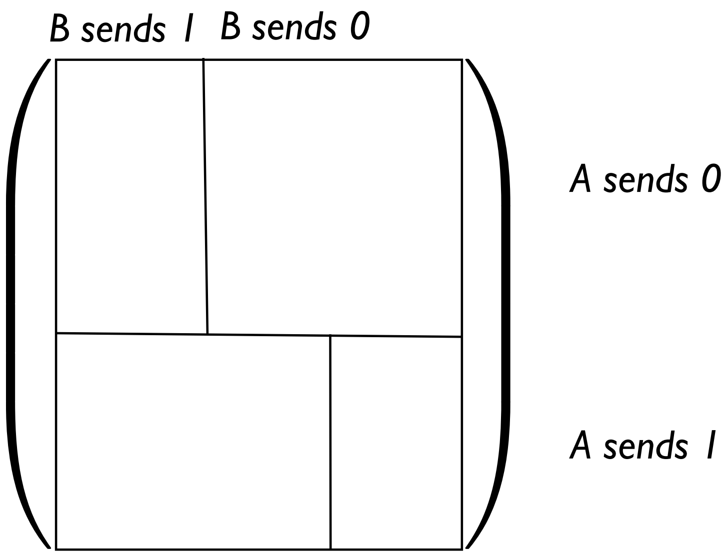

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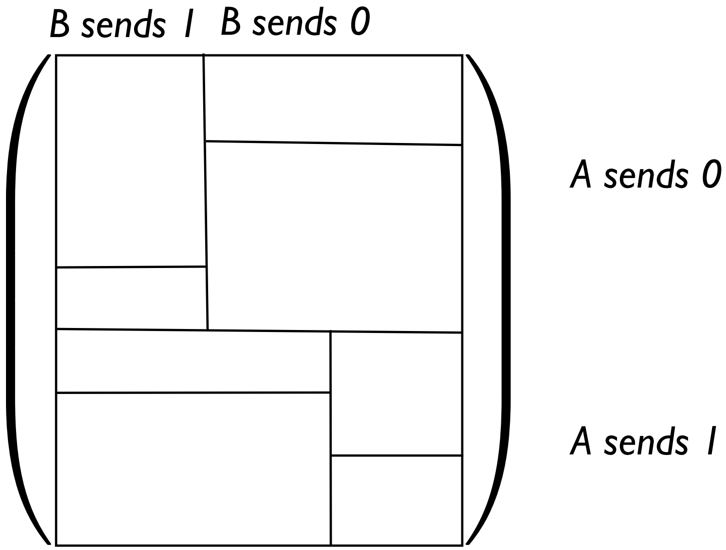

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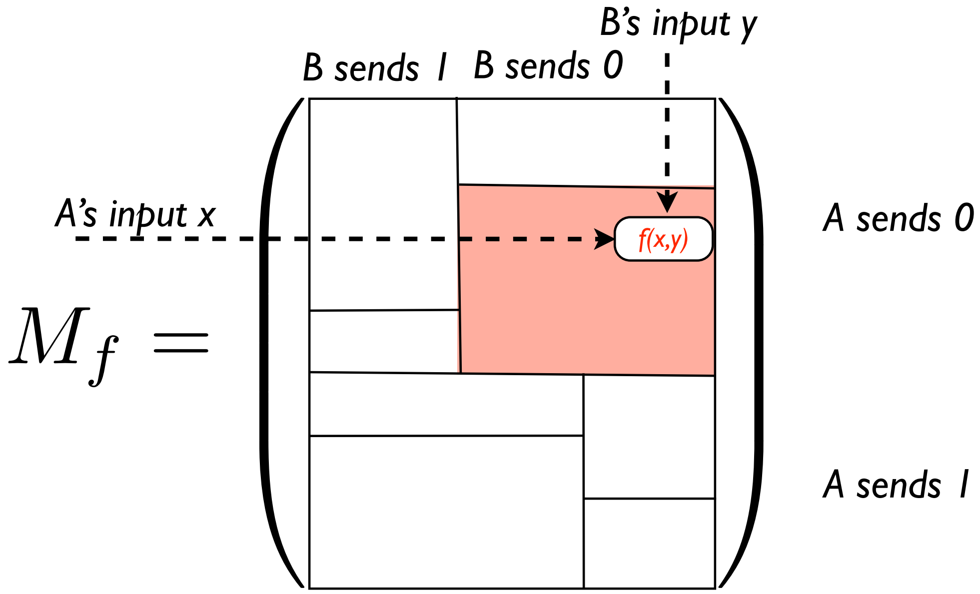

- $D^{Rect}(f)$ = smallest number of disjoint monochromatic rectangles needed to cover M_f
- $L(f) \geq D^{Rect}(R_f)$ [KW88]

Given x, y for which $f(x) \neq f(y)$, find i s.t. $x_i \neq y_i$

Formula size lower bound, spectral formulation

Theorem [LLS05] Formula size lower bound

$$L(f) \geq \max_A \frac{\|A\|^2}{\max_i \|A_i\|^2}$$

$$M_f = \left(\quad \right)$$

$$\|A\| = \max_{u,v} \frac{u^* A v}{|u||v|}$$

Formula size lower bound, spectral formulation

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$$M_f = \begin{pmatrix} \boxed{A} \end{pmatrix}$$

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Formula size lower bound, spectral formulation

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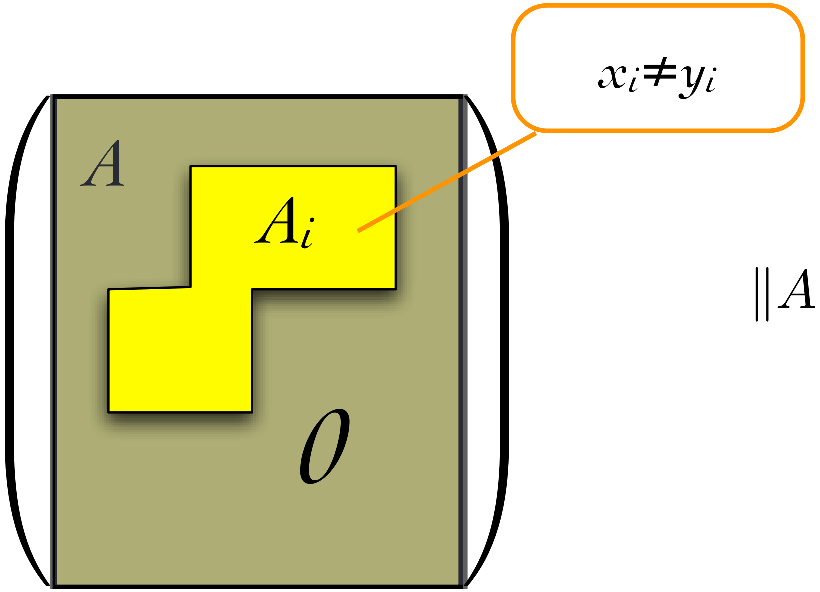

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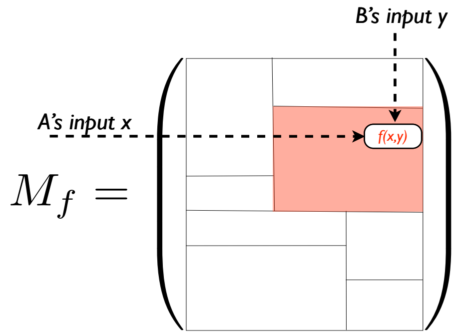

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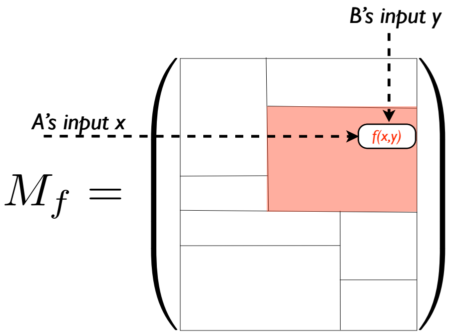


- If $\mathcal{R}$ is an optimal partition $R_1,\ldots,R_N$, then if μ is subadditive

$$L(f) \geq D^{Rect}(R_f) = \#\mathcal{R} \geq \frac{\mu(X \times Y)}{\max_i \mu(R_i)}$$

- If $\mathcal{S}$ is a covering with $\mathcal{R} < \mathcal{S}$ (refinement) then if μ is monotone,

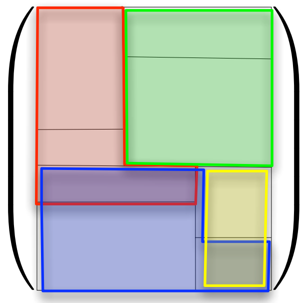
$$L(f) \geq \frac{\mu(X \times Y)}{\max_{S \in \mathcal{S}} \mu(S)}$$

- <u>Key lemma</u> [LLS05] $\|M\|^2$ is subadditive and monotone

Formula size lower bound

<u>Theorem</u> [LLS05]

$$L(f) \geq \max_A \frac{\|A\|^2}{\max_i \|A_i\|^2}$$

$M_f =$ 

- If $\mathcal{R}$ is an optimal partition $R_1,\ldots,R_N,$ then if μ is subadditive

$$L(f) \geq D^{Rect}(R_f) = \#\mathcal{R} \geq \frac{\mu(X \times Y)}{\max_i \mu(R_i)}$$

- If $\mathcal{S}$ is a covering with $\mathcal{R} < \mathcal{S}$ (refinement) then if μ is monotone,

$$L(f) \geq \frac{\mu(X \times Y)}{\max_{S \in \mathcal{S}} \mu(S)}$$

- <u>Key lemma</u> [LLS05] $\|M\|^2$ is subadditive and monotone

Relation to other methods

- Closely related to the quantum spectral method

$$L(f) \geq \max_A \frac{\|A\|^2}{\max_i \|A_i\|^2}$$

$$Q_\varepsilon(f) \geq \max_A \frac{\|A\|}{\max_i \|A_i\|}$$

- Generalizes many previous methods
  - Khrapchenko's combinatorial method [K71]
  - Koutsoupias' spectral method [K93]
  - A key lemma of Håstad used to prove the current best formula size lower bound (random restrictions) [H98]

Research project

Current projects

- Continue to unify and extend classical and quantum lower bound techniques
  - Combinatorial models
    - Communication complexity, circuits, formula size, decision trees
  - Techniques
    - Fourier analysis
    - Information theory methods

Further projects

- Medium-term
  - Apply quantum Kolmogorov complexity to quantum lower bounds, e.g. quantum information theoretic methods
- Long-term
  - Use Kolmogorov complexity to study derandomization

Thank you