Wait-free Solvability of Equality Negation Tasks

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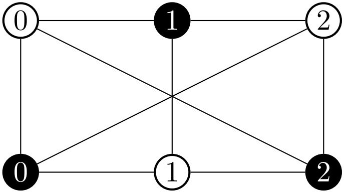
Equality Negation

(Lo and Hadzilacos, *Nondeterministic wait-free hierarchies are not robust*, 2000)

- Two processes P, Q (represented in black and white).
- Three possible inputs values $i_P, i_Q \in \{0, 1, 2\}$.
- Binary decision values $d_P, d_Q \in \{0, 1\}$.
- **Goal:** $i_P = i_Q \iff d_P \neq d_Q$.



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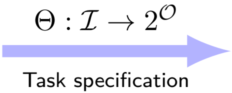

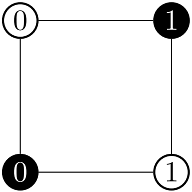

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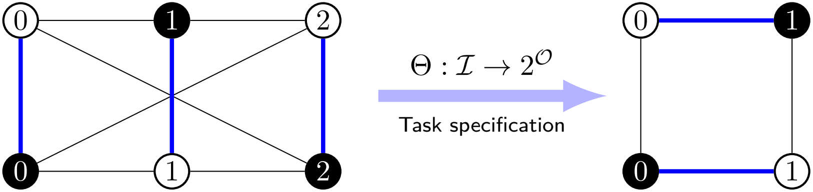

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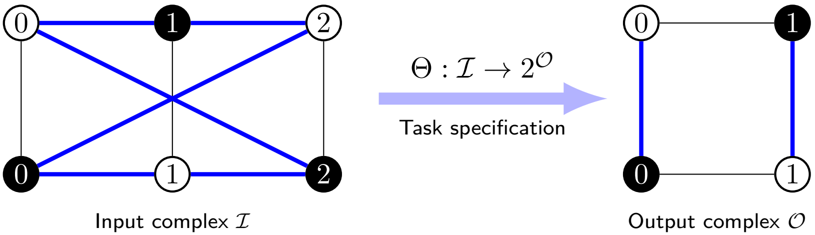
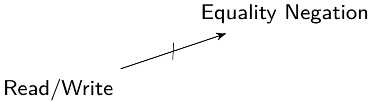


Equality Negation (2)

Facts: (Lo and Hadzilacos, 2000)

(1) EN is not wait-free solvable using read/write registers.



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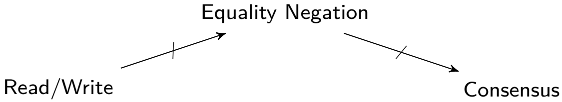

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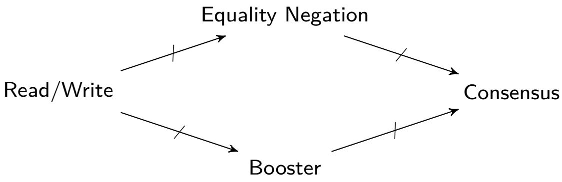

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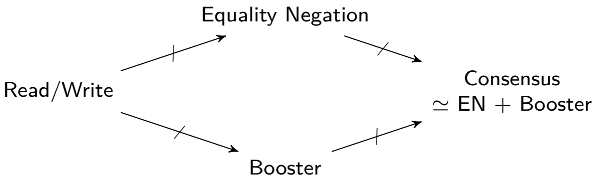
(4) But EN + Booster can implement consensus!



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(2) This talk:
$\longrightarrow$ Extend the task to n processes and study its solvability.

Equality Negation for n processes

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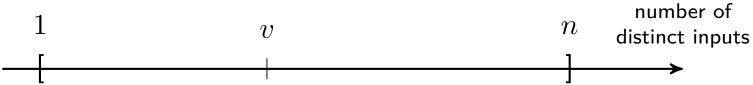

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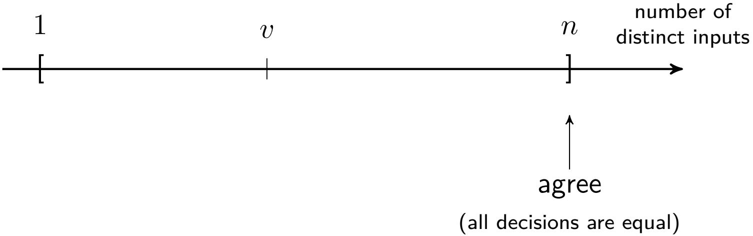

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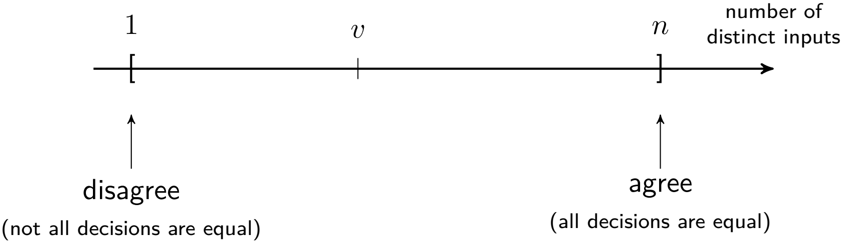

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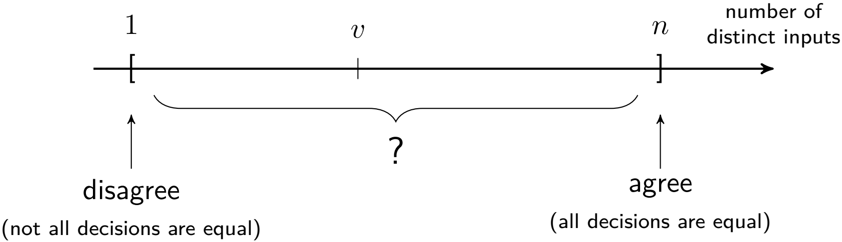

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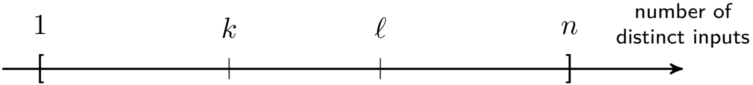

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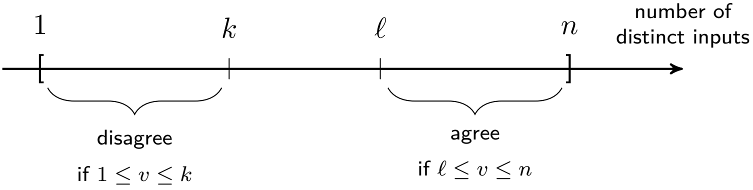

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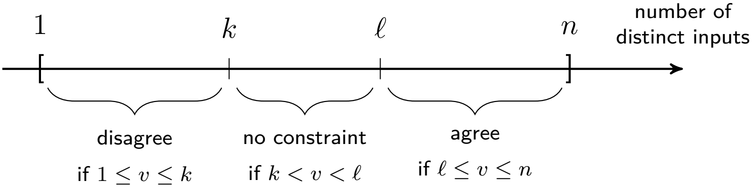

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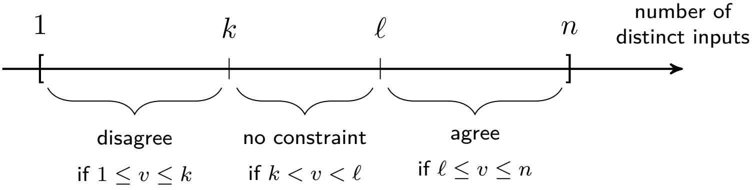
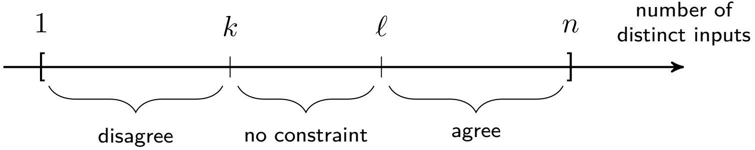


$\longrightarrow$ We get a family of tasks EN(k, ℓ).

Solvable cases

Reminder: parameters $1 \leq k < \ell \leq n$.



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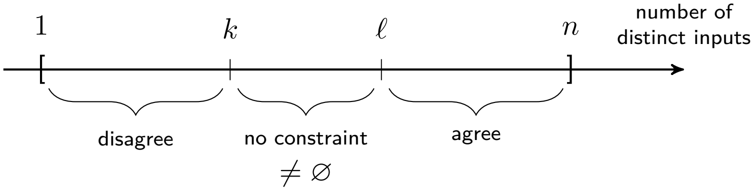

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If $k + 2 \leq \ell$, the task EN(k,ℓ) is wait-free solvable using read/write.

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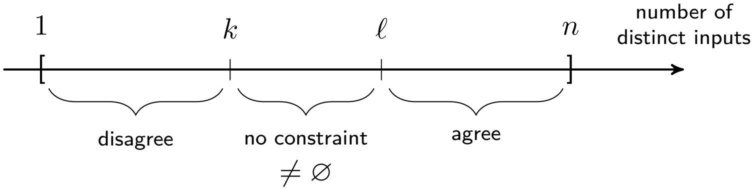

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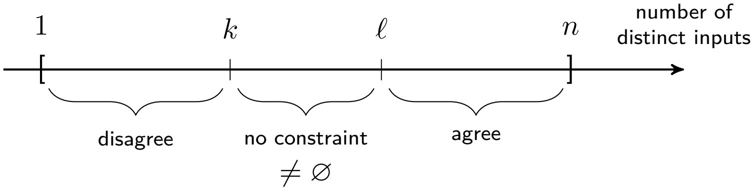

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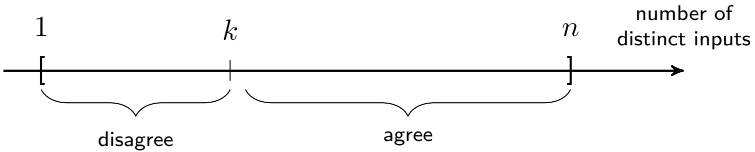

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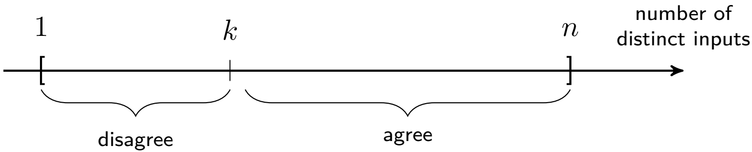

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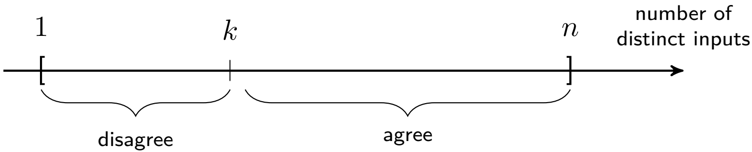

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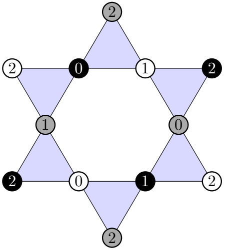
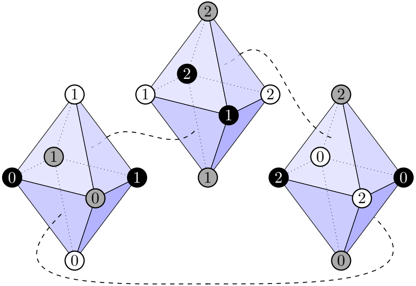
- Uses the Index Lemma

Proof sketch for $n = 3$, $k = 2$

- Three processes: Black, Gray, White.
- Three inputs: 0, 1, 2.

The input complex $\mathcal{I}$ looks like this (exploded view):





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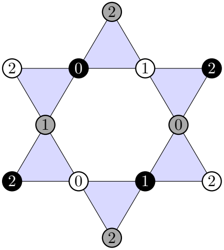

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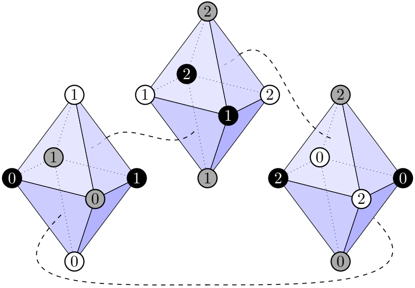

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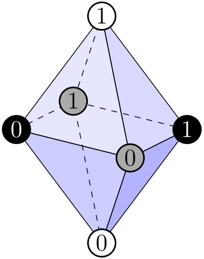

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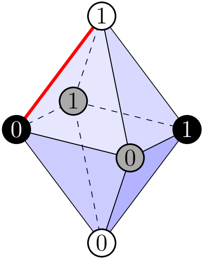

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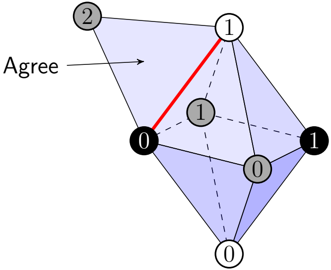

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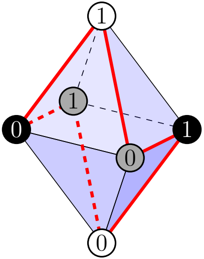

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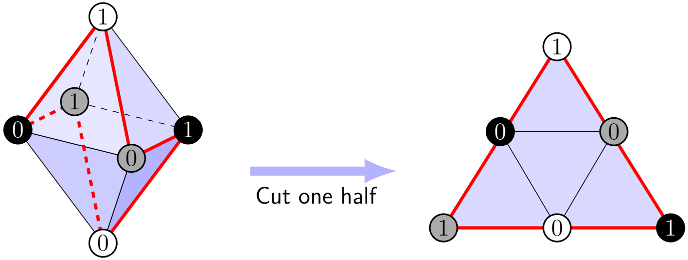

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$H \subseteq S$

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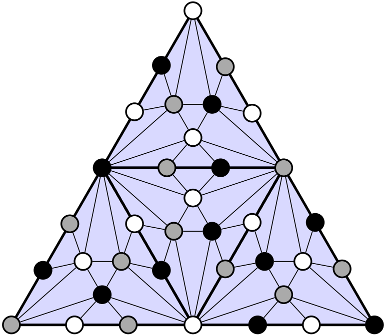
After immediate-snapshot communication (here, one round):



SubDiv$(H) \subseteq$ Protocol Complex

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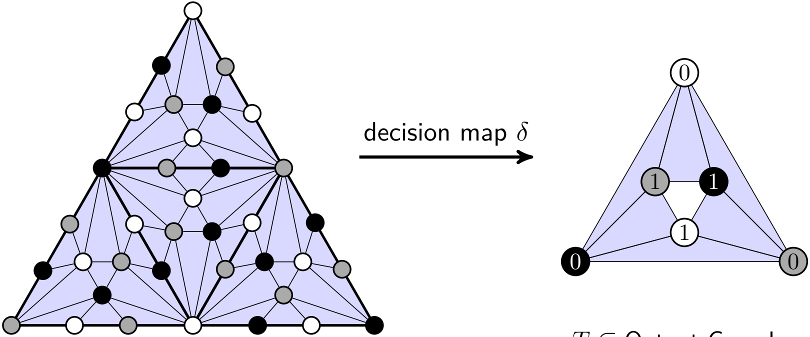

$T \subseteq$ Output Complex

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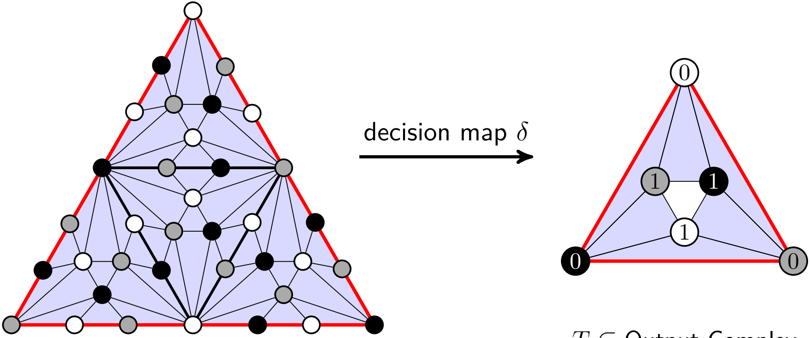

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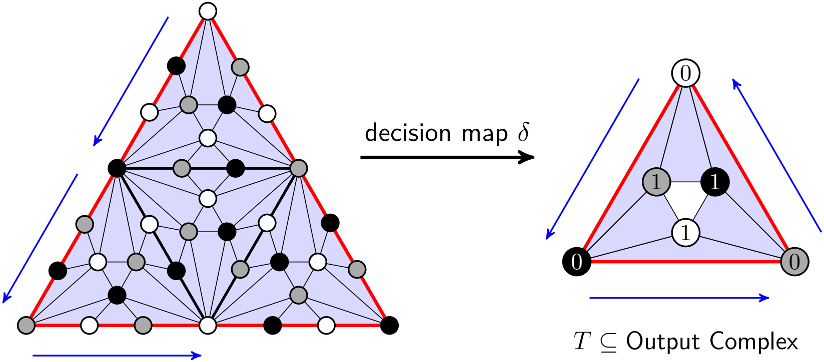

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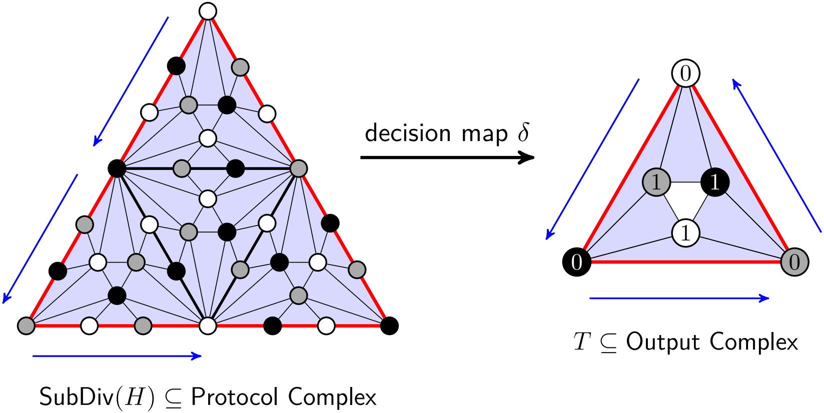

The boundary of SubDiv(H) is winding twice around the boundary of T.

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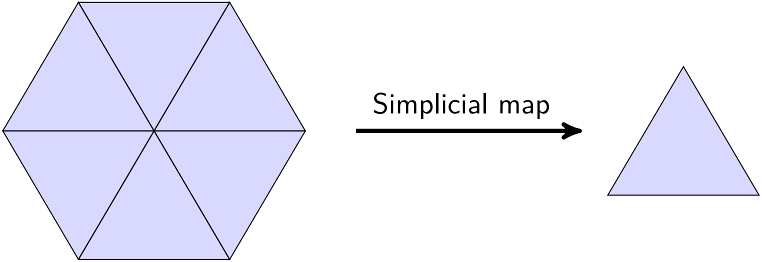

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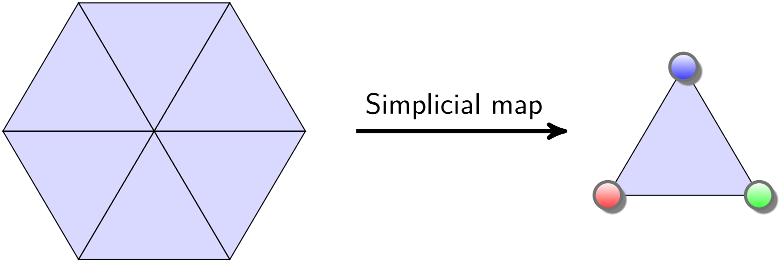

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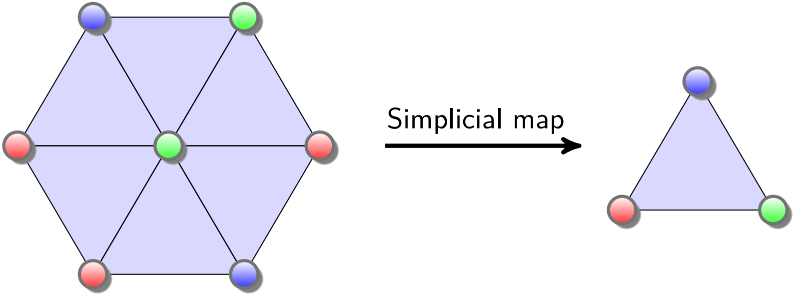

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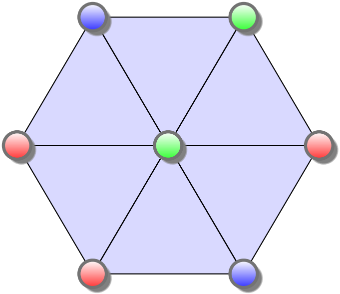

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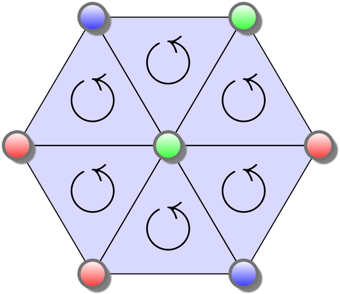

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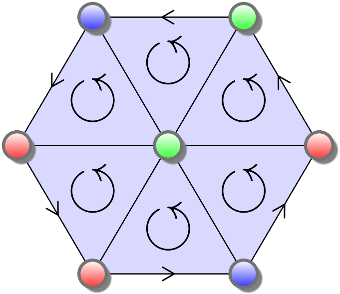

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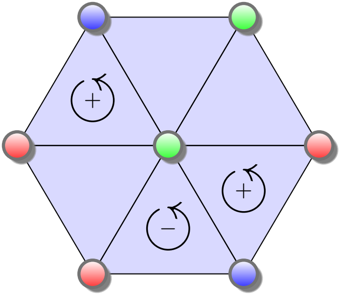

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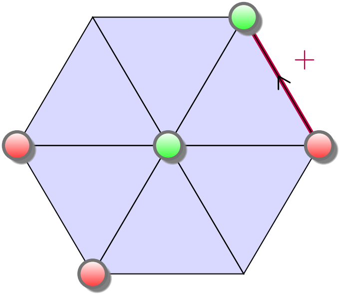

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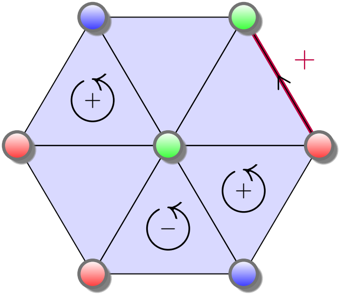


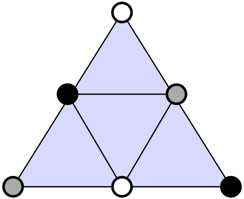
Index Lemma

In a pseudomanifold with boundary, Index $= (-1)^i$ Content.

Proof sketch for $n = 3$, $k = 2$

Back to the subcomplex H of the input complex.
We color the vertices with the value:

$$\text{process number} + \text{decision value} \mod n$$



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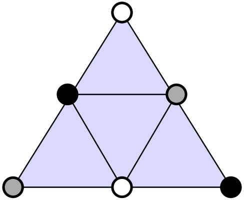

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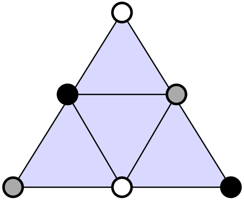

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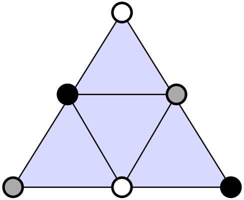

The index of H is 2. Moreover, chromatic subdividions preserve the index, so the index of SubDiv(H) is also 2. By the Index lemma, the content of SubDiv(H) is ± 2. This implies that there are monochromatic triangles w.r.t. decision values. □

Conclusion

We have studied a family of tasks EN(k,ℓ), for $1 \leq k < \ell \leq n$.

- Solvable if $k + 2 \leq \ell$,
- Unsolvable if $k \leq n/2$ and $\ell = k + 1$,
- Unsolvable if $(n - k)$ is odd and $\ell = k + 1$,

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Two key ingredients for impossibility:

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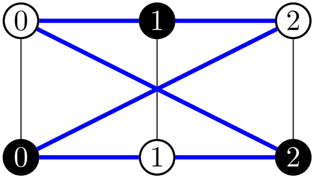
- Connectedness of some subcomplex of the input.



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Thank you!