Simplicial Models: from global states to local states, and what lies in-between

Dagstuhl Seminar

Jérémy Ledent

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Introduction

Worlds and Views (1/3)

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$$W = \{123, 124, 132, 134, 142, 143, 213, 214, 231, 234, \ldots\}$$

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- The local state, a.k.a. view, of an agent a is the card that this agent holds:

$$\text{views}_a = \{1\bot\bot, 2\bot\bot, 3\bot\bot, 4\bot\bot\} \qquad \text{views}_b = \{\bot 1\bot, \bot 2\bot, \bot 3\bot, \bot 4\bot\}$$

$$\text{views}_c = \{\bot\bot 1, \bot\bot 2, \bot\bot 3, \bot\bot 4\}$$

Worlds and Views (2/3)

Example 2 (Muddy Children): Three children a, b, c, can be either clean (0) or dirty (1). Each child can see the state of the other two children, but not themself.

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- The possible worlds are all the possible combinations of clean/dirty:

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- The views of a child are the states of the other two children:

$$\text{views}_a = \{\bot 00, \bot 01, \bot 10, \bot 11\} \qquad \text{views}_b = \{0\bot 0, 0\bot 1, 1\bot 0, 1\bot 1\}$$

$$\text{views}_c = \{00\bot, 01\bot, 10\bot, 11\bot\}$$

Worlds and Views (3/3)

Key idea: Worlds and views can be defined from one another!

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Views $\rightarrow$ Worlds: a world is a set of compatible views.

- **Ex 1**: the world 123 is composed of three views: $1\bot\bot$, $\bot 2\bot$ and $\bot\bot 3$.
- **Ex 2**: the world 010 is composed of three views: $\bot 10$, $0\bot 0$ and $01\bot$.

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- **Ex 2**: the world 010 is composed of three views: $\bot 10$, $0\bot 0$ and $01\bot$.

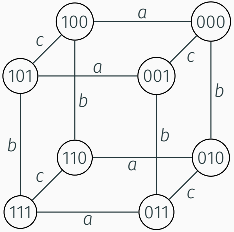
Worlds $\rightarrow$ Views: a view is a set of indistinguishable worlds.

- **Ex 1**: the a-view $2\bot\bot$ corresponds to the set of worlds $\{213, 214, 231, 234, 241, 243\}$.
- **Ex 2**: the b-view $1\bot 0$ corresponds to the set of worlds $\{100, 110\}$.

Kripke Models vs Simplicial Models

Kripke models:

- explicit worlds
- implicit views



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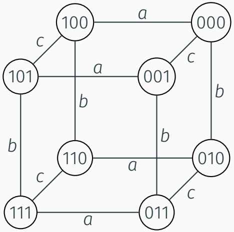
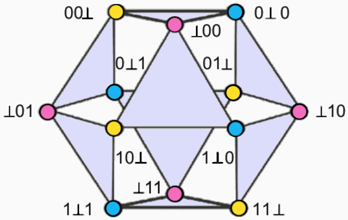


Simplicial models:

- explicit views
- implicit worlds



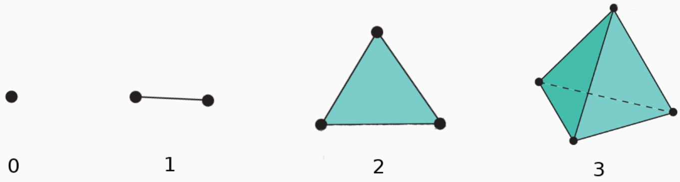
Plan

- Part I – Pure Simplicial Models
  1. Reminders on simplicial complexes
  2. Definition and semantics of simplicial models
  3. Equivalence with Kripke models
- Part II – The ins and outs of Simplicial Models
  4. Variants of simplicial models
  5. Applications to distributed computing
  6. Links between logic and topology

Pure Simplicial Models

Definition

An *n-simplex* is the convex hull of $n+1$ affinely independent points in $\mathbb{R}^{n+1}$.



Crash course on Simplicial Complexes

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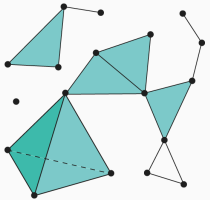
Definition

An (abstract) simplicial complex is a pair (V, S) where:

- V is a set of vertices
- $S \subseteq 2^V$ is a downward-closed family of subsets of V, called simplexes



Epistemic Logic with Distributed Knowledge

Let Ag be a finite set of agents and Prop a set of atomic propositions.

Syntax:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid D_B\,\varphi \qquad\qquad p \in \mathrm{Prop}, B \subseteq \mathrm{Ag}$$

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$D_B\,\varphi$: *"There is distributed knowledge among B that φ is true".*

The usual knowledge operator, $K_a\,\varphi$, can be defined by: $K_a\,\varphi := D_{\{a\}}\,\varphi$.

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Usually (in Kripke models), one defines the group indistinguishability relation $\sim_B = \bigcap_{a \in B} \sim_a$

Chromatic Simplicial Complexes

Definition

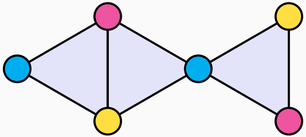
A chromatic simplicial complex is given by (V, S, χ) where:

- (V, S) is a simplicial complex,
- $\chi : V \to \mathrm{Ag}$ is a *coloring* map,

such that every simplex $X \in S$ has all vertices of distinct colors.

A facet is a simplex that is maximal w.r.t. inclusion.
A simplicial complex is pure if all facets have the same dimension.

Example: a pure chromatic simplicial complex of dimension 2.



Pure Simplicial Models

Assume the number of agents is $|\mathrm{Ag}| = n+1$.

Definition (Pure Simplicial Model)

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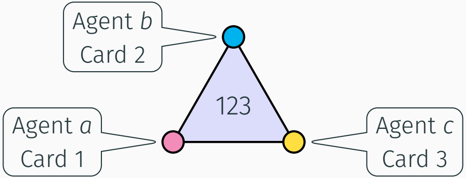

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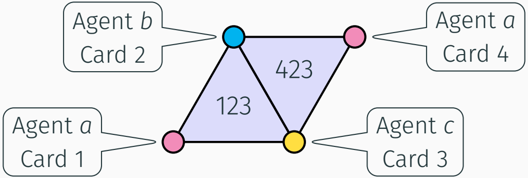

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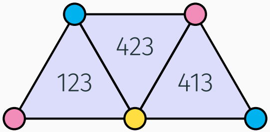

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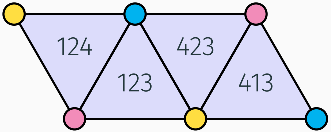

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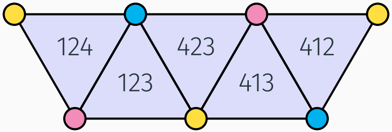

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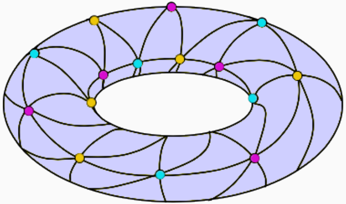

Example 2: the "Muddy Children" puzzle

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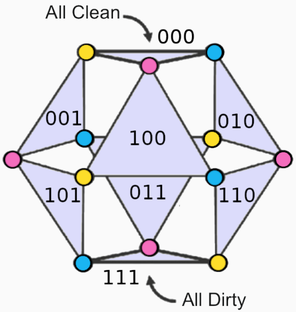

Semantics of simplicial models

We define the satisfaction relation $\mathscr{C}, X \models \varphi$, where:

- $\mathscr{C}$ is a simplicial model,
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- φ is an epistemic logic formula.

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By induction on φ:

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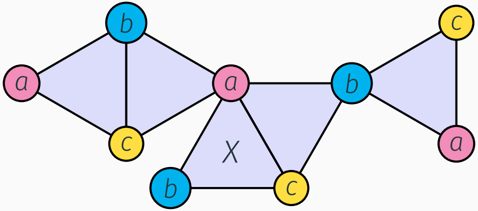

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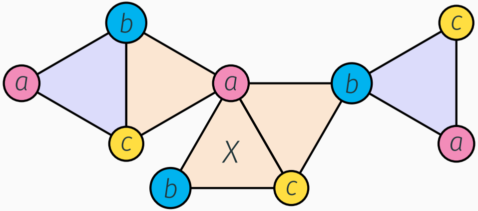

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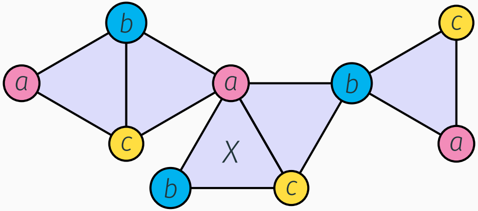

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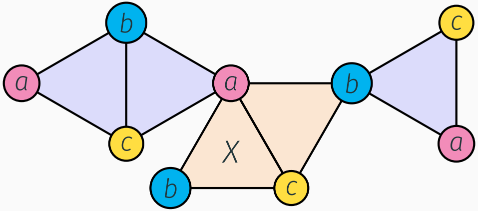
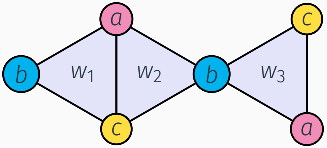


Equivalence with Kripke models

Theorem (Goubault, L., Rajsbaum (2018, 2021))

The category of pure simplicial models is equivalent to the one of proper Kripke models.

Example: with three agents, $\mathrm{Ag} = \{ \text{a}, \text{b}, \text{c} \}$,



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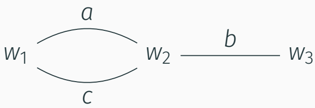

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Variants of Simplicial Models

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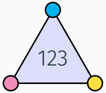

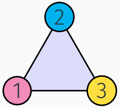

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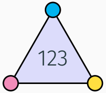

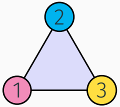

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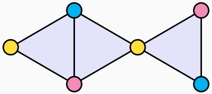
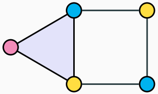
(2) Pure vs. impure simplicial complexes.





- van Ditmarsch (WoLLIC'21)
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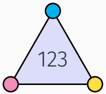
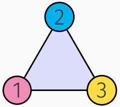
What can we do differently? (1/2)

(1) Atomic propositions on the worlds vs. vertices.





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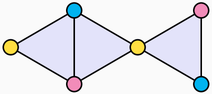

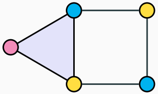

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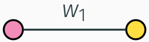
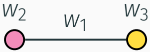
(3) The worlds are facets vs. simplexes.





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What can we do differently? (2/2)

(4) Use Simplicial complexes vs. (Semi)-simplicial sets.

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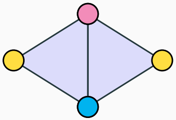

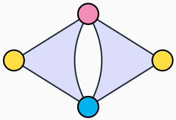

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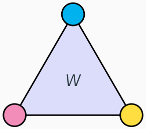
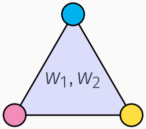
(5) Have several copies of the same world
(a.k.a. non-proper models).





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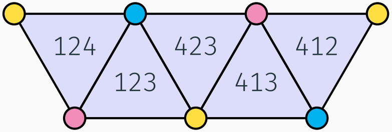
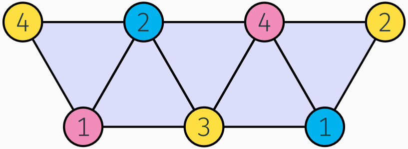
Labelling the worlds vs Labelling the vertices

Example: recall the torus example with cards 1, 2, 3, 4 and agents a, b, c.





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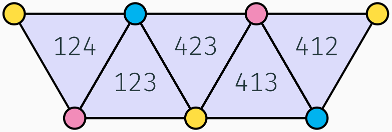
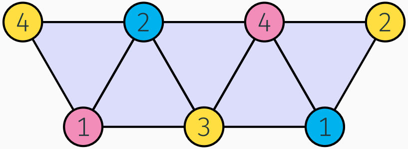




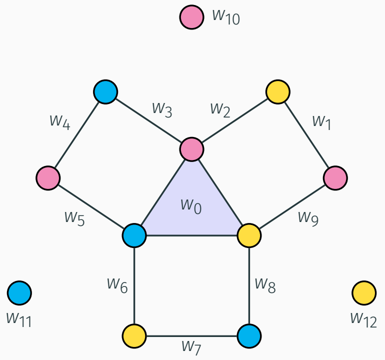
Consequence: Axiom of Locality, for every atomic proposition $p \in \mathrm{Prop}$.

$$\mathbf{Loc_p}: \quad \bigvee_{a \in \mathrm{Ag}} K_a\, p \vee K_a \neg p$$

Impure simplicial models

Impure simplicial complexes.

- Common in distributed computing.
- They model systems with crashes.
- Related to nonrigid sets of agents [FHMV'95].



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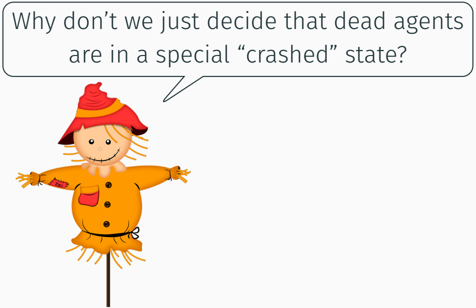

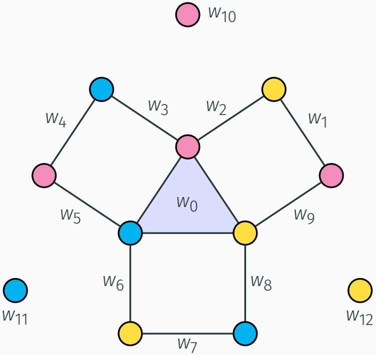

Two approaches for impure simplicial models

The "two-valued" approach. [Goubault, L., Rajsbaum, Kniazev]

- Trust the equivalence with Kripke models
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- Define $\mathscr{C}, w \bowtie \varphi$: "$\varphi$ is well-defined"
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Can we find $\ulcorner\varphi\urcorner$ such that $\mathcal{C}, w \models_3 \varphi \iff \mathcal{C}, w \models_2 \ulcorner\varphi\urcorner$? $\longrightarrow$ Ask Roman Kniazev!

Two-valued approach – Toy example

Recall the definition of the satisfaction relation:

$$\mathscr{C}, X \models K_a\,\varphi \quad \text{iff} \quad \mathscr{C}, Y \models \varphi \quad \text{for all } Y \in \mathrm{Facet}(\mathscr{C}) \quad \text{such that} \quad a \in \chi(X \cap Y)$$

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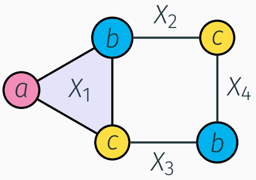

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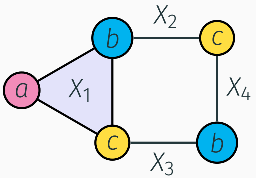

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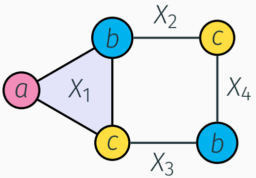

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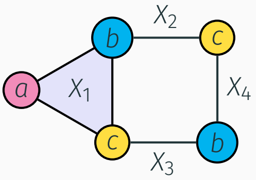

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Definability of “alive” and “dead”

Define the following formulas, for an age‘nt $a \in \mathrm{Ag}$:

$$\mathrm{dead}(a) := K_a \mathrm{false} \qquad \mathrm{alive}(a) := \neg \mathrm{dead}(a)$$

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Example: these formulas are valid in all impure simplicial models

- Dead agents know everything: $\models \mathsf{dead}(a) \implies K_a\varphi.$
- Alive agents know they are alive: $\models \mathsf{alive}(a) \implies K_a\,\mathsf{alive}(a).$
- Alive agents satisfy Axiom **T**: $\models \mathsf{alive}(a) \implies (K_a\varphi \Rightarrow \varphi).$

An equivalent class of Kripke models

Definition (Partial Equivalence Relations)

A partial equivalence relation $R \subseteq X \times X$ is a relation that is symmetric and transitive.
Equivalently: there exists $Y \subseteq X$ such that $R \subseteq Y \times Y$ is an equivalence relation on Y.

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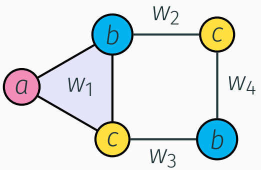
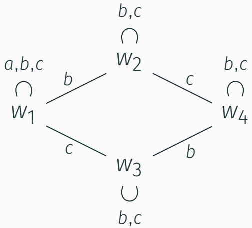
Theorem (Goubault, L., Rajsbaum (2022))

Impure simplicial models are equivalent to proper Kripke models over PERs.

Example:



$\simeq$



Epistemic Covering Models

With each new variant, one usually asks two fundamental questions:

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  Properties of coverings $\iff$ Properties of Kripke models $\iff$ Axioms of the logic.

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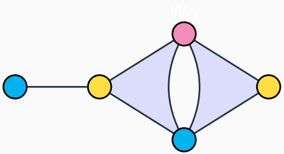
- This solves questions 1 and 2 for all the corresponding sub-classes of models!

New features of Epistemic Coverings

(1) Models are based on semi-simplicial sets, generalizing simplicial complexes.



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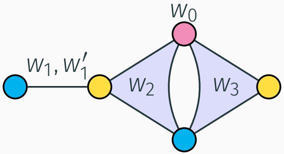
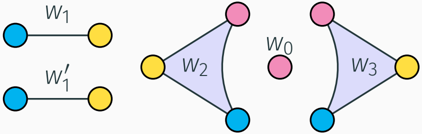


(2) Models are equipped with a discrete covering E,

$$E =$$



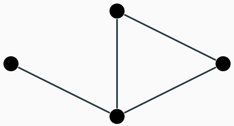
and a map $p : E \to B$, tagging which simplexes are worlds.

Crash course on Semi-Simplicial Sets (1/3)

In Dimension 1:

1-dimensional simplicial complexes
a.k.a. **simple (undirected) graphs**

(V, E) where $E \subseteq \{\{v, v'\} \mid v \neq v' \in V\}$



1-dimensional semi-simplicial sets
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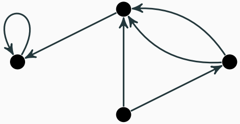
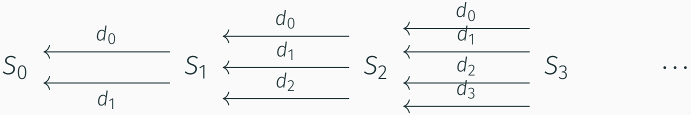
$$V \underset{s}{\overset{t}{\leftleftarrows}} E$$



Crash course on Semi-Simplicial Sets (2/3)

Definition

A semi-simplicial set is given by a sequence of sets $(S_n)_{n \in \mathbb{N}}$, together with face maps $d_i^n : S_n \to S_{n-1}$ for every $n \in \mathbb{N}$ and $0 \leqslant i \leqslant n$,



satisfying the *simplicial identities*: for all $i < j$, $\quad d_i \circ d_j = d_{j-1} \circ d_i$.

Examples: on the board.

Crash course on Semi-Simplicial Sets (3/3)

Now we add colors to the vertices:

Definition

A chromatic semi-simplicial set colored by Ag is given by:

- a set S_A for every $A \subseteq \text{Ag}$,
- a function $d_B : S_A \to S_B$ for every $B \subseteq A$,
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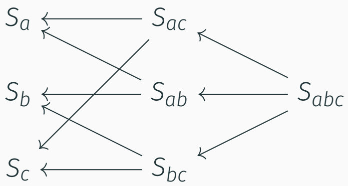

The original definition dates back to:

- Samuel Eilenberg, Joseph A. Zilber
  Semi-simplicial complexes and singular homology, Annals of Mathematics 51:3 (1950)

Introductory papers:

- Greg Friedman, *An elementary illustrated introduction to simplicial sets*, Rocky Mountain J. Math. 42(2): 353-423 (2012) (arXiv:0809.4221, doi:10.1216/RMJ-2012-42-2-353)
- Emily Riehl, *A leisurely introduction to simplicial sets*, 2008, 14 pages (pdf).
- Francis Sergeraert, *Introduction to Combinatorial Homotopy Theory*, July 7, 2013, pdf.
- Christian Rüschoff, *Simplicial Sets*, Lecture Notes 2017 (pdf, pdf)

A tentative example about crypto

Example (Secret Sharing): two agents a, b use a 1-bit One-Time Pad protocol.

- Agent a holds an encrypted message $m \in \{0,1\}$.
- Agent b holds the encryption key $k \in \{0,1\}$.
- The secret is obtained by $s = (m + k) \mod 2$.

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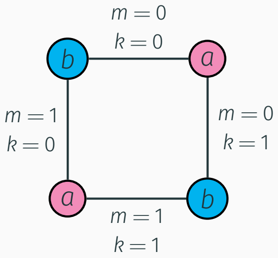

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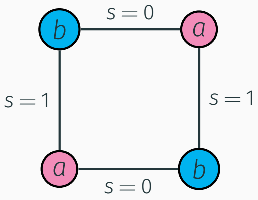

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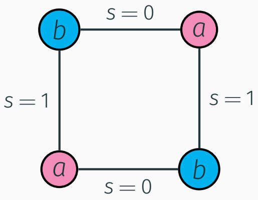

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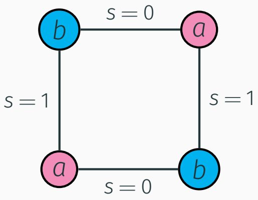

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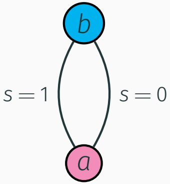

Bisimilarity between simplicial sets and complexes

Theorem

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So what does that mean?

- Semi-simplicial sets are not interesting... :(

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So what does that mean?

- Semi-simplicial sets are not interesting... :(
- or: Semi-simplicial sets can help produce smaller models.
- or: Semi-simplicial sets can help to abstract away implementation details.

Bisimilarity between simplicial sets and complexes

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- or: Semi-simplicial sets can help produce smaller models.
- or: Semi-simplicial sets can help to abstract away implementation details.
- or: We need a stronger logic that can “see” the difference between them.

An equivalent class of Kripke models

Definition (Kripke pseudo-models with PERs)

A Kripke pseudo-model is given by $M = \langle W, \sim, L \rangle$ where:

- W is a set of worlds,
- For every $B \subseteq \mathrm{Ag}$, $\sim_B$ is a partial equivalence relation on W,
- $L : W \to 2^{\mathrm{Prop}}$ is a valuation.

such that:

- for all $B' \subseteq B$, $\quad w \sim_B w' \implies w \sim_{B'} w'$
- for all $B, B' \subseteq \mathrm{Ag}$, $\quad (w \sim_B w \wedge w \sim_{B'} w) \implies w \sim_{B \cup B'} w$

Theorem (Goubault, Kniazev, L., Rajsbaum (2023))

Epistemic Covering models are equivalent to Kripke pseudo-models.

The many sub-classes of Epistemic Coverings

Epistemic Coverings [GKLR'23]

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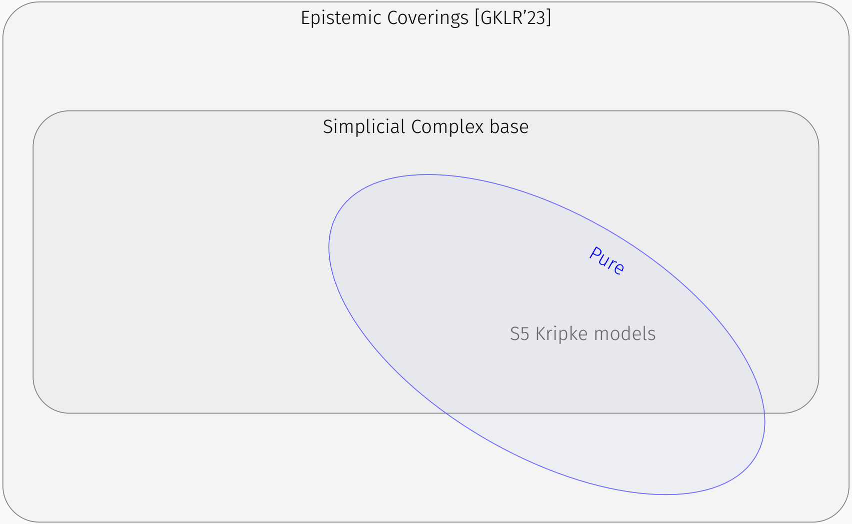
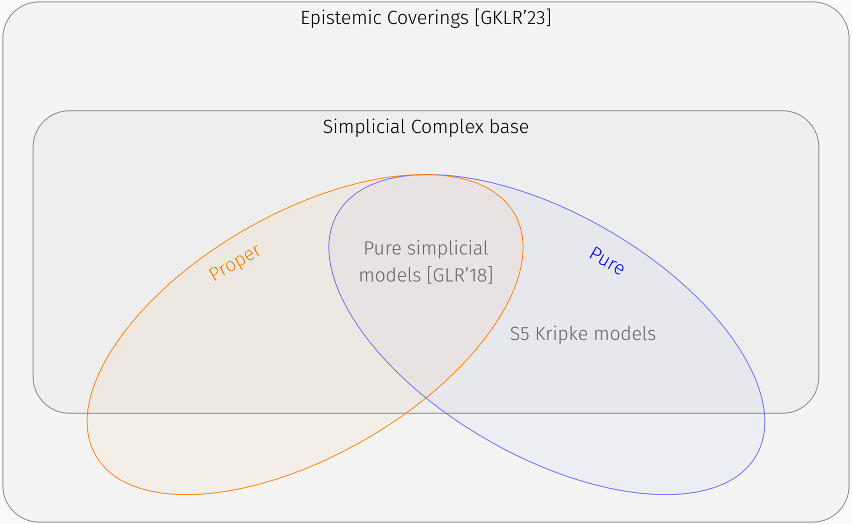
Simplicial Complex base

The many sub-classes of Epistemic Coverings



The many sub-classes of Epistemic Coverings



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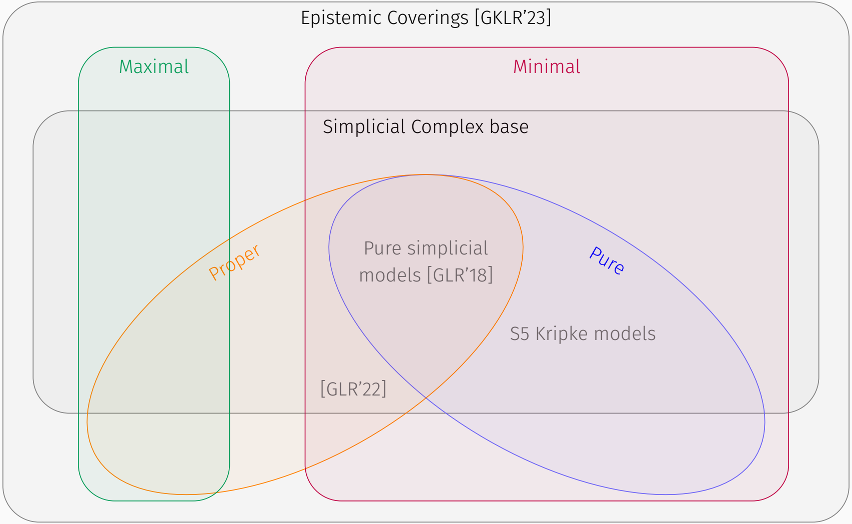
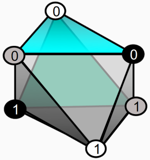


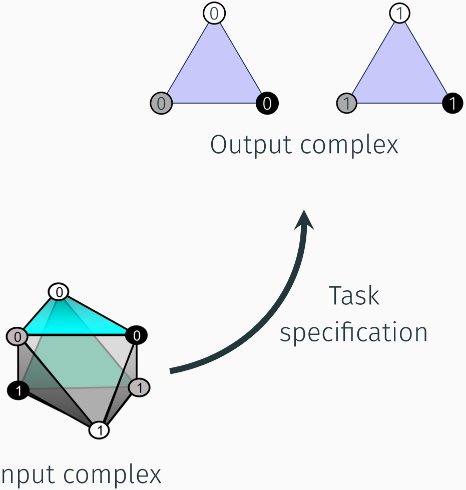
Applications to Distributed Computing

Topological characterization of task solvability (Herlihy et al.)



Input complex

Topological characterization of task solvability (Herlihy et al.)



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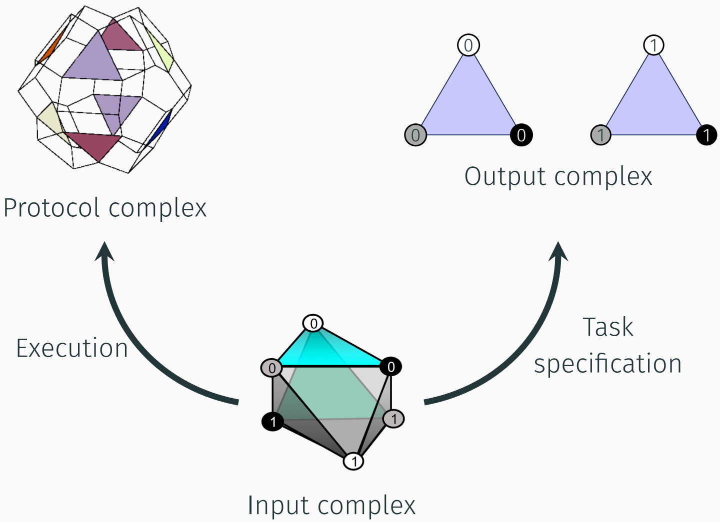

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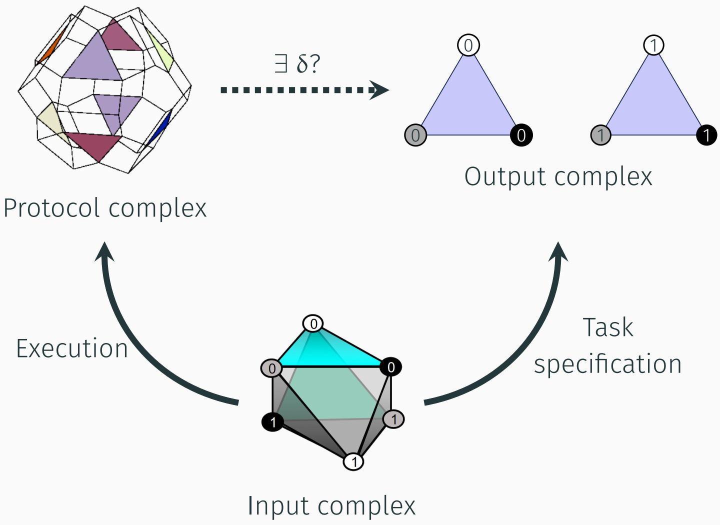

Epistemic proofs of impossibility

Idea: find a logical obstruction to the existence of the simplicial map δ.

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Lemma (Knowledge Gain)

Let $\delta : \mathscr{C} \longrightarrow \mathscr{C}'$ be a morphism of simplicial models, and let φ be a positive formula. Then:

$$\mathscr{C}', \delta(X) \models \varphi \qquad \text{implies} \qquad \mathscr{C}, X \models \varphi$$

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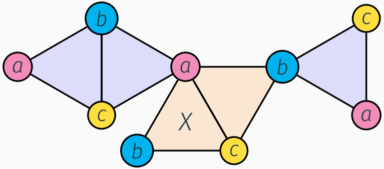
Recipe for impossibility proofs:

- Assume by contradiction that $\delta : \mathscr{P} \longrightarrow \mathscr{O}$ exists.
- Choose a suitable formula φ such that:
- φ is true everywhere in the output model
- φ is false somewhere in the protocol model

Links between Knowledge and Topology

Distributed knowledge = Higher-dimensional connectivity

Recall: $\mathscr{C}, X \models D_{\{a,c\}}\varphi$



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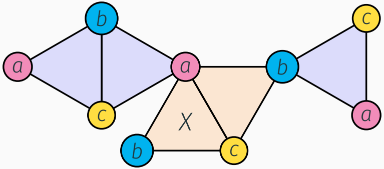
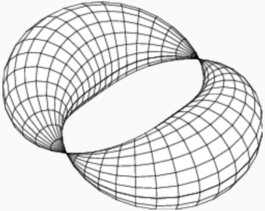


With Common Distributed Knowledge, we can explore the 2-connected component:

$$CD_{\beta}\,\varphi, \text{ where } \beta = \{\{a_1, a_2\} \mid a_1 \neq a_2 \in \mathrm{Ag}\}$$

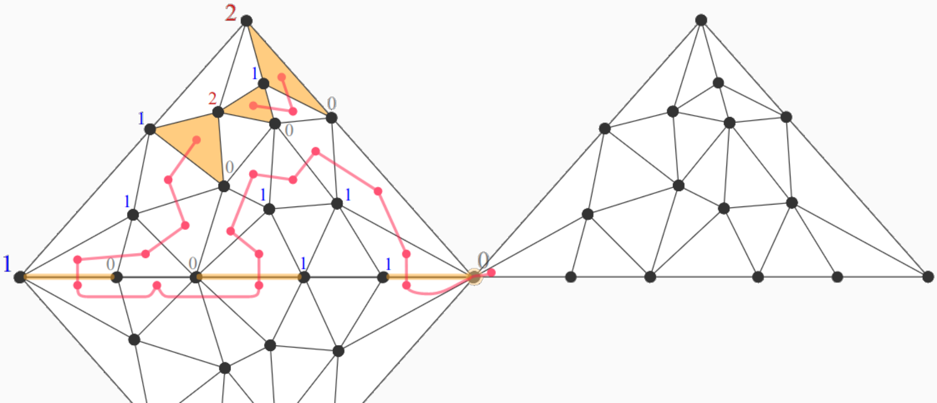


(Fig. by Richard Cushman)

A formula for Sperner’s Lemma

Cf work by Susumu Nishimura:
Proving Unsolvability of Set Agreement Task with Epistemic mu-Calculus (2022).

$$\Phi_k = \nu Z.\left[\mathrm{OFUN} \wedge \mathrm{VALID} \wedge \bigwedge_{\emptyset \subsetneq A \subseteq \Pi} (\mathrm{DEC}_A \Rightarrow \mathrm{D}_A(\mathrm{KNOW} \wedge \mathrm{AGREE}_k \wedge Z))\right]$$



Disaster

Theorem

Every simplicial model $\mathscr{C}$ is bisimilar to its unravelled model $U(\mathscr{C})$.

Consequences?

- $U(\mathscr{C})$ has a very poor topological structure (infinite tree).
- No hope to see features like holes and loops without a radically new logic.

Conclusion

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Key messages:

- Kripke models have a hidden higher-dimensional structure
- Distributed knowledge = higher-dimensional connectivity
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- Lots of research directions!

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Thanks!