Chromatic simplicial complexes are models for epistemic logic

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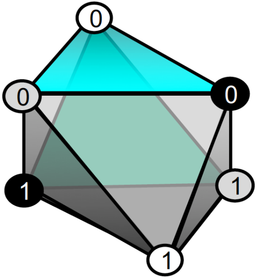
Introduction

Main idea: A close correspondence between two structures.



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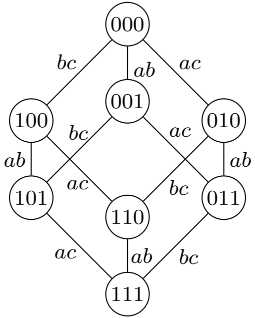

The chromatic simplicial complexes used in fault-tolerant distributed computability.

The Kripke models used in epistemic logic.

Epistemic logic

Multi-agent epistemic logic

Epistemic logic is the logic of knowledge.

Let $\mathcal{A}$ be a finite set of *agents* and AP a set of *atomic propositions*. The syntax of formulas is:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K_a\,\varphi \qquad\qquad p \in AP,\ a \in \mathcal{A}$$

$K_a\,\varphi$ is read *"a knows φ"*.

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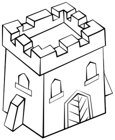
Common knowledge:

$$C_B\,\varphi \;\equiv\; \bigwedge_{\substack{n\in\mathbb{N}\\ a_1,\ldots,a_n\in B}} K_{a_1}\ldots K_{a_n}\,\varphi$$

Example: the two generals problem

Two divisions of the same army, commanded by general A and general B, are surrounding an enemy fortress.

A



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- They must attack simultaneously.

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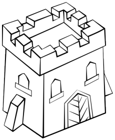

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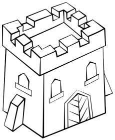

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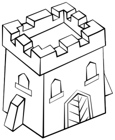

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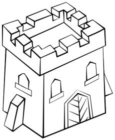

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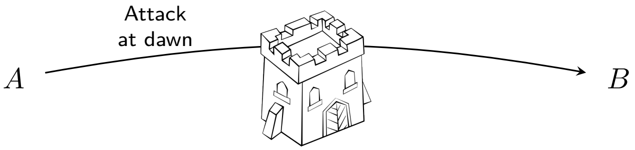

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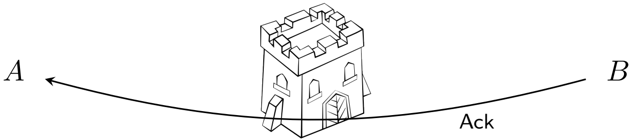

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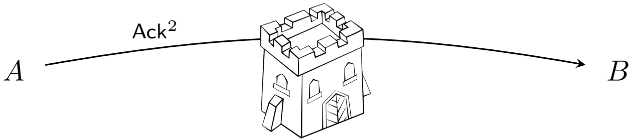
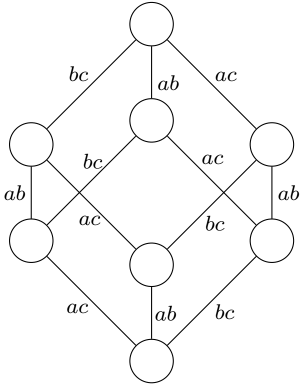


Kripke models

A **Kripke frame** is a tuple $M = \langle W, (\sim_a)_{a \in \mathcal{A}} \rangle$, where:

- W is a set of *worlds*
- For every $a \in \mathcal{A}$, $\sim_a \subseteq W \times W$ is an equivalence relation on W



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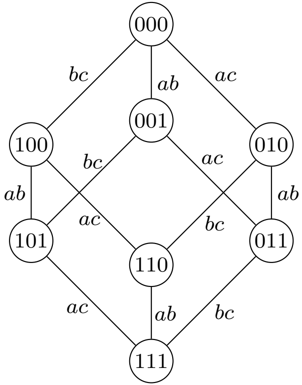
- $L : W \to \mathscr{P}(AP)$

Example: three agents with binary inputs.

- a, b, c are agents.
- $w \sim_a w'$ is represented as an a-labeled edge between w and w'.
- 101 : input values of a, b, c, in that order.



Semantics of epistemic logic formulas

Let $M = \langle W, \sim, L \rangle$ be a Kripke model and $x \in W$ a world of M. We define the **truth** of a formula φ in x, written $M, x \models \varphi$, by induction on φ:

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\begin{array}{lll}
M, x \models p & \text{iff} & p \in L(x) \\
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M, x \models K_a\, \varphi & \text{iff} & \text{for all } y \in W, x \sim_a y \text{ implies } M, y \models \varphi
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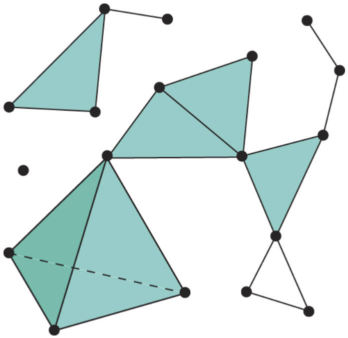
$M, x \models \varphi \land \psi$	iff	$M, x \models \varphi$ and $M, x \models \psi$
$M, x \models K_a\,\varphi$	iff	for all $y \in W, x \sim_a y$ implies $M, y \models \varphi$
$M, x \models C_B\,\varphi$	iff	for all y in the B-connected component of x, $M, y \models \varphi$

Simplicial complexes

Definition

An (abstract) **simplicial complex** is a pair $\langle V, S \rangle$ where V is a set of *vertices* and S is a downward-closed family of subsets of V called *simplices* (i.e., $X \in S$ and $Y \subseteq X$ implies $Y \in S$).



Chromatic simplicial complexes

Fix a finite set $\mathcal{A}$ of agents, represented as *colors*.

Definition

A **chromatic simplicial complex** is given by $\langle V, S, \chi \rangle$ where:

- $\langle V, S \rangle$ is a simplicial complex,
- $\chi : V \to \mathcal{A}$ is a *coloring* map,

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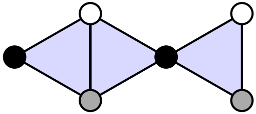
Example: a pure chromatic simplicial complex of dimension 2.

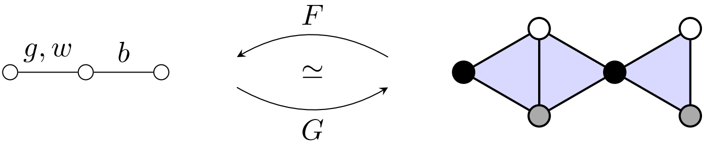


Equivalence

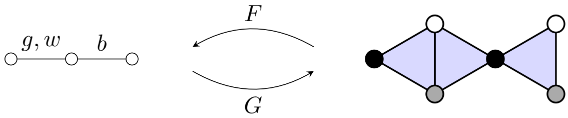
Assume we have $n+1$ agents $\mathcal{A} = \{a_0, \ldots, a_n\}$.

Theorem

There is an equivalence of categories between the category of (proper) Kripke frames and the category of pure chromatic simplicial complexes of dimension n.



Proof of the theorem



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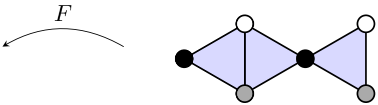

From simplicial complexes to Kripke frames.
Let C be a chromatic simplicial complex. We associate the Kripke frame $F(C) = \langle W, \sim \rangle$, where:

- W is the set of maximal simplices
- For $X, Y \in W$, $X \sim_a Y$ if $X \cap Y$ has an a-colored vertex.

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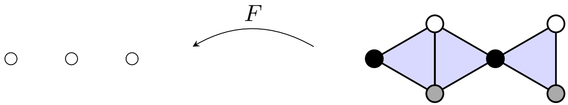

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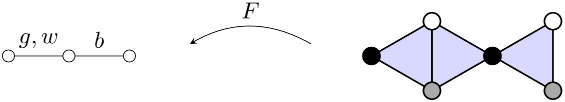

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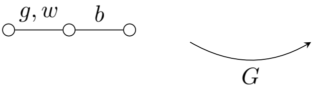


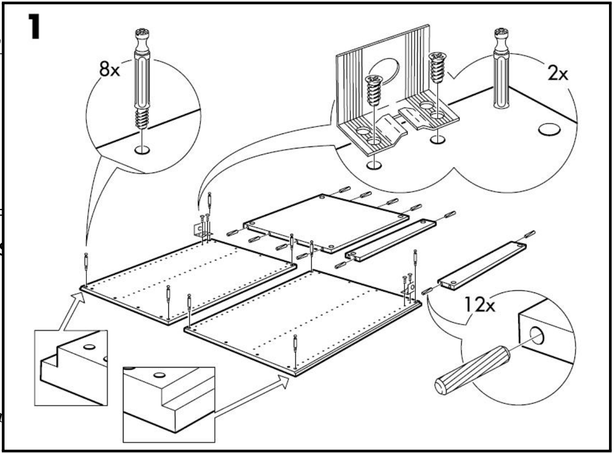
From Kripke frames to simplicial complexes.
Let $M = \langle W, \sim \rangle$ be a Kripke frame and $\mathcal{A} = \{a_0, \ldots, a_n\}$ the set of agents, then:

$$G(M) = \left(\coprod_{x \in W} \{v_0^x, \ldots, v_n^x\} \right) / \equiv$$

where $v_i^x \equiv v_i^y$ iff $x \sim_{a_i} y$.

Proof of the theorem

$g,$



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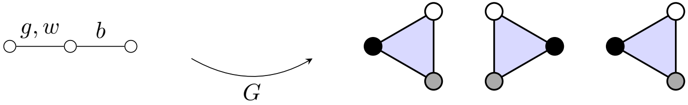

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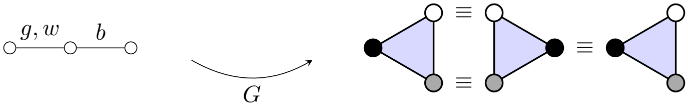

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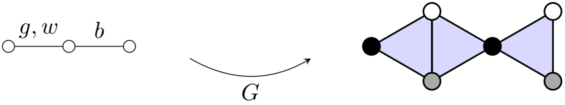

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Simplicial models

Assume we have $n+1$ agents $\mathcal{A} = \{a_0, \ldots, a_n\}$.

Definition

A **simplicial model** is given by $\langle V, S, \chi, \ell \rangle$ where:

- $\langle V, S, \chi \rangle$ is a pure chromatic simplicial complex of dimension n.
- $\ell : V \to \mathscr{P}(AP)$

Example: binary input complex for 3 agents

- Every agent has input value either 0 or 1.
- Every agent knows its value, but not the other values.

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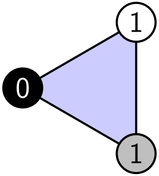

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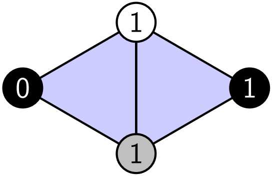

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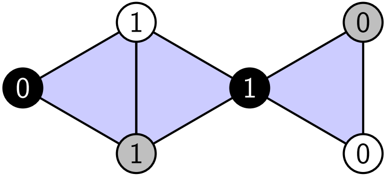

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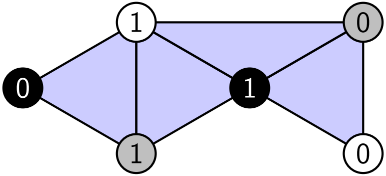

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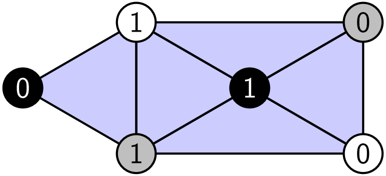

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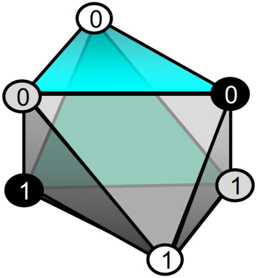

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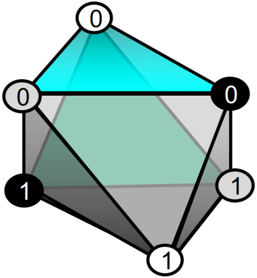

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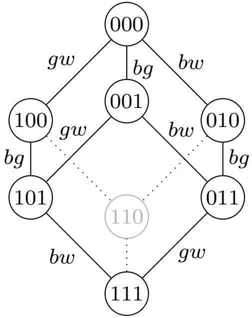

Theorem

The previous theorem still holds for models!

Defining truth in simplicial models

Let $M = \langle V, S, \chi, \ell \rangle$ be a simplicial model and $X \in \mathcal{F}(S)$ a maximal simplex of M.

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M, X \models p & \text{iff} & p \in \ell(X) \\
M, X \models \neg\varphi & \text{iff} & M, X \not\models \varphi \\
M, X \models \varphi \wedge \psi & \text{iff} & M, X \models \varphi \text{ and } M, X \models \psi \\
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M, x \models C_B \varphi & \text{iff} & \ldots
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Theorem

This definition agrees with the usual one:

$$
\begin{array}{lll}
M, X \models_{\mathcal{S}} \varphi & \textit{iff} & F(M), X \models_{\mathcal{K}} \varphi \\
N, x \models_{\mathcal{K}} \varphi & \textit{iff} & G(N), G(x) \models_{\mathcal{S}} \varphi
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Example: card dealing

Consider the following situation: *there are three agents and a deck of four cards* $\{0, 1, 2, 3\}$*. Each agent is given a card at random, and the remaining card is kept hidden.*

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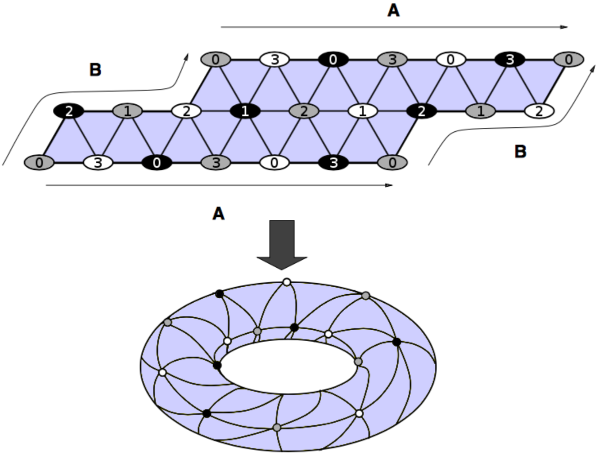

So what?

$\rightarrow$ We have uncovered higher-dimensional topological information which is hidden in Kripke models.

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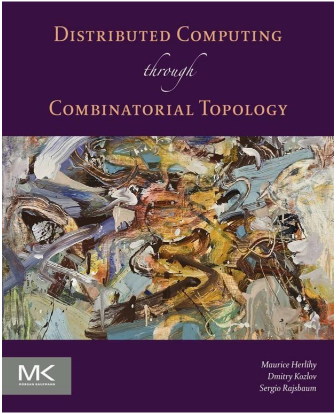
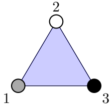
Does it allow us to say anything new about logic?

Yes: examples from distributed computability!



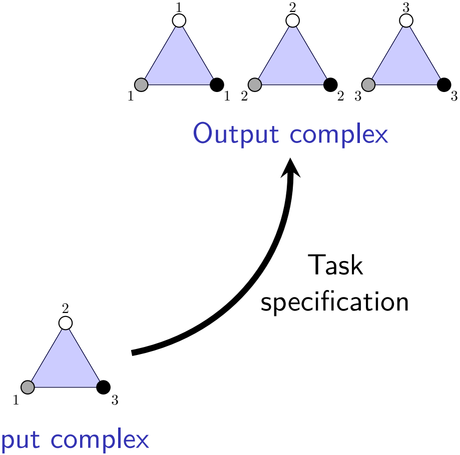
Herlihy, Kozlov, Rajsbaum, 2013

Distributed computability (Herlihy et. al.)



Input complex

Distributed computability (Herlihy et. al.)



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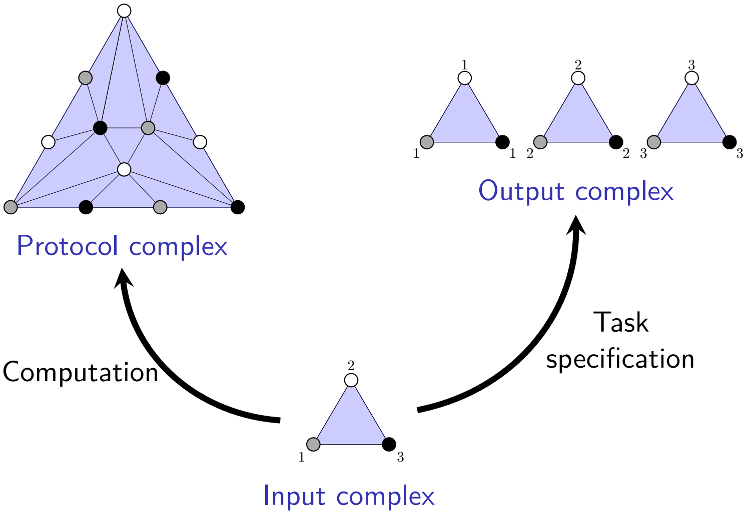

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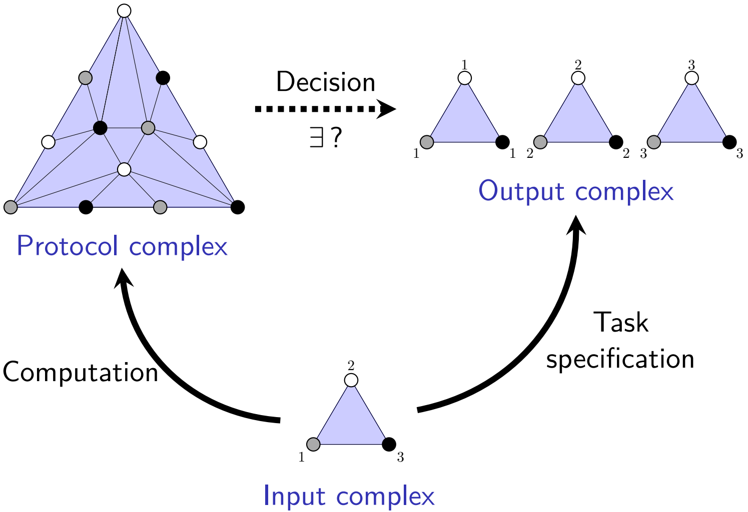

Dynamic epistemic logic

Dynamic Epistemic Logic (DEL)

Syntax:

Let $\mathcal{A}$ be a finite set of *agents* and AP a set of *atomic propositions*. The syntax of formulas is:

$$\begin{array}{lcl} \varphi & ::= & p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K_a\,\varphi \mid C_B\,\varphi \mid [\alpha]\,\varphi \\ \alpha & ::= & \text{(see next slide)} \end{array}$$

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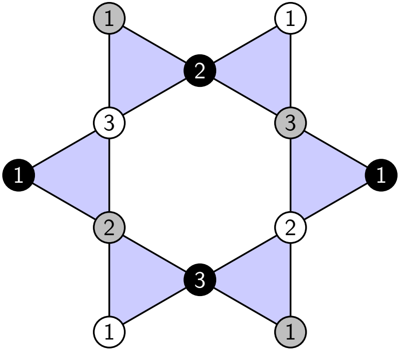
Semantics:

$$\begin{array}{lll} M, x \models p & \text{iff} & p \in L(x) \\ M, x \models \neg\varphi & \text{iff} & M, x \not\models \varphi \\ M, x \models \varphi \wedge \psi & \text{iff} & M, x \models \varphi \text{ and } M, x \models \psi \\ M, x \models K_a\,\varphi & \text{iff} & \text{for all } y \in W, x \sim_a y \text{ implies } M, y \models \varphi \\ M, x \models C_B\,\varphi & \text{iff} & \ldots \\ M, x \models [\alpha]\,\varphi & \text{iff} & M[\alpha], x[\alpha] \models \varphi \end{array}$$

Action models

Three agents, three cards $\{1, 2, 3\}$.



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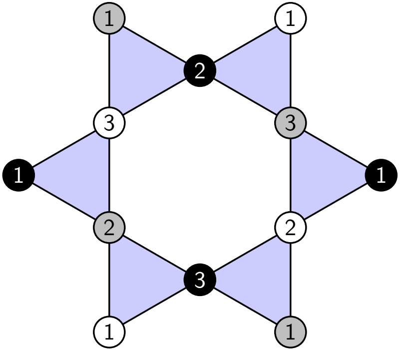

Black announces publicly:
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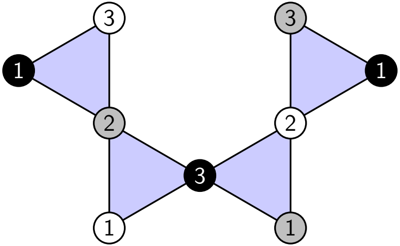

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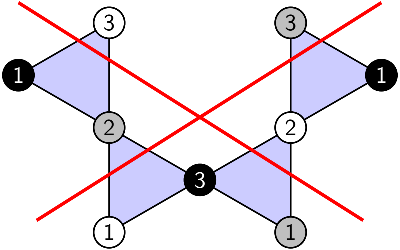

Black announces publicly:
"I do not have card 2".

Black says privately to White:
"I do not have card 2".
$\rightarrow$ this does not work.

Action models

Definition

An **action model** is a tuple $\langle T, (\sim_a)_{a \in \mathcal{A}}, \mathsf{pre} \rangle$ where:

- T is a set of *actions*,
- for each $a \in \mathcal{A}$, $\sim_a$ is an equivalence relation on T,
- for each $t \in T$, $\mathsf{pre}(t) \in \mathcal{L}_{\mathcal{A},AP}$ is a *precondition*.

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Example:

Public announcement

Black: "$\neg 2$"

Black: "$\neg 1$"

Black: "$\neg 3$"

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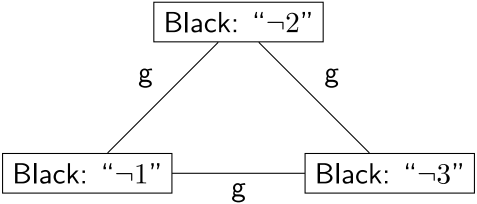
Example:

Private announcement of Black to White



Product update

Let $\mathcal{M} = \langle V, S, \chi, \ell \rangle$ be a simplicial model and $\mathcal{T} = \langle T, \sim, \text{pre}$ an action model. The **product update model** $\mathcal{M}[\mathcal{T}]$ is the following simplicial model:

- its vertices are of the form $(v, t) \in V \times T$,
- $\chi(v, t) = \chi(v)$ and $\ell(v, t) = \ell(v)$,
- the maximal simplices are the (X, t) such that $\mathcal{M}, X \models \text{pre}(t)$

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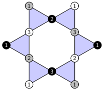
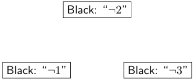
An **action** is $\alpha := (\mathcal{T}, t)$.
The truth of a DEL formula is defined as:

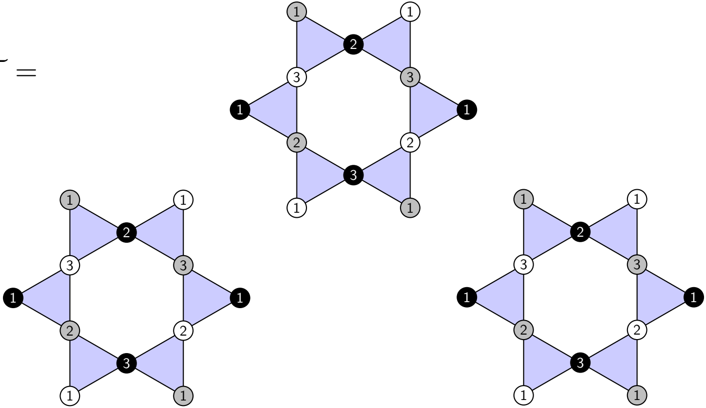
$$\mathcal{M}, X \models [(\mathcal{T}, t)]\,\varphi \quad \text{iff} \quad \mathcal{M}[\mathcal{T}], (X, t) \models \varphi$$

Example: Public announcement

$\mathcal{M} =$ 

$\mathcal{T} =$ 

$\mathcal{M} \times \mathcal{T} =$



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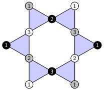
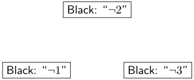
$\mathcal{M} =$ 

$\mathcal{T} =$ 

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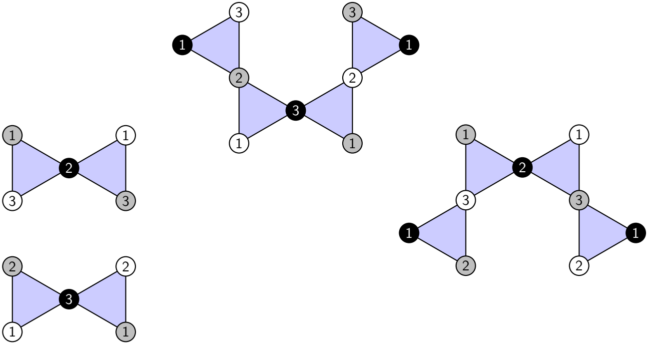

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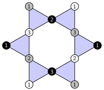
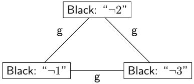
$\mathcal{M} \ =$ 

$\mathcal{T} \ =$ 

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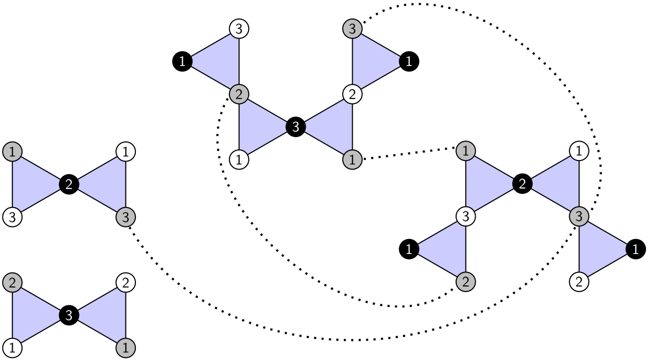
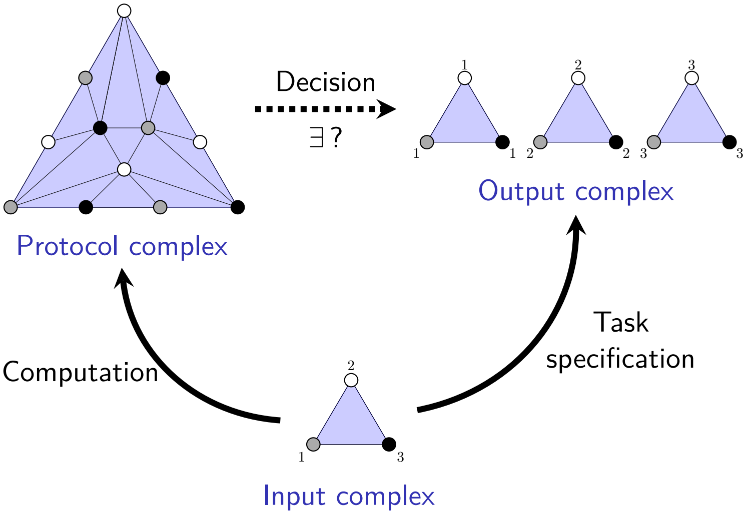


Distributed computability via logic



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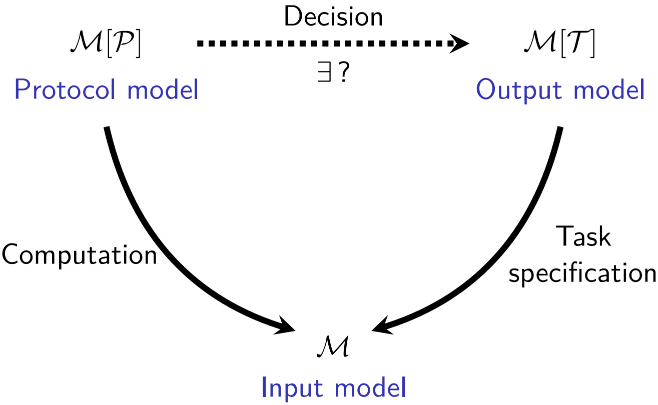

Key Lemma: simplicial maps cannot gain knowledge

Lemma

Consider two simplicial models M and M', and a morphism $f : M \to M'$. Let $X \in \mathcal{F}(M)$ be a maximal simplex of M, a an agent, and φ a positive formula (φ does not contain negations except, possibly, in front of atomic propositions). Then,

$$M', f(X) \models \varphi \quad \textit{implies} \quad M, X \models \varphi$$

Recipe for impossibility proofs:

- Assume $\delta : \mathcal{M}[\mathcal{P}] \longrightarrow \mathcal{M}[\mathcal{T}]$
- Find a suitable formula φ such that:
- φ is true everywhere in the output model
- φ is false somewhere in the protocol model

Conclusions and perspectives

Benefits in both areas

- For computer scientists: we can now understand the abstract topological proofs of impossibility in terms of *knowledge*.

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Future work

- Simplicial complexes that are not pure
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Future work

- Simplicial complexes that are not pure
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- New notions of knowledge?

Distributed computing	Topology	Logic
consensus	connectedness	common knowledge
k-set agreement	k-connectedness	???

Thanks!