Semi-simplicial Set Models for Distributed Knowledge

LICS 2023, Boston, USA

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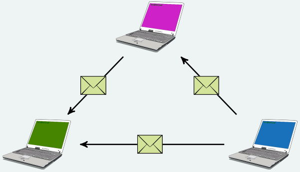
Tuesday 27 June, 2023

Simplicial Models for Epistemic Logic

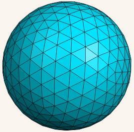
Epistemic Logic

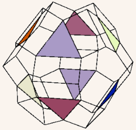
"I know that you know that I know"

Distributed computing



Combinatorial Topology





Multi-Agent Epistemic Logic

Let Ag a finite set of agents and At a set of atomic propositions.

Syntax :

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K_a\,\varphi \mid \dots \qquad\qquad p \in \mathrm{At},\ a \in \mathrm{Ag}$$

- $K_a\,\varphi$ is read: “agent a knows φ”.

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- $D_{\{a\}} \varphi \iff K_a \varphi$.

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In distributed computing:

Agents	$\longleftrightarrow$	Processes
Atomic propositions	$\longleftrightarrow$	Facts about the system

Simplicial Models

Definition (Goubault, L., Rajsbaum – 2018, 2021)

A simplicial model is given by $\mathscr{C} = (V, S, \chi, \ell)$, where:

- (V, S) is a simplicial complex.
- $\chi : V \to \mathsf{Ag}$ assigns an agent to each vertex.
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We deal one card to each agent, and keep the remaining card hidden.

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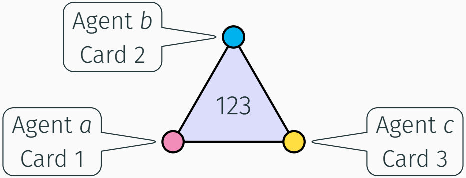

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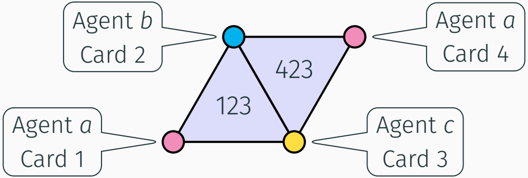

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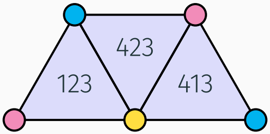

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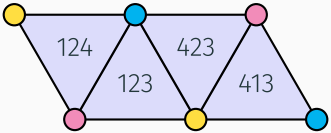

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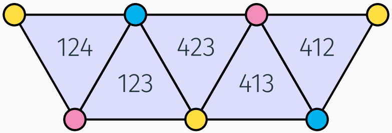

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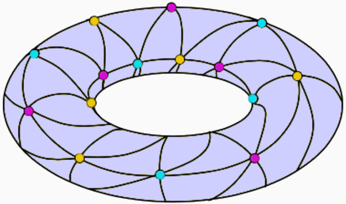

Semantics of simplicial models

We define the satisfaction relation $\mathscr{C}, X \models \varphi$, where:

- $\mathscr{C}$ is a simplicial model,
- $X \in \mathrm{Facet}(\mathscr{C})$ is a world of $\mathscr{C}$,
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By induction on φ:

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\mathscr{C}, X \models p & \text{iff} & p \in \ell(X) \\
\mathscr{C}, X \models \neg\varphi & \text{iff} & \mathscr{C}, X \not\models \varphi \\
\mathscr{C}, X \models \varphi \wedge \psi & \text{iff} & \mathscr{C}, X \models \varphi \ \text{ and } \ \mathscr{C}, X \models \psi \\
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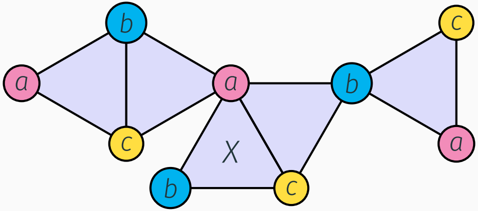

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X and Y share an a-colored vertex

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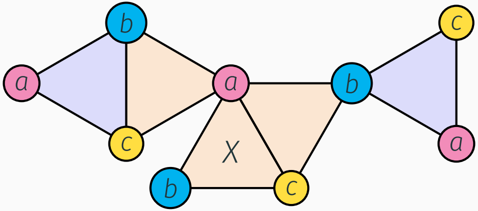

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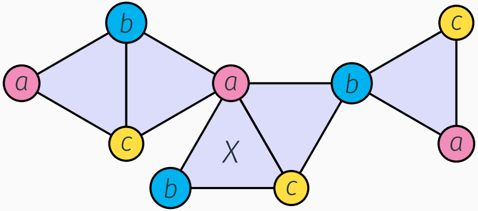

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X and Y share an A-colored face

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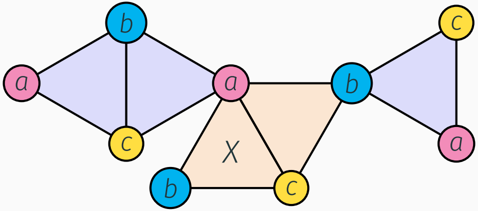
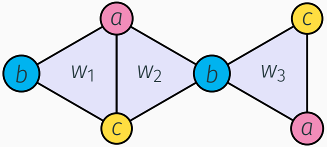


Equivalence with Kripke models

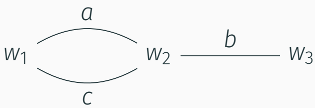
Theorem (Goubault, L., Rajsbaum (2018, 2021))

The category of pure simplicial models is equivalent to the one of proper Kripke models.

Example: with three agents, Ag = { a , b , c },



$\simeq$



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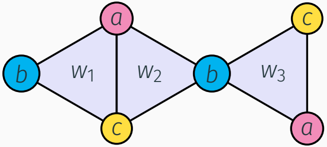

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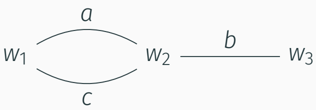


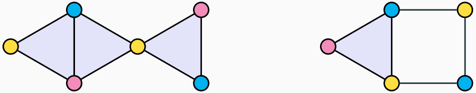
Corollary (Conservation of satisfiability)

$\mathcal{C}, w \models \varphi$ *in a pure simplicial model* *iff* $M, w \models \varphi$ *in the associated Kripke model.*

Variants of Simplicial Models

What can we do differently?

(1) Pure vs. impure simplicial complexes.



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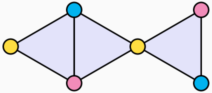

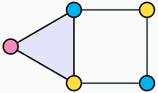

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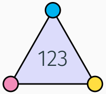
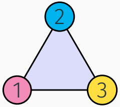
(2) Atomic propositions on the worlds vs. vertices.





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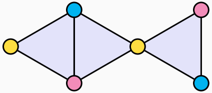

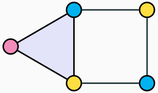

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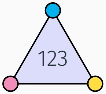

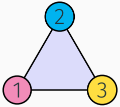

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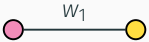
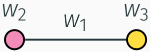
(3) The worlds are facets vs. simplexes.





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Our Contribution

With each new variant, one usually asks two fundamental questions:

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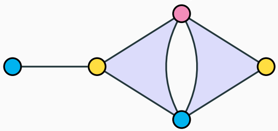
Our contribution:

- We define a very general class of simplicial models: epistemic coverings.
- We establish a dictionary:
  Properties of coverings $\iff$ Properties of Kripke models $\iff$ Axioms of the logic.
- This solves questions 1 and 2 for all the corresponding sub-classes of models!

New features of Epistemic Coverings

(1) Models are based on semi-simplicial sets, generalizing simplicial complexes.



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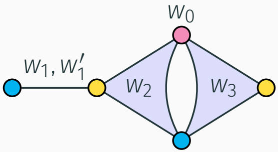
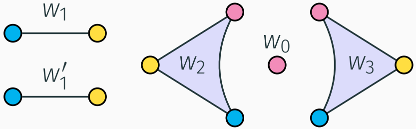


(2) Models are equipped with a discrete covering E,

$E =$



and a map $p : E \to B$, tagging which simplexes are worlds.

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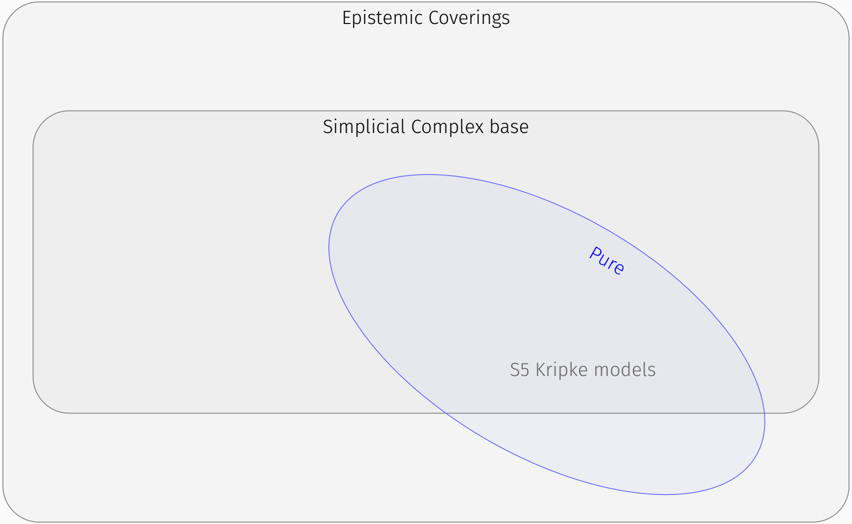

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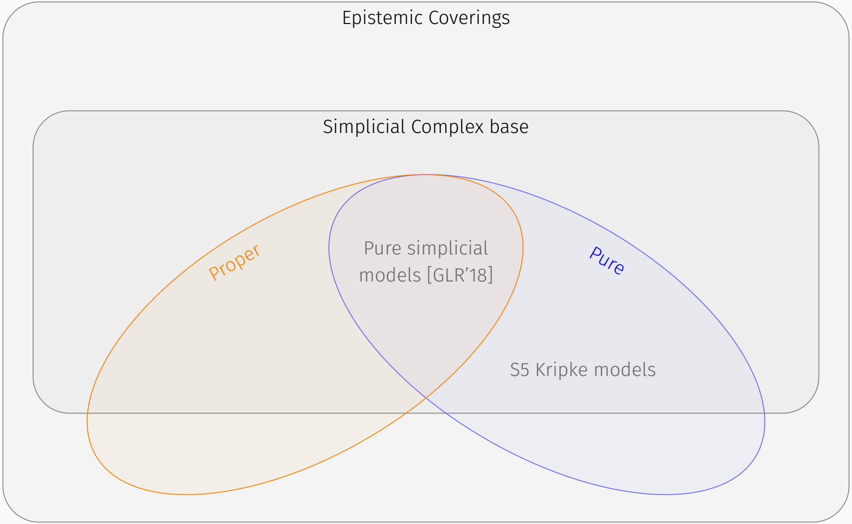

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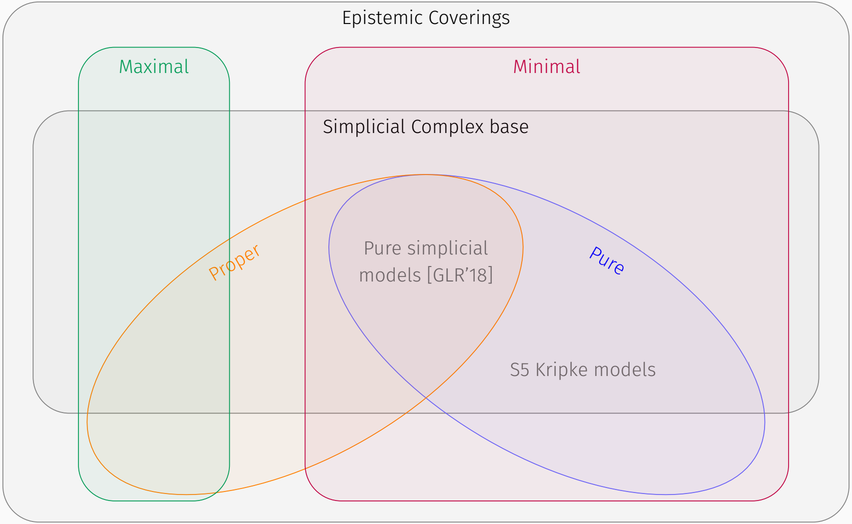

Conclusion

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Key messages:

- S5 Kripke models have an underlying higher-dimensional structure.
- Distributed knowledge = higher-dimensional connectivity.
- We introduced epistemic coverings, a very large class of simplicial models.
- Many interesting variants occur as sub-classes defined by various properties.
- Each sub-class has (1) a Kripke counterpart; (2) a sound and complete axiomatization.

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- S5 Kripke models have an underlying higher-dimensional structure.
- Distributed knowledge = higher-dimensional connectivity.
- We introduced epistemic coverings, a very large class of simplicial models.
- Many interesting variants occur as sub-classes defined by various properties.
- Each sub-class has (1) a Kripke counterpart; (2) a sound and complete axiomatization.

Thanks!