Brief Announcement: Variants of Approximate Agreement on Graphs and Simplicial Complexes

Jérémy Ledent

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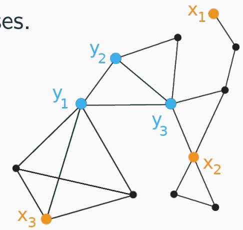
MSP Group, University of Strathclyde, Glasgow

Graph Approximate Agreement

Fix a connected graph $\mathcal{G} = (V, E)$, and let $n \geq 3$ be the number of processes.
The graph approximate agreement task on $\mathcal{G}$ is defined as follows.



Input: Each process is given a vertex $x_i \in V$.

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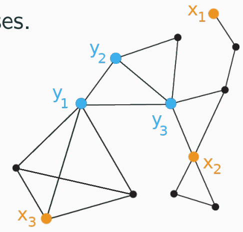

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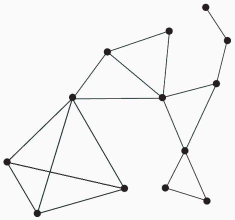
⚠ $\mathcal{G}$ is **not** the network topology of the system.

Wait-free solvability of Graph Apprixomate Agreement

Nowak and Rybicki, *Byzantine Approximate Agreement on Graphs*, DISC '19:

- ✓ Solvable if $\mathscr{G}$ is chordal.



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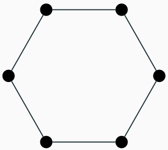

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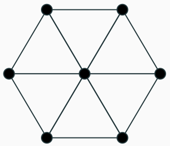

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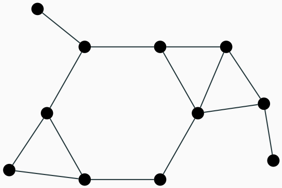
Alistarh, Ellen, Rybicki, *Wait-free approximate agreement on graphs*, SIROCCO '21:

- ✓ Solvable if $\mathscr{G}$ is nicely bridged.
- ✗ Unsolvable if $\mathscr{G}$ admits a *lower bound labelling*.



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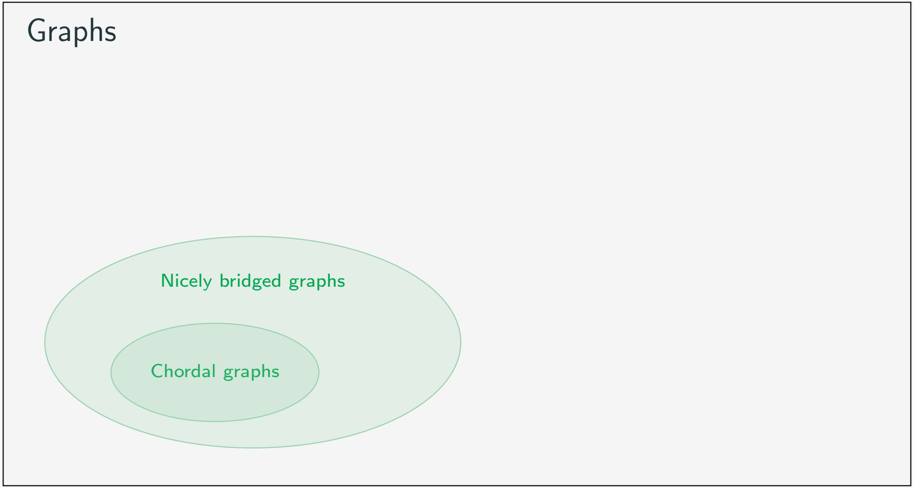

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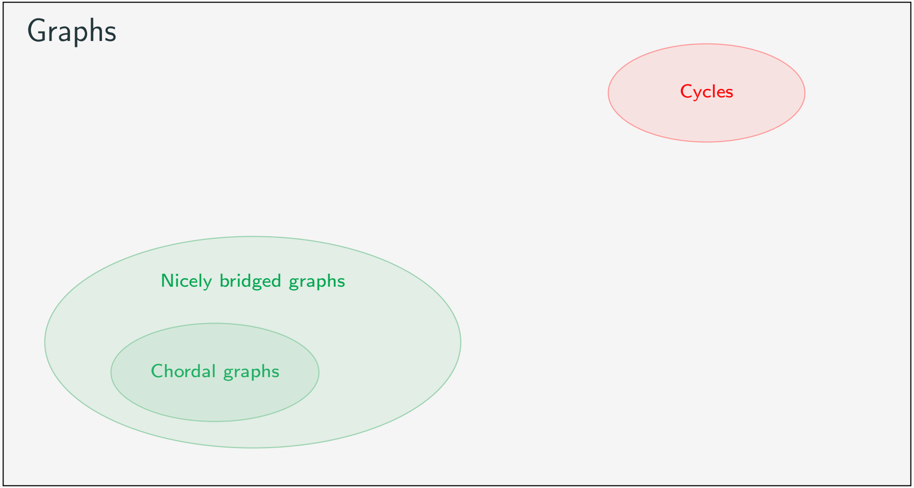

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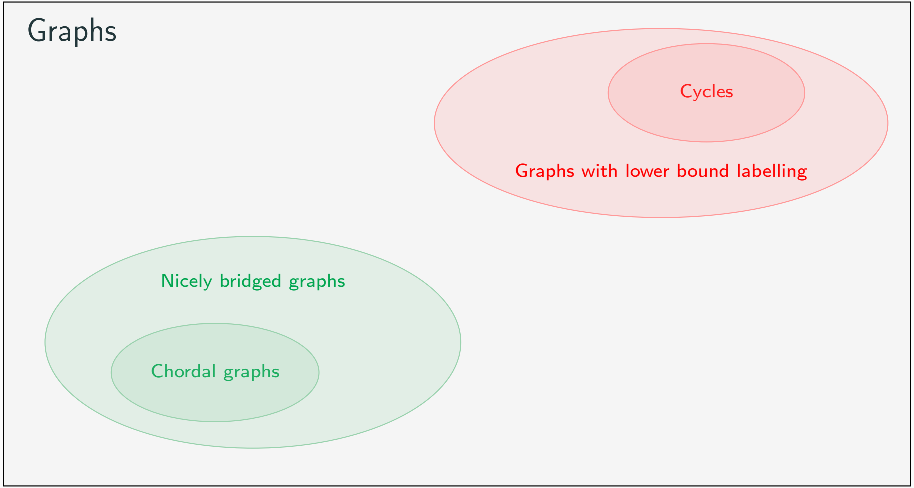

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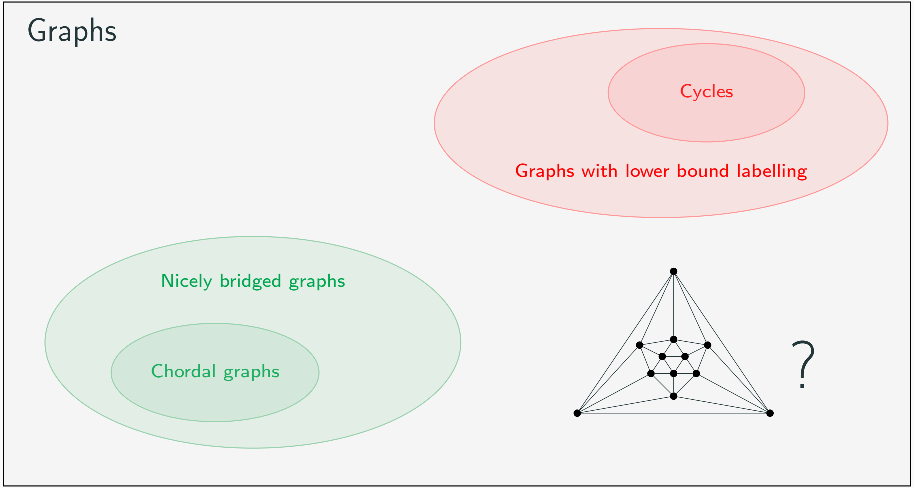

Contribution

Main Goal: show a more topological point of view on Graph Approximate Agreement.

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Purpose:

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- Replace the graph $\mathscr{G}$ by a simplicial complex, $\kappa(\mathscr{G})$.

Purpose:

- Better understanding of the results from the literature.
- Conjecture the exact class of graphs where the task is wait-free solvable.

The complex of cliques

Let $\mathscr{G} = (V, E)$ be a graph.

Definition

The complex of cliques of $\mathscr{G}$ is the simplicial complex $\kappa(\mathscr{G}) = (V, S)$, where:

$$S = \{X \subseteq V \mid X \text{ is a clique of } \mathscr{G}\}$$

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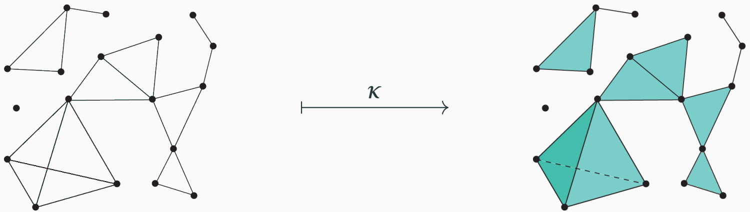

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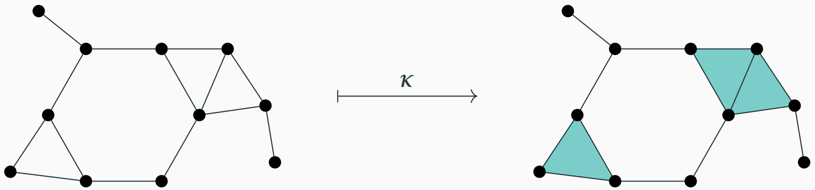

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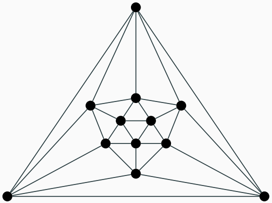

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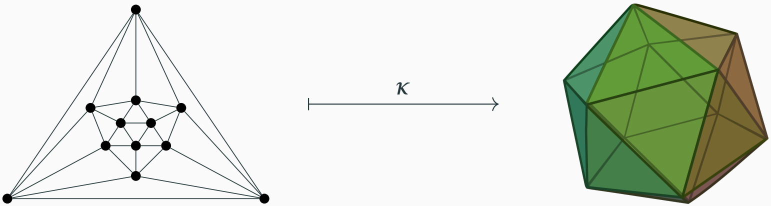

Simplex Agreement task

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⚠ $\mathscr{K}$ is **not** the input complex of the task.

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Theorem

Graph Approximate Agreement on $\mathscr{G}$ is the same task as Simplex Agreement on $\kappa(\mathscr{G})$.

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Fact: The dimension of the complex of cliques is $\dim(\kappa(\mathscr{G})) = \omega(\mathscr{G}) - 1$.

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Now compare the following results:

Theorem (Nowak and Rybicki, DISC 2019)

For a chordal graph $\mathscr{G}$, graph agreement is solvable with f Byzantine faults if $n > (\omega(\mathscr{G}) + 1)f$.

Theorem (Mendes, Helihy, Vaidya, Garg, 2015)

Multidimensional approx. agreement in $\mathbb{R}^d$ is solvable with f Byzantine faults iff $n > (d + 2)f$.

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Motto: *Simplex Agreement is the discrete counterpart of Multidimensional Agreement.*

Consequences (2)

In the wait-free shared memory model:

Theorem (Herlihy and Shavit, STOC 1993)

Multidimensional ε-approximate agreement is unsolvable in a subspace $\mathcal{V} \subseteq \mathbb{R}^d$ that contains holes of radius $\geq \varepsilon$.

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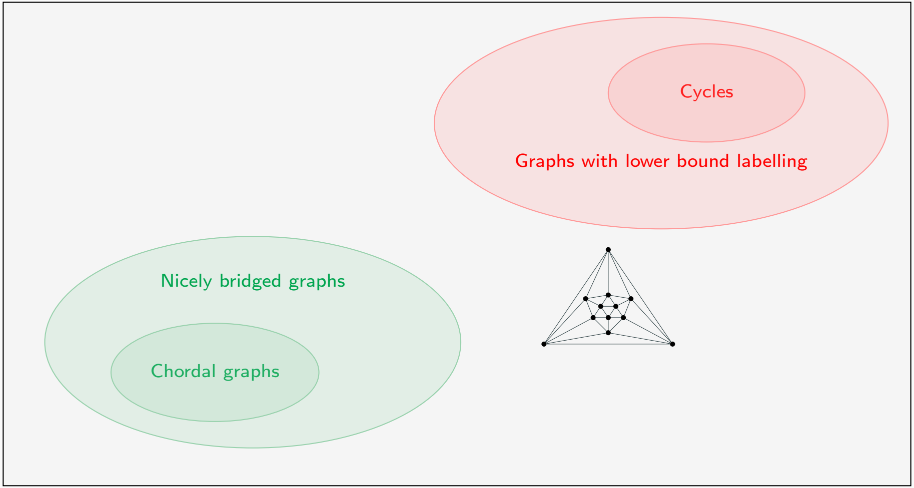
By analogy with the continuous case, we can conjecture the following:

Conjecture

- *Simplex agreement on $\mathcal{K}$ is solvable iff $\mathcal{K}$ is contractible.*
- *Graph approximate agreement on $\mathcal{G}$ is solvable iff $\kappa(\mathcal{G})$ is contractible.*

Consequences (3)



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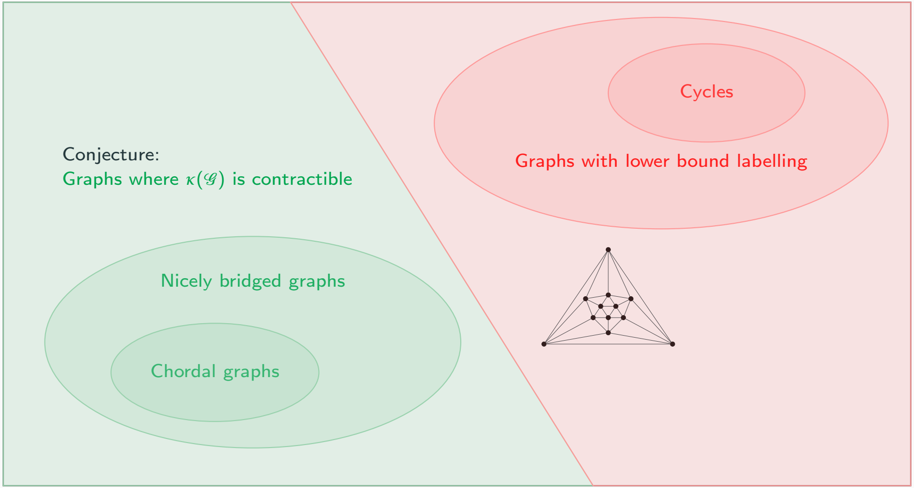

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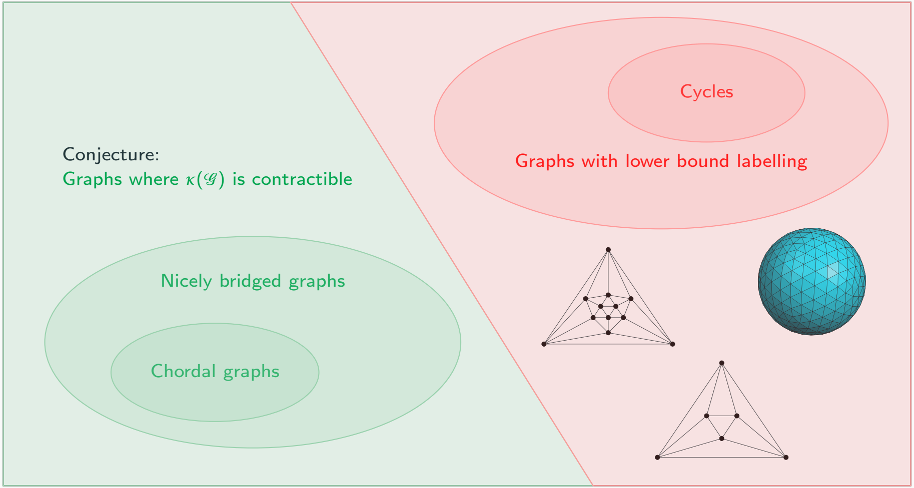

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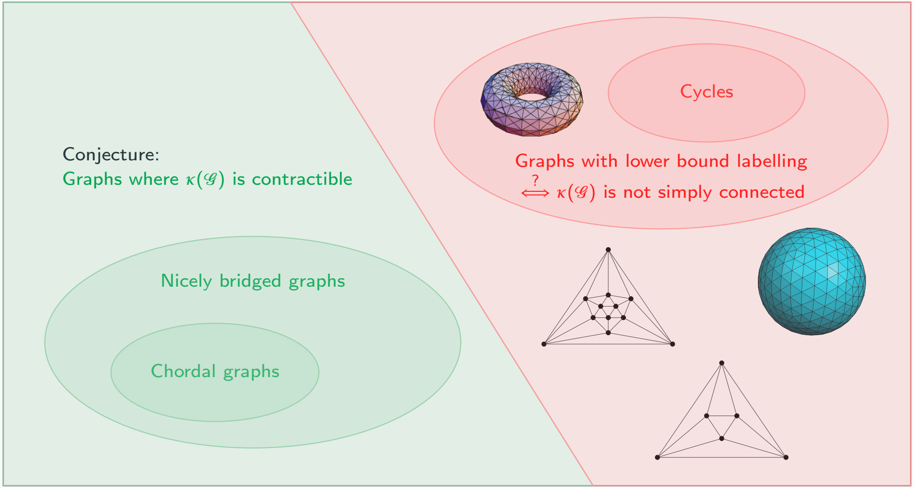

Thanks!