A Simplicial Model for KB4: Epistemic Logic with Agents that May Die

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Friday 18 March, 2022

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Introduction

Epistemic Logic: Syntax

Let Ag be a finite set of agents and Prop a set of atomic propositions.

Syntax:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K_a \varphi \qquad p \in \text{Prop},\ a \in \text{Ag}$$

Example formula: $K_a \neg K_b \varphi$ where $a, b \in \text{Ag}$

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In distributed computing:

Agents	$\longleftrightarrow$	Processes
Atomic propositions	$\longleftrightarrow$	Facts about the system

Epistemic Logic : Semantics

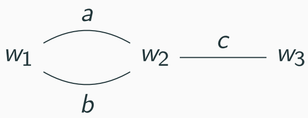
Kripke semantics : Based on Hintikka's idea of *"possible worlds"*.

Definition

An epistemic Kripke model $M = (W, \sim, L)$ is given by:

- a set W of possible worlds,
- for each $a \in \mathrm{Ag}$, an equivalence relation $\sim_a \subseteq W \times W$, called indistinguishability,
- a function $L : W \to \mathscr{P}(\mathrm{Prop})$ assigning atomic propositions to each world.

Example:



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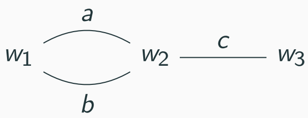

Satisfaction relation : $M, w \models K_a \varphi$ iff $M, w' \models \varphi$ for all w' such that $w \sim_a w'$.

Simplicial complexes

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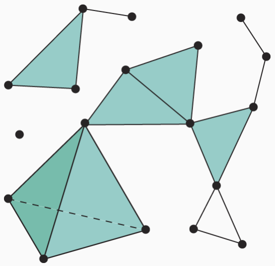
A simplicial complex is a pair (V, S) where:

- V is a set of vertices,
- S is a downward-closed family of subsets of V, called simplexes.



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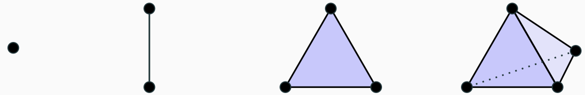
simplex



dimension	0	1	2	3	$\cdots$

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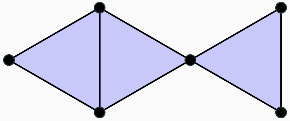
A facet is a simplex which is maximal w.r.t. inclusion.

A simplicial complex is pure if all facets have the same dimension.



Pure Simplicial Models

Definition (Goubault, Ledent, Rajsbaum (2018, 2021))

A pure simplicial model is given by $\mathscr{C} = (V, S, \chi, \ell)$ where:

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We deal one card to each agent, and the remaining card is kept hidden.

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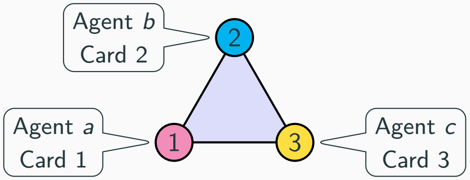

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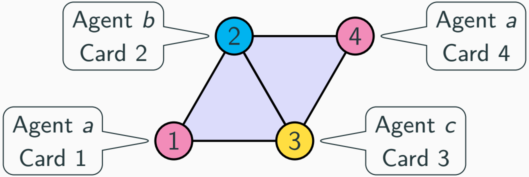

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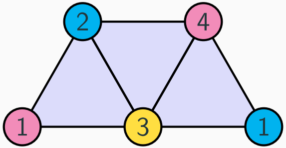

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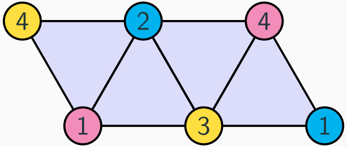

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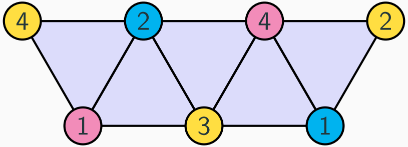

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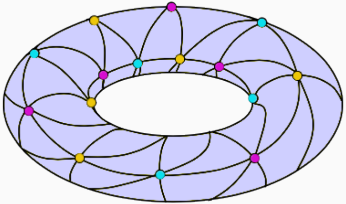

Equivalence with Kripke models

Suppose the number of agents is $|\text{Ag}| = n + 1$.

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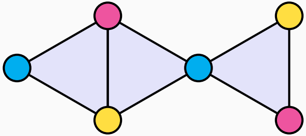

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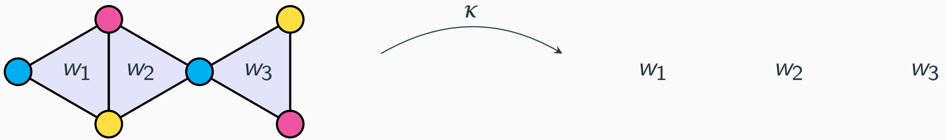

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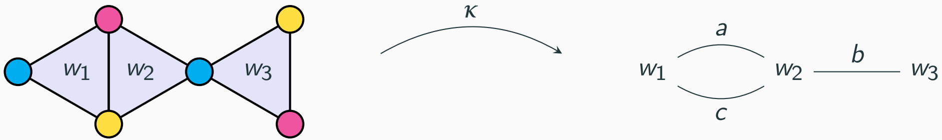

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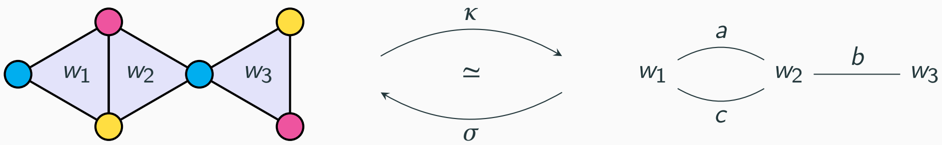

Contribution

What about impure simplicial models?

- Previous paper: pure simplicial models, i.e., all worlds must have the same dimension.
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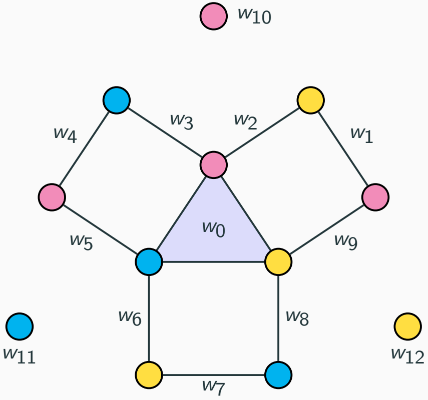

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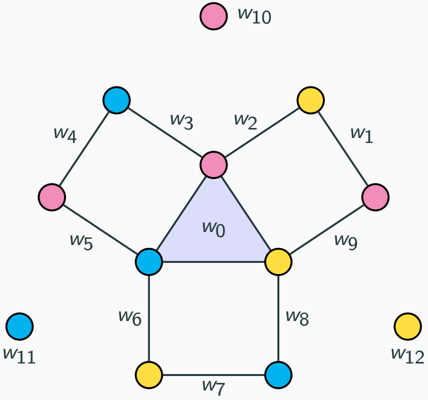

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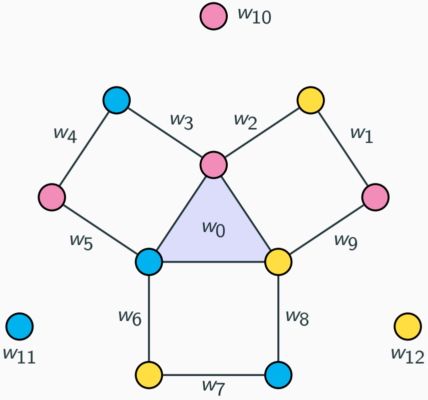

Satisfaction relation

- Define $\mathscr{C}, w \models \varphi$, where w is a facet of $\mathscr{C}$:

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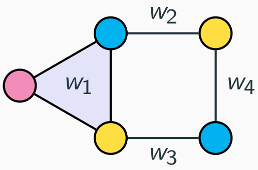

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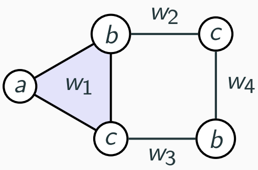

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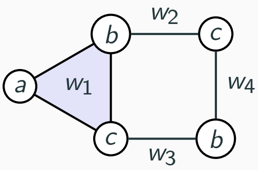

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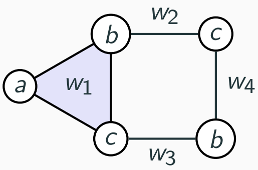

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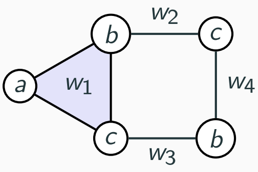

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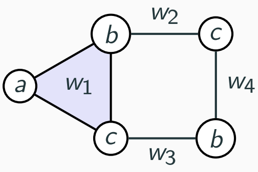

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Dead or Alive?

Define the following formulas:

$$\mathrm{dead}(a) = K_a\,\mathrm{false} \qquad\qquad \mathrm{alive}(a) = \neg\,\mathrm{dead}(a)$$

One can check that:

$$\mathscr{C}, w \models \mathrm{alive}(a) \quad \text{iff} \quad a \in \chi(w)$$

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In every simplicial model:

- Dead agents know everything: $\models \mathrm{dead}(a) \implies K_a\varphi$.
- Alive agents satisfy Axiom **T**: $\models \mathrm{alive}(a) \implies (K_a\varphi \Rightarrow \varphi)$.
- Alive agents know they are alive: $\models \mathrm{alive}(a) \implies (K_a\,\mathrm{alive}(a))$.

Axiomatisation

KB4: the following axioms are valid in all simplicial models.

$$\mathbf{K} : K_a\varphi \land K_a(\varphi \Rightarrow \psi) \implies K_a\psi \qquad \mathbf{B} : \varphi \implies K_a\neg K_a\neg\varphi \qquad \mathbf{4} : K_a\varphi \implies K_aK_a\varphi$$

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$$\mathbf{K} : K_a\varphi \wedge K_a(\varphi \Rightarrow \psi) \implies K_a\psi \qquad \mathbf{B} : \varphi \implies K_a\neg K_a\neg\varphi \qquad \mathbf{4} : K_a\varphi \implies K_aK_a\varphi$$

Two extra axioms: not provable in **KB4**, but valid in all simplicial models.

- **NE**: there is at least one alive agent.
- **SA**: if an agent alone, then this agent knows that it is alone.

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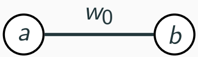
Theorem (Completeness)

The axiom system ***KB4*** *+* ***NE*** *+* ***SA*** *is sound and complete w.r.t. (impure) simplicial models.*

Extensions (Journal version)

Worlds are facets vs. Worlds are simplexes.



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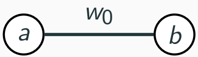

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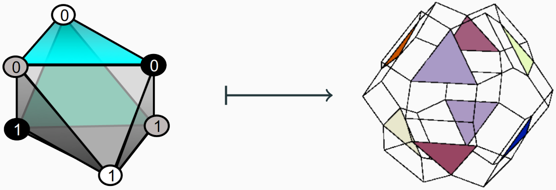
Dynamic Epistemic Logic.

Synchronous message-passing model with crashes



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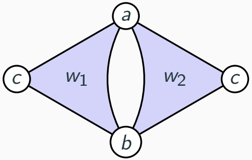

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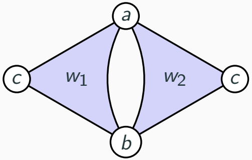

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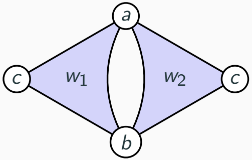

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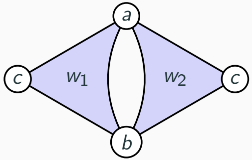

Thanks for listening!