Geometric Semantics for Asynchronous Computability

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PhD defense – École Polytechnique

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The distributed computing setting

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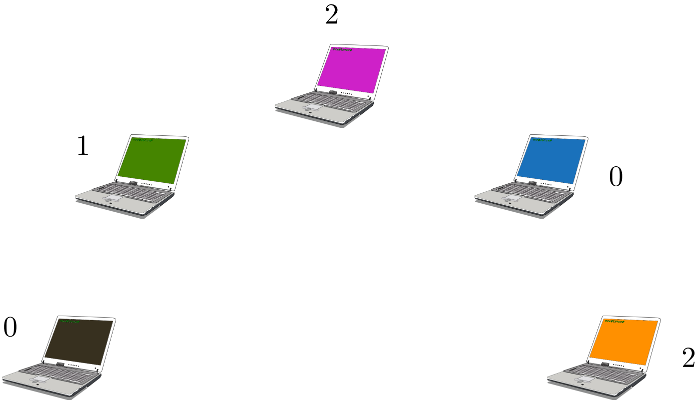

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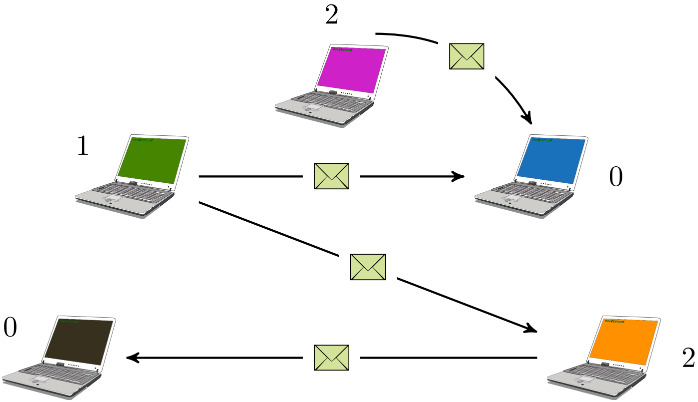

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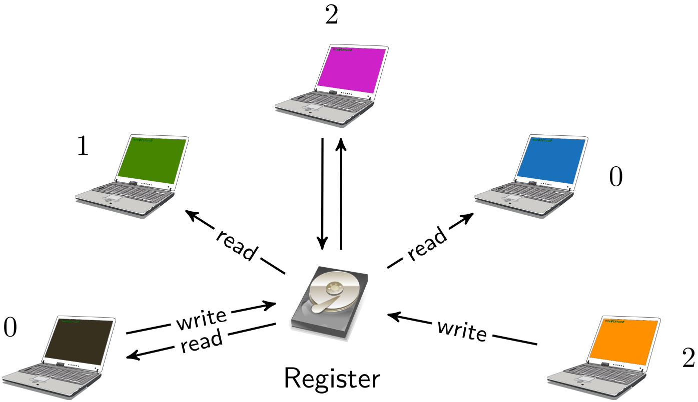

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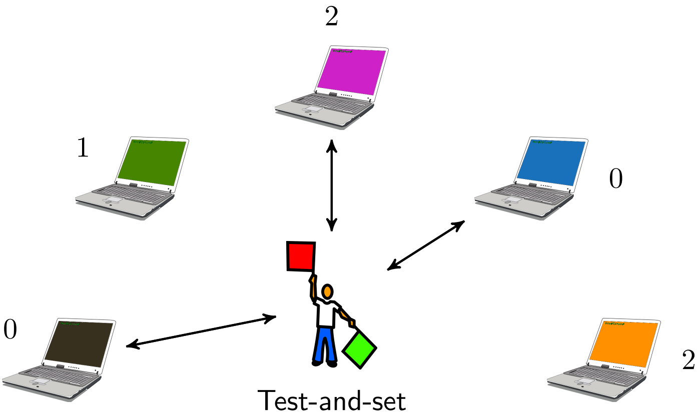

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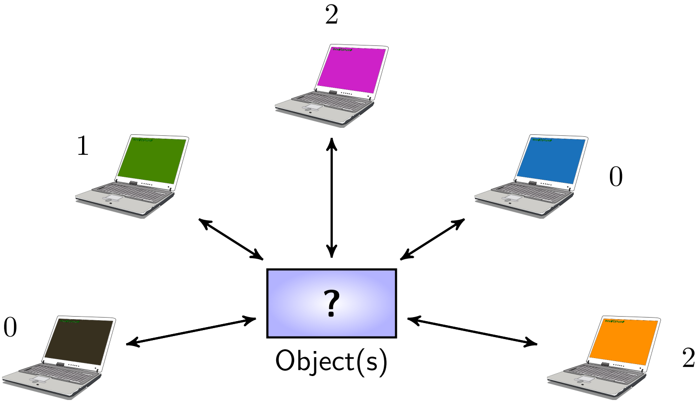

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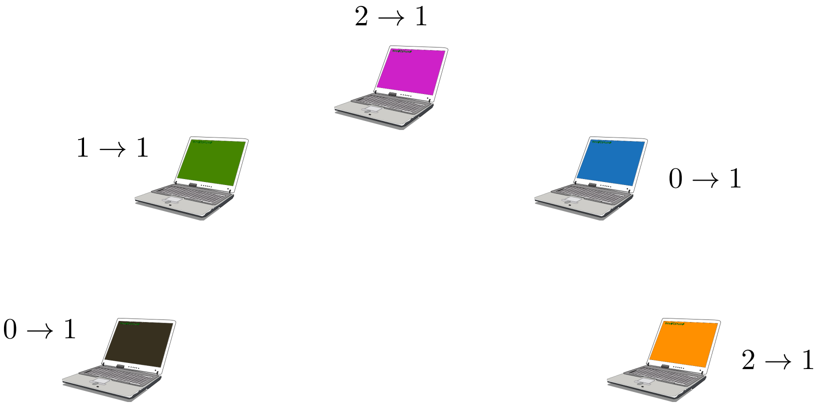

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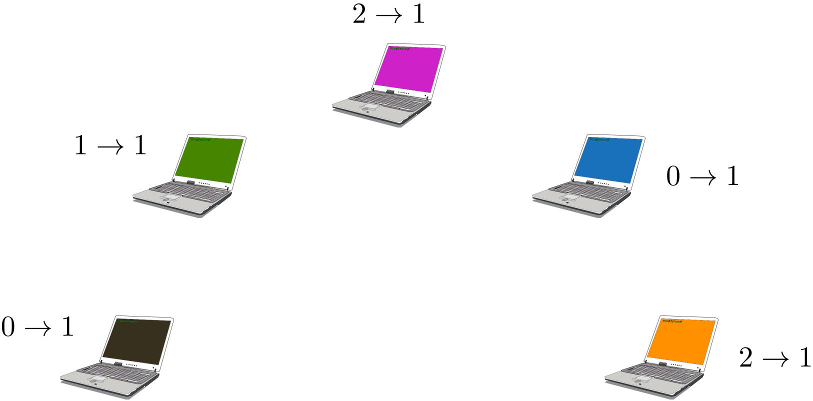

Task specification: $(0, 1, 2, 0, 2) \to (1, 1, 1, 1, 1)$ ✓ or ✗ ?

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a.k.a. Fault-tolerant distributed computing

Goal: Study which concurrent tasks are solvable in various computational models.

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Our assumptions: Asynchronous and wait-free.

A topological approach

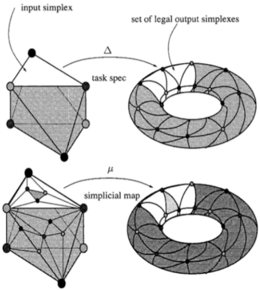

FIG. 13. Asynchronous computability theorem.

THEOREM 3.1 (ASYNCHRONOUS COMPUTABILITY THEOREM). *A decision task* $\langle \mathcal{I}, \mathcal{O}, \Delta \rangle$ *has a wait-free protocol using read-write memory if and only if there exists a chromatic subdivision* σ *of* $\mathcal{I}$ *and a color-preserving simplicial map*

$$\mu: \sigma(\mathcal{I}) \to \mathcal{O}$$

such that for each simplex S *in* $\sigma(\mathcal{I})$, $\mu(S) \in \Delta(carrier(S, \mathcal{I}))$.

Herlihy and Shavit, 1999
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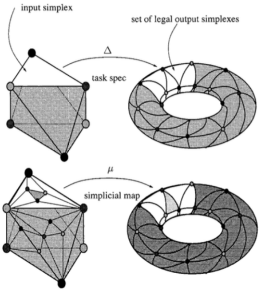

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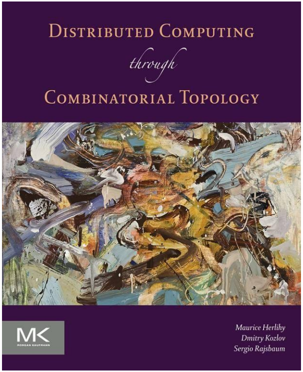

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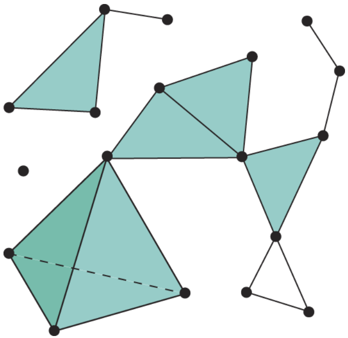
Simplicial complexes

Definition

An (abstract) **simplicial complex** is a pair $\langle V, S \rangle$ where V is a set of *vertices* and S is a downward-closed family of subsets of V called *simplices* (i.e., $X \in S$ and $Y \subseteq X$ implies $Y \in S$).



Example: binary input complex for 3 processes

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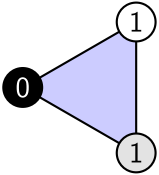

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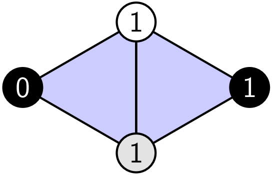

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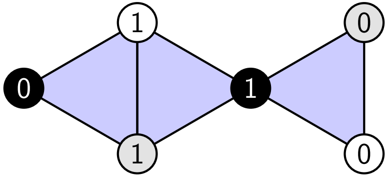

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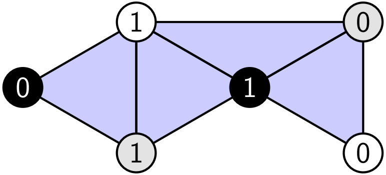

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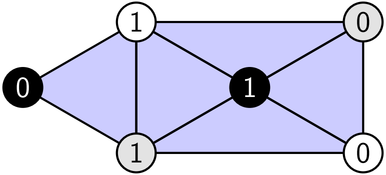

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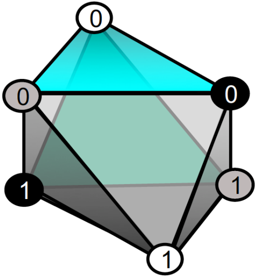

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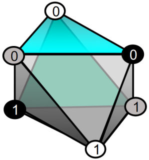

Input complex

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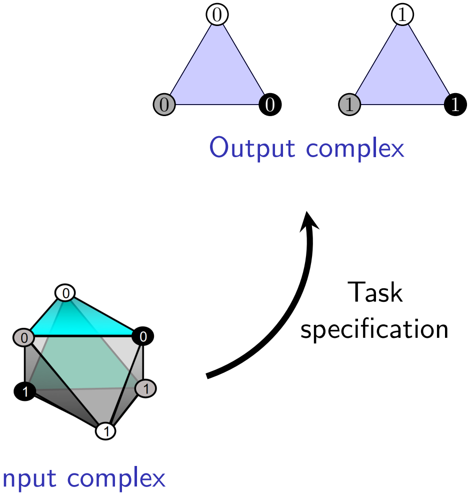

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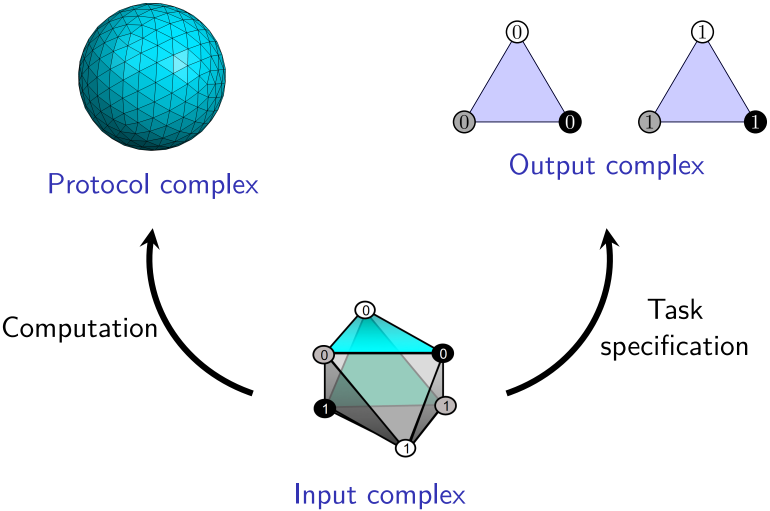

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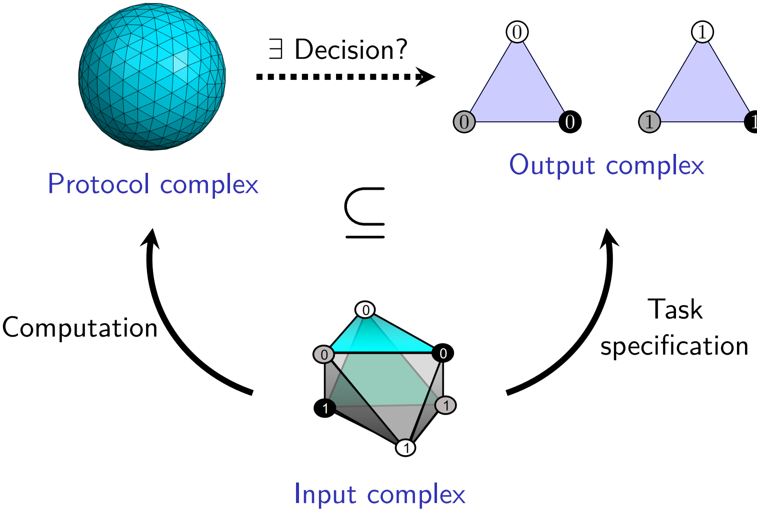

Part I: Operational Semantics

Asynchronous Computability Theorem (ACT)

Theorem (Herlihy and Shavit, 1999)

A task is solvable by a **wait-free** protocol using **read/write registers** if and only if there is a decision map from the protocol complex into the output complex such that [...].

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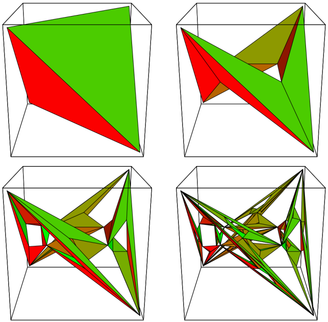
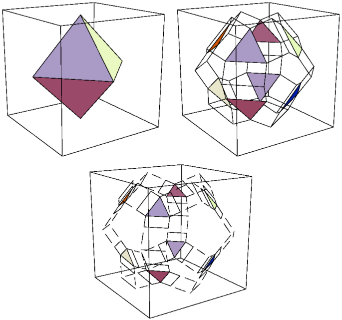
- we use other objects instead of read/write registers?

  $\longrightarrow$ Our goal here.

Protocol complexes for other objects

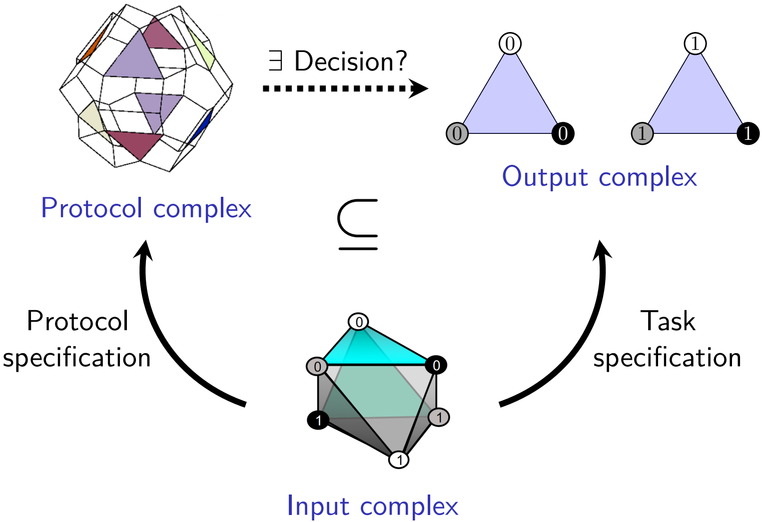


For test-and-set protocols
Herlihy, Rajsbaum, PODC'94



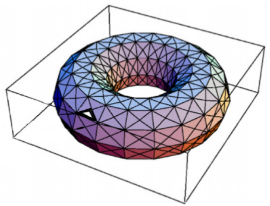
For synchronous message-passing
Herlihy, Rajsbaum, Tuttle, 2001

Topological **definition** of solvability



Benefits and drawbacks

✓ We can prove very general abstract results:



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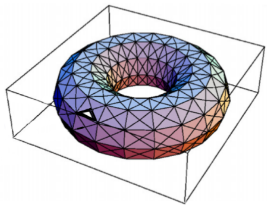

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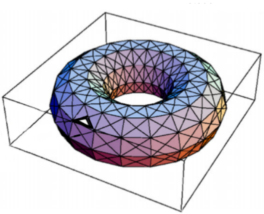

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✗ How do we know our protocol is correctly modeled?

Goal: Give a concrete meaning to "solving a task" using arbitrary objects, and prove that it agrees with the topological definition.

Outline

(1) Define a notion of concurrent object specification which is as general as possible. It should include non-linearizable objects.

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Generalized ACT

A wait-free protocol *solves* a task if and only if there is a simplicial map from the protocol complex to the output complex which is carried by the task specification.

A Sound Foundation for the Topological Approach to Task Solvability. L., Mimram (CONCUR'19)

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Idea: the specification of an object is the set of all the correct execution traces (Lamport, 1986).

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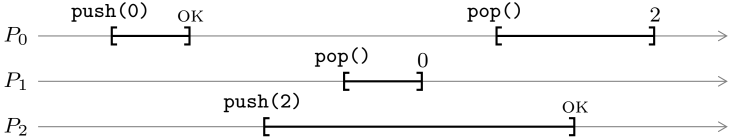

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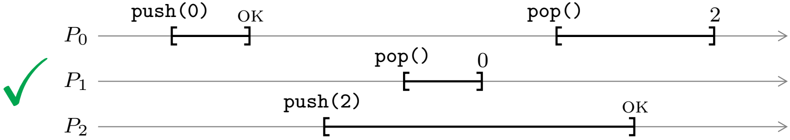

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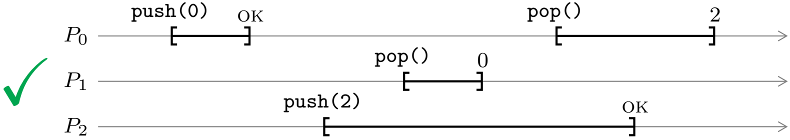
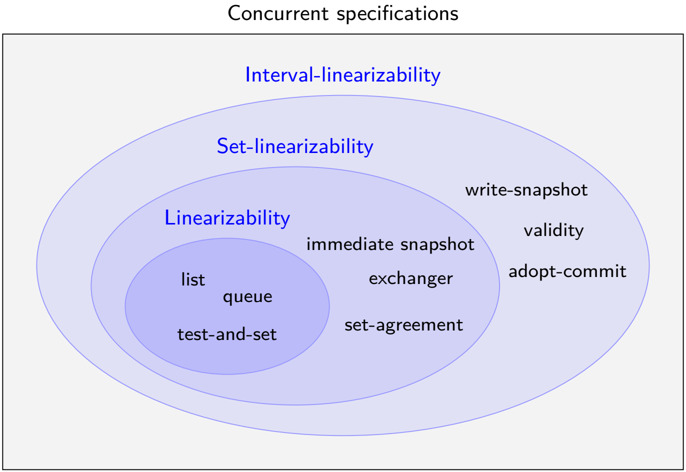


Write $\mathcal{T}$ for the set of all execution traces.

Definition

A *concurrent specification* is a subset $\sigma \subseteq \mathcal{T}$.

Concurrent specifications (2)



Concurrent Specifications Beyond Linearizability. Goubault, L., Mimram (OPODIS'18)

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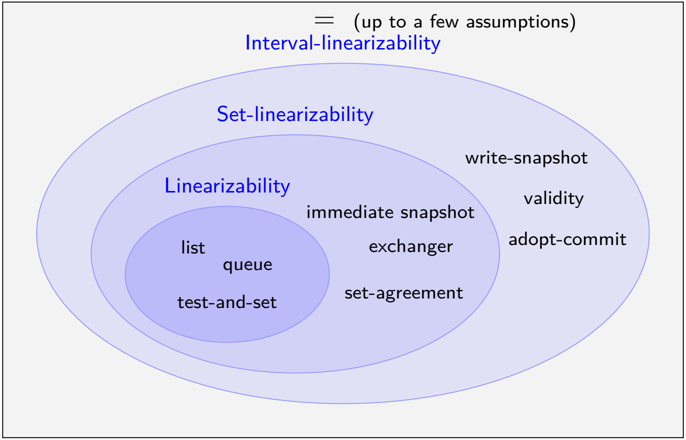

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Tasks vs Objects

Recall that a task for n processes is a relation $\Theta \subseteq \mathsf{Val}^n \times \mathsf{Val}^n$.

Tasks are less expressive than objects:

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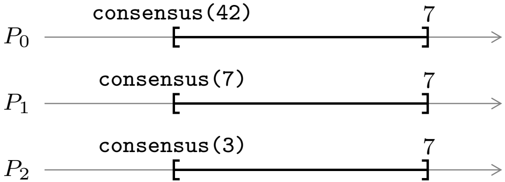
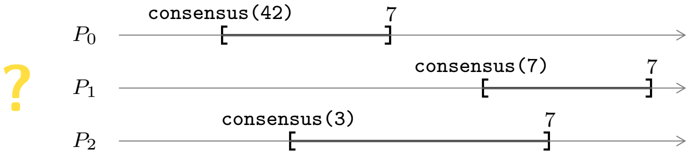


Turning a task into an object

How do we specify a consensus object?



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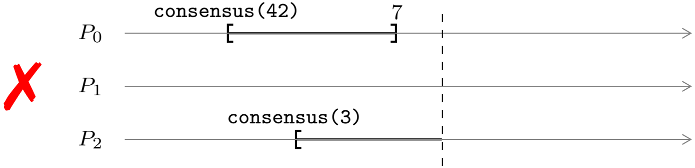

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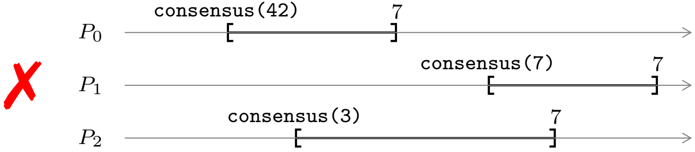

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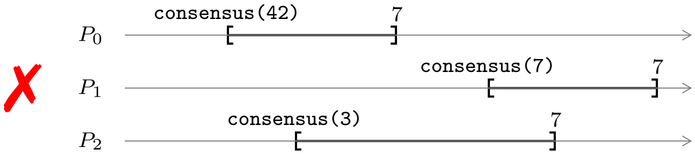

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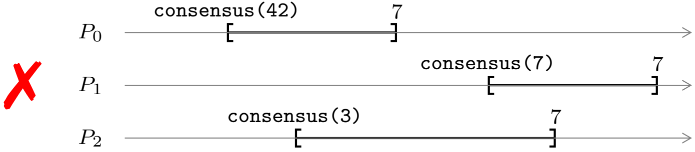

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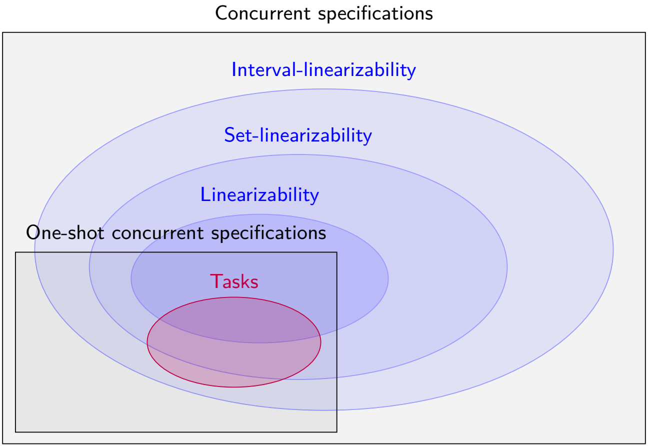
Theorem

The functions F and G form a Galois connection:

$$\sigma \subseteq G(\Theta) \iff F(\sigma) \subseteq \Theta$$

Tasks vs Objects (2)



Unifying Concurrent Objects and Distributed Tasks: Interval-Linearizability.

Castañeda, Rajsbaum, Raynal (2018).

Generalized Asynchronous Computability Theorem

Solving a task Θ simply means implementing the object $F(\Theta)$.

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Work in progress

- **Compositionality:**

  "*If A solves B and B solves C, then A solves C.*"

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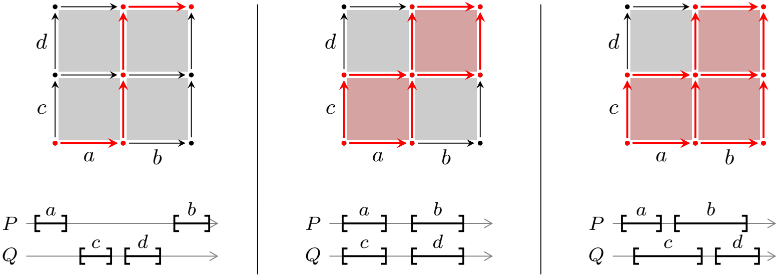
- **Directed Topology:**



Part II: Geometric Models for Epistemic Logic

Multi-agent epistemic logic

Epistemic logic is the modal logic of knowledge.

Let $\mathcal{A}$ be a finite set of *agents* and At a set of *atomic propositions*.
The syntax of formulas is:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K_a\,\varphi \qquad\qquad p \in \mathsf{At},\ a \in \mathcal{A}$$

$K_a\,\varphi$ is read *"a knows φ"*.

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Common knowledge:

$$C_B\,\varphi \;\equiv\; \bigwedge_{\substack{n\in\mathbb{N}\\ a_1,\dots,a_n\in B}} K_{a_1}\dots K_{a_n}\,\varphi$$

Kripke models

A Kripke model is a tuple $M = \langle W, \sim, L \rangle$, where:

- W is a set of *worlds*
- For every $a \in \mathcal{A}$, $\sim_a \subseteq W \times W$ is an equivalence relation on W
- $L : W \to \mathscr{P}(\mathsf{At})$

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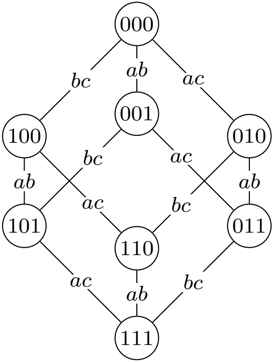
A Kripke model is a tuple $M = \langle W, \sim, L \rangle$, where:

- W is a set of *worlds*
- For every $a \in \mathcal{A}$, $\sim_a \subseteq W \times W$ is an equivalence relation on W
- $L : W \to \mathscr{P}(\mathsf{At})$

Example: three agents with binary inputs.

- a, b, c are agents.
- $w \sim_a w'$ is represented as an a-labeled edge between w and w'.
- 101: input values of a, b, c, in that order.

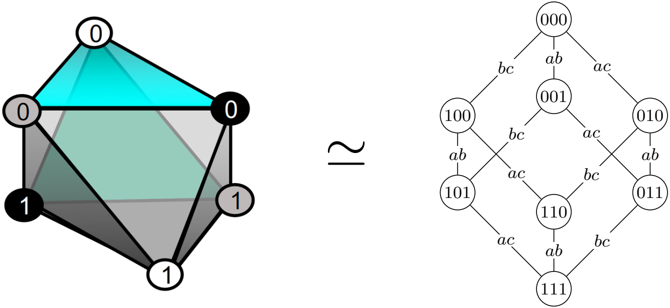


Semantics of epistemic logic formulas

Let $M = \langle W, \sim, L \rangle$ be a Kripke model and $x \in W$ a world of M. We define the truth of a formula φ in x, written $M, x \models \varphi$, by induction on φ:

$$
\begin{array}{lll}
M, x \models p & \text{iff} & p \in L(x) \\
M, x \models \neg\varphi & \text{iff} & M, x \not\models \varphi \\
M, x \models \varphi \wedge \psi & \text{iff} & M, x \models \varphi \text{ and } M, x \models \psi \\
M, x \models K_a\,\varphi & \text{iff} & \text{for all } y \in W, x \sim_a y \text{ implies } M, y \models \varphi
\end{array}
$$

An equivalence of categories



Theorem

The category of labeled pure chromatic simplicial complexes is equivalent to the category of local proper Kripke models.

A Simplicial Complex Model for Dynamic Epistemic Logic, Goubault, L., Rajsbaum (GandALF'18)

Example: card game

Consider the following situation: *there are three agents and a deck of four cards* $\{0, 1, 2, 3\}$. *Each agent is given a card at random, and the remaining card is kept hidden.*

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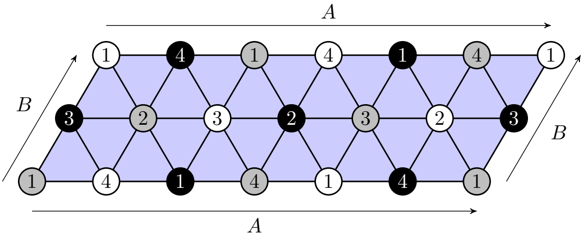

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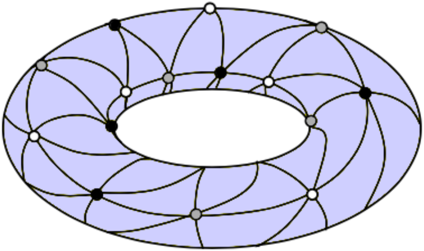

Dynamic Epistemic Logic (DEL)

Syntax:
Let $\mathcal{A}$ be a finite set of *agents* and At a set of *atomic propositions*.
The syntax of formulas is:

$$\begin{array}{lcl} \varphi & ::= & p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K_a\,\varphi \mid [\alpha]\,\varphi \\ \alpha & ::= & \text{“action”} \end{array}$$

$[\alpha]\,\varphi$ intuitively means *“φ will be true after the action α occurs”*.

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Actions in distributed computing

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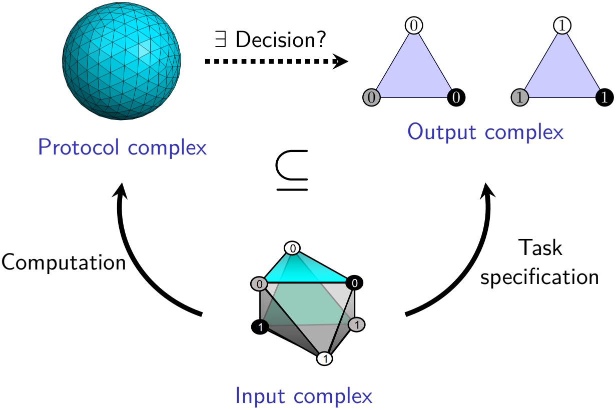

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$$
\begin{tikzcd}
\begin{array}{c}\mathcal{M}[\mathcal{P}] \\ \text{Protocol model}\end{array} \arrow[rr, dashed, "\exists\ \text{Decision?}"] \arrow[dr, bend right, "\pi_M"'] & & \begin{array}{c}\mathcal{M}[\mathcal{T}] \\ \text{Output model}\end{array} \arrow[dl, bend left, "\pi_M"] \\
& \begin{array}{c}\mathcal{M} \\ \text{Input model}\end{array} &
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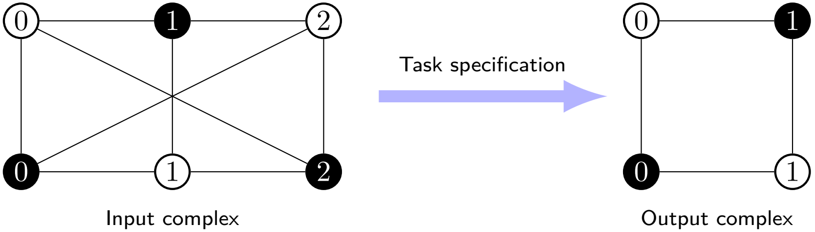
Case study: the Equality Negation task

(*Nondeterministic wait-free hierarchies are not robust*, Lo and Hadzilacos, 2000)

- Two processes P, Q (represented in black and white).
- Three possible inputs values $i_P, i_Q \in \{0, 1, 2\}$.
- Binary decision values $d_P, d_Q \in \{0, 1\}$.
- **Goal:** $i_P = i_Q \iff d_P \neq d_Q$.



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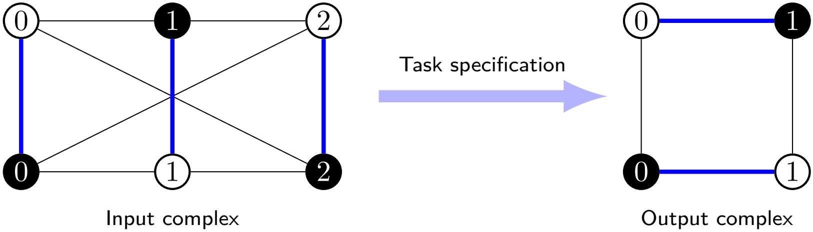

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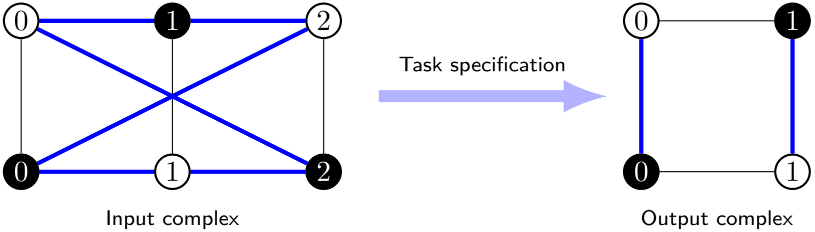

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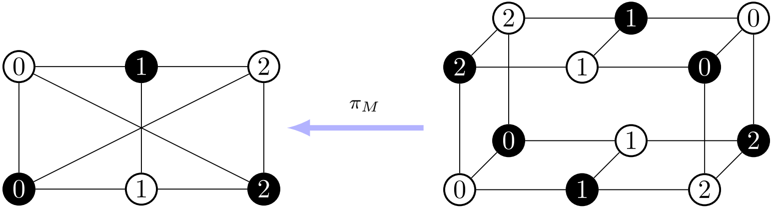
Using DEL, the task is modeled as follows:



Input model M

Output model $M[T]$

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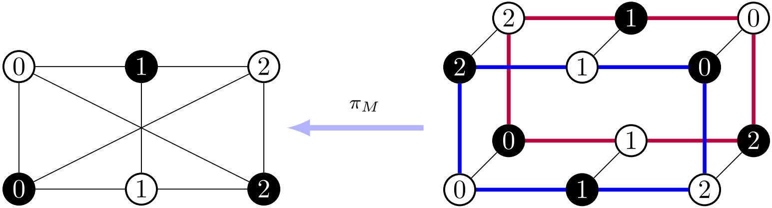

Input model M Output model $M[T]$

In blue: processes decide 0
In red: processes decide 1

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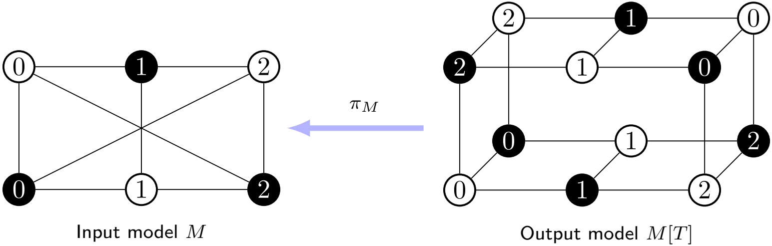

Intuitively, $M[T]$ describes the knowledge that the processes should acquire in order to solve the task.

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- *A Dynamic Epistemic Logic Analysis of the Equality Negation Task*, Goubault, Lazić, L., Rajsbaum (DaLi'19).
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- *Wait-free solvability of Equality Negation Tasks*, Goubault, Lazić, L., Rajsbaum (DISC'19).
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Thanks!