Higher Inductive Types as Homotopy-Initial Algebras

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Introduction

In Extensional Type Theory we have a well-known correspondence (Dybjer 1996) between

1. Inductive types: finite types 0, 1, 2, . . ., natural numbers \( \mathbb{N} \), lists \( \text{List}[A] \), well-founded trees \( W_{x:A}B(x) \), etc.

2. Initial algebras of a certain form

\( (\mathbb{N}, 0, \text{suc}) \) is initial among algebras of the form \( (C, z, c) \), where \( z : C \) and \( c : C \rightarrow C \).

*Initial:* there is a unique function \( h : \mathbb{N} \rightarrow C \) which preserves the constructors (a *homomorphism*).
Introduction

In Intensional Type Theory this correspondence breaks down: we cannot prove (definitional) uniqueness.

In Homotopy Type Theory, we can prove *propositional* uniqueness, and more: we have a correspondence (Awodey et al, 2012) between

1. Inductive types: 0, 1, 2, $\mathbb{N}$, List[$A$], $W_{x:A}B(x)$, etc. with *propositional computation rules*

2. *Homotopy-initial* algebras of a certain form

($\mathbb{N}, 0, \text{suc}$) is homotopy-initial among algebras of the form ($C, z, c$).

*Homotopy-initial*: the type of homomorphisms from ($\mathbb{N}, 0, \text{suc}$) to any other algebra ($C, z, c$) is contractible.
A powerful tool in HoTT are *Higher-Inductive Types* (HITs):

1. HITs extend ordinary inductive types by allowing constructors involving *path spaces* of $X$ (e.g., $c : a \equiv_X b$) rather than just points of $X$ (e.g., $c : X$).

E.g., the circle $S^1$ is a HIT generated by four constructors:

$$
\begin{align*}
\text{north} & : S^1 \\
\text{south} & : S^1 \\
\text{east} & : \text{north} \equiv_{S^1} \text{south} \\
\text{west} & : \text{north} \equiv_{S^1} \text{south}
\end{align*}
$$
2. Many interesting constructions arise as HITs: spheres $S^n$, interval, torus $T$, quotients, pushouts, suspensions, integers $\mathbb{Z}$, truncations $\|A\|$ (aka squash types), . . .

3. **Open question:** Which computation rules should be propositional vs. definitional? *Here we assume the former.*

3. **Open problem:** finding a unifying schema for HITs (not a subject of this talk).

The subject of this talk: *Can a manageable class of HITs be characterized by a universal property - as homotopy-initial algebras?*
Higher Inductive Types

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3. **Open question**: Which computation rules should be propositional vs. definitional? *Here we assume the former.*

3. **Open problem**: finding a unifying schema for HITs (**not** a subject of this talk).

The subject of this talk: *Can a manageable class of HITs be characterized by a universal property - as homotopy-initial algebras? Yes!*
W-suspensions

Martin-Löf’s well-founded trees $W_{x:A}B(x)$: nontrivial induction on point constructors; no higher-dimensional constructors.
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+ “Generalized suspensions”: vacuous induction on point constructors; arbitrary number of path constructors between any two point constructors.

*Induction and higher-dimensionality remain orthogonal, which gives W-suspensions a well-behaved elimination principle.*
W-suspensions: point constructors

The W-suspension type $W$ is a HIT generated by

$$\text{point} : \Pi_a : A (B(a) \to W) \to W$$

$$\text{path} : \ldots$$

where, just like for well-founded trees,

- $A$ is the type of point constructors
- $B : A \to \text{type}$ gives the arity of each point constructor
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Example: The type $\mathbb{N}$ has two point constructors: one for zero and one for successor. Thus, $\mathbb{N}$ is a $W$-suspension with $A := 2$ and $B$ given by $\top \mapsto 0, \bot \mapsto 1$. 
The $W$-suspension type $W$ is a HIT generated by

\[
\begin{align*}
\text{point} : & \Pi_{a:A} (B(a) \to W) \to W \\
\text{path} : & \Pi_{c:C} \Pi_{b_F:B(F(c))} W \Pi_{b_G:B(G(c))} W \\
\text{point}(F(c), b_F) = & W \text{ point}(G(c), b_G)
\end{align*}
\]

where

- $C$ is the type of path constructors
- $F : C \to A$ and $G : C \to A$ give the left and right endpoints of each path constructor
Example: the circle $S^1$ as a W-suspension

Revisiting the circle:

![Diagram of a circle with points labeled north, east, south, and west]
Example: the circle $S^1$ as a $W$-suspension

Revisiting the circle:

we see that $S^1$ is a $W$-suspension with

- $A := 2$
- $B$ is given by $\top, \bot \mapsto 0$
- $C := 2$
- $F$ is given by $\top, \bot \mapsto \text{north}$
- $G$ is given by $\top, \bot \mapsto \text{south}$
Main Theorem

Theorem

In HoTT, the existence of $W$-suspensions is equivalent to the existence of a suitable algebra $(W, \text{point}, \text{path})$ which is homotopy-initial.

Corollary

In HoTT, the existence of the circle $S^1$ is equivalent to the existence of a suitable algebra $(S^1, \text{north}, \text{south}, \text{east}, \text{west})$ which is homotopy-initial.

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In HoTT, the existence of the natural numbers $\mathbb{N}$ is equivalent ... and so on
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... and so on
Proof Idea

We show that for any algebra \((W, \text{point}, \text{path})\), the induction principle is equivalent to the simpler \textit{recursion principle} plus a \textit{uniqueness condition}, which are in turn equivalent to homotopy-initiality:

\[
\text{Induction} = \text{Recursion} + \text{Uniqueness} = \text{Homotopy-Initiality}
\]

\textit{Recursion principle}: for any algebra \((C, p, r)\), we have a homomorphism from \((W, \text{point}, \text{path})\) to \((C, p, r)\).

\textit{Uniqueness condition}: any two homomorphisms from \((W, \text{point}, \text{path})\) to \((C, p, r)\) are propositionally equal.
Proof Idea

For the circle $S^1$:

**Definition**

A homomorphism from $(C, n_C, s_C, e_C, w_C)$ to $(D, n_D, s_D, e_D, w_D)$ is a map $f : C \rightarrow D$ together with paths

$$\alpha : f(n_C) = n_D$$

$$\beta : f(s_C) = s_D$$

and higher paths $\theta, \phi$:

\[
\begin{array}{c}
\begin{array}{ccc}
& f(e_C) & \\
\alpha & \theta & \beta \\
\hline
n_D & e_C & s_D \\
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{ccc}
& f(w_C) & \\
\alpha & \phi & \beta \\
\hline
n_D & w_C & s_D \\
\end{array}
\end{array}
\]
Proof Idea

The uniqueness condition for $S^1$ thus says that any two homomorphisms $(f, \alpha_f, \beta_f, \theta_f, \phi_f)$ and $(g, \alpha_g, \beta_g, \theta_g, \phi_g)$ from $(S^1, \text{north}, \text{south}, \text{east}, \text{west})$ to $(C, n_C, s_C, e_C, w_C)$ are equal.

This is the same as saying that

1. There is a path $p : f = g$ (a propositional $\eta$-rule).
2. The (higher) paths $\alpha_f, \beta_f, \theta_f, \phi_f$ and $\alpha_g, \beta_g, \theta_g, \phi_g$ are suitably related over $p$. 

Conclusion

We have

- Introduced a class of higher inductive types, which is relatively simple and subsumes types like
  - well-founded trees $W_{x:A} B(x)$, hence the types of natural numbers $\mathbb{N}$, lists $\text{List}[A]$, ...
  - the interval $I$
  - all the spheres $S^n$
  - ordinary suspensions $\text{susp}(A)$

with propositional computational rules.
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  - the interval \( I \)
  - all the spheres \( S^n \)
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with propositional computational rules.

- Shown that this class can be characterized as a homotopy-initial algebra of a certain form; *thus equating the proof-theoretic concept of a higher-inductive type with a particular universal property.*
Conclusion

Open questions:
  ▶ What other HITs arise naturally as $W$-suspensions?
  ▶ Does homotopy-initiality scale to other HITs such as set and groupoid quotients, higher-level truncations, the torus, .... ?

References:
  ▶ P. Dybjer, Representing Inductively Defined Sets by Well-orderings in Martin-Löf’s Type Theory, 1996.