Incompletness of classes of models for the λ-calculus

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Introduction

This report presents the results of my internship at the PPS lab, under the supervision of Thomas Ehrhard. The subject of this internship was motivated by the temporary presence of Antonino Salibra, who worked on the incompleteness of some classes of λ-models.

The purpose was to study and compare two kinds of λ-models: algebraic models, and more specifically lambda abstraction algebras, and categorical models. The first ones have a simple structure and are easier to study, and for that reason there are many known results about them. But the drawback of this simplicity is that actual examples of algebraic models are not easy to find. On the other hand, categorical models can be built quite easily, but these models are complex and cannot be studied in a global way. Works on categorical models are most often the study of specific constructions.

In the first part of this report we define the objects we will study. We explain what is the λ-calculus, what are a theory and a model of a theory. Then we eplicit the constructions of lambda abstraction algebra and categorical models.

In a second part we will present the results obtained by Antonino Salibra about algebraic models, before trying to apply the to the categorical approach.

Part I
Basic notions

1 Models of the λ-calculus

1.1 λ-terms and λ-calculus

The λ-calculus is perfomed on particular terms, the *λ-terms*. The set of λ-terms Λ is defined inductively, given an infinite set of variables Var:

$$t, u = x \in \mathrm{Var} \mid \lambda x.t \mid tu$$

A λ-term is to be thought as a function: the term tu is the *application* of the function t to the argument u, and the *abstraction* $\lambda x.t$ is the function which to x associates t. This is formalized by the core rule of the λ-calculus, the β-rule:

$$(\lambda x.t)u =_\beta t[x := u]$$

where $t[x := u]$ is the term t where the occurences of x are replaced by the term u. For instance $(xyz)[y := t] = xtz$. This is called a *substitution*.

The formal definition of the substitution must be made carefully: we cannot accept for instance that $(\lambda x.\lambda y.xy)y =_\beta \lambda y.yy$. This because in a λ-term, every occurence of a variable is either related to a specific abstraction symbol, or to none: in the term $(\lambda x.(\lambda x.x)x)x$, the first occurence of x refers to the second abstraction, its second occurence refers to the first abstraction, and its last occurence is not related to any abstraction. The first two occurences are said to be *bound* while the last one is *free*. And when we perform a substitution, we must not change the signification of the variables. In the example above, the last y which is free in the right term is bound after the substitution, and for this reason the equality does not hold.

Intuitively, the name of the variable used to define a function does not matter: the functions $\lambda x.zxz$ and $\lambda y.zyz$ are the same. On the other hand, one cannot change the name of a free variable. So we introduce a new rule, the α-rule:

$$\lambda x.t =_\alpha \lambda y.t[x := y]$$

where the susbtitution $[x := y]$ is easier to define than in the general case $[x := u]$.

Then to use the β-rule on a term $(\lambda x.t)u$, we can change the names in t of the variables both used in abstractions in t and free in u, then perform the substitution.

The α- and β-rules form the core of the λ-calculus, but we are interested in what other equalities we can add to this calculus and their consequences. This is what the study of λ-theories is about.

1.2 Theories in the λ-calculus

Definition 1.1. A λ-*theory* is a set of equations $\mathcal{T}$ between λ-terms that defines an equality, meaning such that for all $t, u, v, w \in \Lambda$:

- $t =_\mathcal{T} t$
- $t =_\mathcal{T} u \Rightarrow u =_\mathcal{T} t$
- $t =_\mathcal{T} u \wedge u =_\mathcal{T} v \Rightarrow t =_\mathcal{T} v$
- $t =_\mathcal{T} u \Rightarrow \lambda x.t =_\mathcal{T} \lambda x.u$ for all $x \in \mathrm{Var}$
- $t =_\mathcal{T} u \wedge v =_\mathcal{T} w \Rightarrow tu =_\mathcal{T} vw$

where $t =_\mathcal{T} u$ means $t = u \in \mathcal{T}$. To simplify the description of a theory, for any set $\mathcal{T}$ of equations, we will call "the theory $\mathcal{T}$" the least set that contains $\mathcal{T}$ and verify the above conditions.

To be a λ-theory, $\mathcal{T}$ must also respect the α and β rules: if from these rules we have $t =_{\alpha\beta} u$ then $t =_\mathcal{T} u$.

The least λ-theory is then given by only the α and β rules. We note $\lambda\beta$ this theory.

The theory that equates every pair of terms is called the *inconsistent theory*, and noted ∇.

Working on λ-theories is difficult, and we lack tools to study the structure of the class of all λ-theories. For this reason we consider well-known objects and transpose the semantic of the *lambda*-calculus to them: this is what we call a model.

1.3 Models of a theory

Definition 1.2. A *model* $\mathcal{M}$ of a theory $\mathcal{T}$ is a set of elements M given with a function $[.]_{\mathcal{M}} : \Lambda \to M$ such that if $t =_{\mathcal{T}} u$ then $[t]_{\mathcal{M}} = [u]_{\mathcal{M}}$. We call *model of the λ-calculus* a model of $\lambda\beta$.

Conversely, for any model $\mathcal{M}$ of the λ-calculus, we call the theory of $\mathcal{M}$ the theory $Th(\mathcal{M}) = \{t =_{\mathcal{T}} u \mid [t]_{\mathcal{M}} = [u]_{\mathcal{M}}\}$.

Our goal is to build a particular class of model that is both simple to study and complete, meaning that for every $\mathcal{T} \neq \nabla$ there is a model $\mathcal{M}$ in this class such that $\mathcal{T} = Th(\mathcal{M})$. It would then have a structure similar to the one of the class of λ-theories and provide us with tools to study it.

Many models are build using either universal algebras or categories. We will see how these particular models work.

2 Algebraic models

2.1 Universal algebras

A *universal* algebra is defined over a type Σ, also called a *signature*. A type is a finite set of symbols, each of them being given an arity.

Definition 2.1. An algebra $\mathcal{A}$ of type Σ is a couple (A, F) where A is a set of elements and F is a set of functions such that there is a function $.^{\mathcal{A}} : \Sigma \to F$ and for every $f \in \Sigma$ of arity n, $f^{\mathcal{A}}$ is a function $A^n \to A$.

If the arity of $f \in \Sigma$ is 0 then $f^{\mathcal{A}}$ is a function $A^0 \to A$: we rather say that $f^{\mathcal{A}}$ is an element of A.

We write $\mathcal{A} = (A, f_1, ..., f_n)$ when $\mathcal{A}$ is an algebra of type $\{f_1, ..., f_n\}$. This notation does not explicit the arity of the functions.

2.2 Lambda Abstraction Algebras

Definition 2.2. A *Lambda Abstraction Algebra (LAA)* is an algebra $(A, \cdot, \{\lambda_x \mid x \in \text{Var}\}, \{x \mid x \in \text{Var}\})$ for a set of variables Var, where the arity of $\cdot$ is 2 and for all $x \in \mathrm{Var}$, the arity of λ_x is 1 and the arity of x is 0. Besides it must satisfy for any $a, b, c \in A$, $x, y, z \in \mathrm{Var}$:

- $(\lambda_x.x)a = a$
- $(\lambda_x.y)a = y$ if $y \neq x$
- $(\lambda_x.a)x = a$
- $(\lambda_x.(\lambda_x.a))b = \lambda_x.a$
- $(\lambda_x.ab)c = ((\lambda_x.a)c)((\lambda_x.b)c)$
- $(\lambda_y.b)z = b \Rightarrow (\lambda_x.(\lambda_y.a))b = \lambda_y.((\lambda_x.a)b)$ $(y \neq z, x \neq y)$
- $(\lambda_y.a)z = a \Rightarrow \lambda_x.a = \lambda_y.((\lambda_x.a)y)$

The first five rules explicit how the substitution works. The last one is the α-conversion. Note that in a LAA, the elements are not terms, so we cannot define inductively the set of free variables like we do in Λ. That is why we say here that a variable y is free in $a \in A$ if there exists $z \neq y$ such that $(\lambda_y.a)z = a$.

Note that we use here two different kinds of variables. a, b, c represent any element in the algebra $\mathcal{A}$. On the other hand, as the type of $\mathcal{A}$ is $\cup_{x \in \mathrm{Var}}\{\lambda_x, x\}$, what we note x is actually $x^{\mathcal{A}}$ the particular element in $\mathcal{A}$ associated with the constant symbol x.

We have then an immediate way to represent λ-terms:

- $[x] = x$
- $[\lambda x.t] = \lambda_x([t])$
- $[tu] = [t] \cdot [u]$

As said before, we deal here only with LAAs. However the following constructions exists for all algebras, not just LAAs, and are useful to study any algebraic model of rhe λ-calculus.

2.3 Product of algebras

Definition 2.3. Given a type $\Sigma = \{f_1, ..., f_n\}$ and two algebras $\mathcal{A}, \mathcal{B}$ of type Σ, the product $\mathcal{A} \times \mathcal{B}$ is $(A \times B, f_1^{\mathcal{A}\times\mathcal{B}}, ..., f_n^{\mathcal{A}\times\mathcal{B}}\}$ with $f^{\mathcal{A}\times\mathcal{B}}((a_1, b_1), ..., (a_p, b_p)) = (f^{\mathcal{A}}(a_1, ..., a_p), f^{\mathcal{B}}(b_1, ..., b_p))$.

The product of two LAAs $\mathcal{A} \times \mathcal{B}$ is clearly still a LAA, and $Th(\mathcal{A} \times \mathcal{B}) = Th(\mathcal{A}) \cap Th(\mathcal{B})$.

2.4 Congruences

Definition 2.4. An *equivalence relation* on a set X is a subset R of $X \times X$ such that for all $x, y, z \in X$:

- $(x, x) \in R$
- if $(x, y) \in R$ then $(y, x) \in R$
- if $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$

Definition 2.5. A *congruence* θ on an algebra (A, Σ) is an equivalence relation on A such that for all $f \in \Sigma$, if $\forall i \leq p, (a_i, b_i) \in\theta$ then $(f(a_1, ..., a_p), f_i(b_1, ..., b_p)) \in\theta$.

We note $\nabla = A \times A$ and $\Delta = \{(a, a) \mid a \in A\}$.

Given a congruence θ, $\mathcal{A}/\theta$ is the algebra defined by $A/\theta = \{cl(a) \mid a \in A\}$, with $cl(a) = \{b \in A \mid (a, b) \in\theta\}$, and $f^{\mathcal{A}/\theta}(cl(a)) = cl(f^{\mathcal{A}}(a))$.

We then see that $\lambda\beta$ defines a congruence on the LAA $(\Lambda, \cdot, \lambda_x, x)$ and that every theory is a congruence on $\Lambda/\lambda\beta$.

3 Categorical models

3.1 Categories and CCC

Definition 3.1. A *category* C is defined by:

- a class of *objects* $Obj(C)$
- for every $X, Y \in Obj(C)$ a class $C(X, Y)$, whose elements are called *maps*
- for every $X \in Obj(C)$ a particular map $\mathrm{Id}_X \in C(X, X)$, the *identity*
- for every $X, Y, Z \in Obj(C)$, a *composition operator* $\circ : C(X, Y) \times C(Y, Z) \to C(X, Z)$

such that for every $X, Y, Z, T \in Obj(C)$, $f \in C(X, Y)$, $g \in C(Y, Z)$, $h \in C(Z, T)$,

$$f \circ \mathrm{Id}_X = \mathrm{Id}_Y \circ f = f \qquad (h \circ g) \circ f = h \circ (g \circ f)$$

The idea behind that definition is that the objects play the role of sets and the elements of $C(X, Y)$ behave like actual maps between X and Y. For this reason we note $f : X \to Y$ when $f \in C(X, Y)$. However the comparison is not always accurate, and can sometimes be misleading.

Definition 3.2. A category C is *cartesian* if there is:

- a *terminal object*, i.e. an object T such that for every $X \in Obj(C)$, $\#C(X,T) = 1$
- an operation $\& : Obj(C) \times Obj(C) \to Obj(C)$
- an operation $\langle .,. \rangle : C(Y,X_1) \times C(Y,X_2) \to C(Y,X_1\&X_2)$
- particular maps $\pi_i : X_1\&X_2 \to X_i$

such that

$$\pi_i \circ \langle f_1, f_2 \rangle = f_i \qquad \langle f_1, f_2 \rangle \circ g = \langle f_1 \circ g, f_2 \circ g \rangle \qquad \langle \pi_1, \pi_2 \rangle = \mathrm{Id}_{X_1\&X_2}$$

Definition 3.3. A cartesian category C is a *cartesian closed category (CCC)* if for every $X, Y \in Obj(C)$ there is for all $X, Y \in Obj(C)$:

- a map object $X \Rightarrow Y$
- an evaluation map $\mathrm{Ev}_{X,Y} : (X \Rightarrow Y)\&X \to Y$
- a function $\Lambda : C(Z\&X, Y) \to C(Z, X \Rightarrow Y)$ for all $Z \in Obj(C)$

such that forall $f : Z\&X \to Y$, $g : Z' \to Z$

$$\mathrm{Ev} \circ (\Lambda(f)\&\mathrm{Id}) = f \qquad \Lambda(f) \circ g = \Lambda(f \circ (g\&\mathrm{Id}_X)) \qquad \Lambda(\mathrm{Ev}) = \mathrm{Id}_{X\Rightarrow Y}$$

where for $f_1 : X_1 \to Y_1$, $f_2 : X_2 \to Y_2$, we define $f_1\&f_2 : X_1\&X_2 \to Y_1\&Y_2$ as $\langle f_1 \circ \pi_1, f_2 \circ \pi_2 \rangle$.

Intuitively, the elements of the map object $X \Rightarrow Y$ represent the maps of $C(X,Y)$, and the evaluation applies a map $X \to Y$ to an element of X. Once again, this intuition does not give an accurate understanding of the situation, and we will see later an exemple where the objects of C are sets but there is no bijection between $X \Rightarrow Y$ and $C(X,Y)$.

For any object X and any set I, the object X^I and the projections $\pi_i : X^I \to X$ for $i \in I$ are intuitively defined. Note that while the finite product $X^{\{i_1,\ldots,i_p\}} = X^p = ((X\&X)\&...)\&X$ is always defined in a cartesian category, X^I does not always exists for I infinite.

Remember that even though we do not use the symbol "&", when we perform the product of objects, this is always the product of the category. When our objects are sets, X^p is to be understood as $((X\&X)\&...)\&X$ and not $((X \times X) \times ...) \times X$.

A cartesian category is not a category to which we add some structure: being cartesian is universal property, meaning there is a unique way to define the product. More precisely, if there are two different products in a cartesian category, then they are isomorphic, and the corresponding isomorphism is unique.

The same holds for cartesian closed categories.

3.2 Reflexive object and model of λ-calculus

In the λ-calculus, the λ-terms represent functions $\Lambda/\lambda\beta \to \Lambda/\lambda\beta$, so in a CCC, we can build a model of the λ-calculus if we find an object U that is somehow linked to $U \Rightarrow U$.

Definition 3.4. A *reflexive objet* in a CCC C is a triple $(U, \text{app}, \text{lam})$ with $U \in Obj(C)$, $\text{app} : U \to U \Rightarrow U$ and $\text{lam} : U \Rightarrow U \to U$ such that $\text{app} \circ \text{lam} = \text{Id}_{U \Rightarrow U}$.

Intuitively, what we want to do is, given a reflexive object U, interpret U as the set of all closed λ-terms, and for every finite subset of Var $\Gamma = \{x_1, ..., x_n\}$ interpret any terms t whose free variables are all in Γ as the map $U^\Gamma \to U$ such that $[t]u_1...u_n = t[x_1 := u_1, ..., x_n := u_n]$.

Here we build our model assuming we can perform a countable cartesian product, so the object $\&_{\text{Var}}U$ is defined. Then we define $M = C(U^{\text{Var}}, U)$ and:

- $[x] = \pi_x$
- $[\lambda x.t] = \text{lam} \circ \Lambda([t] \circ \eta_x)$
- $[tu] = \text{Ev} \circ \langle \text{app} \circ [t], [u] \rangle$

where $\eta_x : (U^{\text{Var}})\&U \to U^{\text{Var}}$ is defined by $\pi_y \circ \eta_x = \pi_2$ if $y = x$, $\pi_y \circ \pi_1$ otherwise.

Any categorical model can be seen as a LAA. We just extend the definition above to any map $f, g \in C(U^{\text{Var}}, U)$:

- $x = \pi_x$
- $\lambda_x.f = \text{lam} \circ \Lambda(f \circ \eta_x)$
- $g \cdot f = \text{Ev} \circ \langle \text{app} \circ g, f \rangle$

Part II
Study of classes of models

4 Completeness criterion for algebraic models

Antonino Salibra has shown that a complete class of algebraic models must contain decomposable models.

4.1 Factor congruences

The decomposability of an algebra into a product is equivalent to the existence of particular congruences, that correspond to the equalities on each member of the product.

Definition 4.1. A pair of *complementary factor congruences* is a pair $(\theta, \overline{\theta})$ such that

$$\theta \cap \overline{\theta} = \Delta \qquad \overline{\theta} \circ \theta = \nabla$$

where $\overline{\theta} \circ \theta = \{(a,b) \mid \exists c \in A : (a,c) \in \theta \wedge (c,b) \in \overline{\theta}\}$

Lemma 4.1. *If θ and $\overline{\theta}$ are complementary factor congruences, then for all $a, b \in A$ there is a unique $c \in A$ such that $a\ \theta\ c\ \overline{\theta}\ b$.*

Proof. Since $\overline{\theta} \circ \theta = \nabla$, for all $a, b \in A$ there is such a c.

For $c, c' \in A$, if $a\ \theta\ c\ \overline{\theta}\ b$ and $a\ \theta\ c'\ \overline{\theta}\ b$ then $c\ \theta\ a\ \theta\ c'$ and $c\ \overline{\theta}\ b\ \overline{\theta}\ c'$, so $c(\theta \cap \overline{\theta})c'$, $c = c'$.

Theorem 4.2. *If θ and $\overline{\theta}$ are complementary factor congruences, then $\mathcal{A} \simeq \mathcal{A}/\theta \times \mathcal{A}/\overline{\theta}$.*

Proof. Let $\varphi : a \mapsto (cl_\theta(a), cl_{\overline{\theta}}(a))$, from the definition of congruences φ is a morphism $A \to A/\ \theta \times A/\ \overline{\theta}$. For any $a, b \in A$, let c such that $a\ \theta\ c\ \overline{\theta}\ b$, then c does not depoend on the choice of a and b in $cl_\theta(a)$ and $cl_{\overline{\theta}}(b)$, and c is the only element of A such that $\varphi(c) = (cl_\theta(a), cl_{\overline{\theta}}(b))$.

Conversely for any product algebra $\mathcal{A} \times \mathcal{B}$, the congruences $\theta = \{((a, b_1), (a, b_2)) \mid a \in \mathcal{A}, b_1, b_2 \in \mathcal{B}\}$ and $\overline{\theta} = \{((a_1, b), (a_2, b)) \mid a_1, a_2 \in \mathcal{A}, b \in \mathcal{B}\}$ are factor congruences.

4.2 Central elements in Church algebras

Factor congruences are easier to study in algebras where there exists an "if then else" structure, i.e. a term function q and two terms 1 and 0 such that

$$q(1, a, b) = a \qquad q(0, a, b) = b$$

Such algebras are called *Church algebras*.

Every LAA is a Church algebra. Indeed we can define $q(a,b,c) = (a \cdot b) \cdot c$, $1 = \lambda_{xy}.x$ and $0 = \lambda_{xy}.y$. Thus we can restrict our study of factor congruences to the case of Church algebras.

Then a pair of factor congruences $\theta, \overline{\theta}$ is fully defined by the element $e \in A$ such that $1\ \theta\ e\ \overline{\theta}\ 0$. Indeed we have $a = q(1,a,b)\ \theta\ q(e,a,b)\ \overline{\theta}\ q(0,a,b) = b$, so by lemma 5.1 $\theta = \{(a,b) \mid q(e,a,b) = b\}$ and $\overline{\theta} = \{(a,b) \mid q(e,a,b) = a\}$.

Definition 4.2. A *central element* is an element $e \in A$ such that for all $a,b,c \in A$:

- $q(e,a,a) = a$
- $q(e,q(e,a,b),c) = q(e,a,c) = q(e,a,q(e,b,c))$
- $q(e, f^{\mathcal{A}}(a_1,...,a_p), f^{\mathcal{A}}(b_1,...,b_p)) = f^{\mathcal{A}}(q(e,a_1,b_1),...,q(e,a_p,b_p))$ for all $f \in \Sigma$
- $q(e,1,0) = e$

Theorem 4.3. *The function $e \mapsto (\{(a,b) \mid q(e,a,b) = b\}, \{(a,b) \mid q(e,a,b) = a\})$ is a bijection from the set of central elements of $\mathcal{A}$ to the set of pairs of factor congruences of $\mathcal{A}$.*

Proof. For any central element e, let $\theta_e = \{(a,b) \mid q(e,a,b) = b\}$, $\overline{\theta}_e = \{(a,b) \mid q(e,a,b) = a\}$. We show that θ_e is a congruence, the same proof holding for $\overline{\theta}_e$. For $a,b,c \in A$:

- $q(e,a,a) = a$ so $a\ \theta_e\ a$
- if $q(e,a,b) = b$ then $q(e,b,a) = q(e,q(e,a,b),a) = q(e,a,a) = a$, so $b\ \theta_e\ a$
- if $q(e,a,b) = b, q(e,b,c) = c$ then $q(e,a,c) = q(e,a,q(e,b,c)) = q(e,q(e,a,b),c) = q(e,b,c) = c$, so $a\ \theta_e\ c$

θ_e is an equivalence relation, and the third property of central elements proves that it is a congruence.

Now if $a\ \theta_e\ b$ and $a\ \overline{\theta}_e\ b$ then $q(e,a,b) = a$ and $q(e,a,b) = b$, so $\theta_e \cap \overline{\theta}_e = \Delta$. And $q(e,q(e,a,b),a) = a$, $q(e,b,q(e,a,b)) = b$ so $a\ \theta_e\ q(e,a,b)\ \overline{\theta}_e\ b$, $\theta_e \circ \overline{\theta}_e = \nabla$. θ_e and $\overline{\theta}_e$ are factor congruences.

We have seen that every pair of factor congruences is of the form $(\theta_e, \overline{\theta}_e)$. If $(\theta_e, \overline{\theta}_e) = (\theta_{e'}, \overline{\theta}_{e'})$ then $1\ \theta_e\ q(e,1,0)\ \overline{\theta}_e\ 0$ and $1\ \theta_e\ q(e',1,0)\ \overline{\theta}_e\ 0$ so $e = q(e,1,0) = q(e',1,0) = e'$.

From this and theorem 5.2 we deduce that if there exists a central element different from 0 and 1, then there is a nontrivial pair of factor congruences, and $\mathcal{A} \simeq \mathcal{A}/\theta \times \mathcal{A}/\overline{\theta}$ is the product of smaller algebras.

4.3 Completeness criterion

Some theories imply the existence of nontrivial central elements. Actually the properties an element must have to be central are exactly the common propeties of 1 and 0, so such a theory is easy to build. Given a λ-term t, let $\mathcal{T}_1 = \{\lambda\beta + t = 1\}$, $\mathcal{T}_2 = \{\lambda\beta + t = 0\}$, then t is central in both $\mathcal{T}_1$ and $\mathcal{T}_2$, and t is central in the theory $\mathcal{T}_1 \cap \mathcal{T}_2$.

Now we want t to be different from 1 and 0 in $\mathcal{T}_1 \cap \mathcal{T}_2$. To ensure that we need in $\mathcal{T}_1$, $t \neq 0$ and in $\mathcal{T}_2$, $t \neq 1$. It is possible to find such a term t, the most simple being $\Omega = (\lambda x.xx)(\lambda x.xx)$. Thus:

Theorem 4.4. *In every complete class of algebraic models of the λ-calculus there exists a decomposable model.*

Antonino Salibra has then shown that the models in the Scott-continuous semantics, the stable semantics or the strongly stable semantics are all simple, i.e. they only have two congruences, Δ and ∇, and thus are undecomposable and, according to this theorem, incomplete.

5 Application to a categorical model: $\mathbf{Rel}_!$

Categorical models differ from algebraic models in that they are more explicit about the working of the operations of the λ-calculus. The drawback is that there is no notion such as the one of congruence, since the abstraction and application are complex constructions and not part of the base structure of the model, and it is much more difficult to deduce anything about a category from its behaviour as a model.

What we can easily do is, given a categorical model, building the corresponding LAA and studying it using the known algebraic results. That is what I have done with the models of a particular category, $\mathbf{Rel}_!$, to prove that the class of models built from its reflexive objects is incomplete.

5.1 The category Rel

Definition 5.1. **Rel** is the category of the sets and relations:

- its objects are all the sets
- $\mathbf{Rel}(X, Y) = \mathcal{P}(X, Y)$, the maps $X \to Y$ are the relations between X and Y
- the composition is $R \circ S = \{(\alpha, \gamma) \mid \exists\beta : (\alpha, \beta) \in S \wedge (\beta, \gamma) \in R\}$
- the identities are $\mathrm{Id}_X = \{(\alpha, \alpha) \mid \alpha \in X\}$.

We define the product of objects by $\&_{i\in I} X_i = \cup_{i\in I}(\{i\}\times X_i)$, with projections $\pi_i = \{((i,\alpha),\alpha) \mid \alpha \in X_i\}$, and the product of maps by

$$\langle R_i\rangle_{i\in I} = \{(\beta,(i,\alpha)) \mid (\beta,\alpha) \in R_i\}_{i\in I}$$

In particular $X^I = I \times X$.

Rel is fundamentally linear. To get over this linearity we introduce an exponential and extend **Rel** to a bigger category $\mathbf{Rel}_!$.

5.2 The category $\mathbf{Rel}_!$

We define the exponential of a set X as $!X = M_{\text{fin}}(X)$ the set of all finite multisets of X. A multiset of X is a function $m : X \to \mathbb{N}$, and a finite multiset is a multiset m such that the support of m $supp(m) = \{\alpha \in X \mid m(\alpha) > 0\}$ is finite.

We note $[\alpha_1,...,\alpha_p]$ the multiset $\alpha \mapsto \sum_{i\le p} \delta_{\alpha,\alpha_i}$ and $m+n$ the multiset $\alpha \mapsto m(\alpha)+n(\alpha)$.

To the exponential are associated a *promotion* and a *dereliction*:

$d_X : !X \to X = \{([\alpha],\alpha) \mid \alpha \in X\}$ $\qquad$ $p_X : !X \to !!X = \{(m,[m_1,...,m_p]) \mid m = m_1+...+m_p\}$

We also define the exponential of a relation $!R : !X \to !Y = \{([\alpha_1,...,\alpha_p],[\beta_1,...,\beta_p]) \mid \forall i \le p, (\alpha_i,\beta_i) \in R\}$.

Definition 5.2. The category $\mathbf{Rel}_!$ is defined by:

- $Obj(\mathbf{Rel}_!) = Obj(\mathbf{Rel})$
- $\mathbf{Rel}_!(X,Y) = \mathbf{Rel}(!X,Y)$
- $R \circ_! S = R \circ !S \circ p_X$
- $\mathrm{Id}^!_X = \mathrm{Id}_X \circ d_X$

We define the same product as in **Rel**, with projections $\pi_i \circ d_{X_1\&X_2}$, and $\emptyset$ is a terminal object, so $\mathbf{Rel}_!$ is cartesian.

Let

- $X \Rightarrow Y = !X \times Y$
- $\Lambda(f) = \{([\gamma_1,...,\gamma_p],([\alpha_1,...,\alpha_q],\beta)) \mid ([(1,\gamma_1),...(1,\gamma_p),(2,\alpha_1),...,(2,\alpha_q)],\beta) \in f\}$ for all $f \in \mathbf{Rel}_!(Z\&X,Y)$
- $\mathrm{Ev} = \{(([(m,\beta)],m),\beta) \mid m \in !X \wedge \beta \in Y\}$

We can easily show that the equations of a cartesian closed category are satisfied, so $\mathbf{Rel}_!$ is a CCC.

5.3 Incompleteness of $\mathbf{Rel}_!$

I have shown that for any reflexive object $(U, \mathrm{app}, \mathrm{lam})$ in $\mathbf{Rel}_!$, the corresponding model is algebraically undecomposable.

The LAA associated to a reflexive object is $\mathcal{A} = (A, \cdot, \lambda_x.x)_{x\in\mathrm{Var}}$ with:

- $A = \mathbf{Rel}_!(U^{\mathrm{Var}}, U) = M_{\mathrm{fin}}(\mathrm{Var} \times U) \times U$
- $x = \{([(x, \alpha)], \alpha) \mid \alpha \in U\}$
- $R \cdot S = \mathrm{Ev} \circ_! \langle \mathrm{app} \circ_! R, S\rangle$ for all $R, S \in A$
- $\lambda_x.R = \mathrm{lam} \circ_! \Lambda(R \circ_! \eta_x)$ for all $R \in A$.

Theorem 5.1. *$\mathcal{A}$ is undecomposable.*

The "if then else". We will show that for the usual "if then else" structure, i.e. $q(R, S, T) = R \cdot S \cdot T$, the only central elements in $\mathcal{A}$ are trivial. First we explicit the definition of $R \cdot S \cdot T$.

In order to simplify the notations, we will omit the symbol "$\cdot$".

If we explicit the definition of RS, we obtain

$$RS = \{(\rho, \alpha) \mid \exists \rho_1, ..., \rho_p, \sigma_1, ..., \sigma_q \in U^{\mathrm{Var}}, \alpha_1, ..., \alpha_p, \beta_1, ..., \beta_q \in U :$$
$$\rho = \sum_{i\le p} \rho_i + \sum_{i\le q} \sigma_i \wedge ([\alpha_1, ..., \alpha_p], ([\beta_1, ..., \beta_q], \alpha)) \in \mathrm{app}$$
$$(\forall i \le p, (\rho_i, \alpha_i) \in R) \wedge (\forall i \le q, (\sigma_i, \beta_i) \in S)\}$$

We introduce the notation $R^! = !R \circ p_X = \{(\rho_1 + ... + \rho_p, [\alpha_1, ..., \alpha_p] \mid \forall i, (\rho_i, \alpha_i) \in R\}$. Then

$$RS = \{(\rho + \sigma, \alpha) \mid \exists m, n \in !U : (\rho, m) \in R^! \wedge (\sigma, n) \in S^! \wedge (m, (n, \alpha)) \in \mathrm{app}\}$$

$$(RS)^! = \{(\sum_{i=1}^{p} \rho_i, \sum_{i=1}^{p} [\alpha_i]) \mid \exists m_1, ..., m_p, n_1, ..., _p \in !U :$$
$$\forall i \le p, (\rho_i, m_i) \in R^! \wedge (\sigma_i, n_i)\ S^! \wedge (m_i, (n_i, \alpha_i)) \in \mathrm{app}\}$$

$$= \{(\rho + \sigma, \sum_{i=1}^{p} [\alpha_i]) \mid \exists m, n \in !U : (\rho, m) \in R^! \wedge (\sigma, n) \in S^!$$
$$\exists m_1, ..., m_p, n_i, ...n_p \in !U : \sum_{i=1}^{p} m_i = m \wedge \sum_{i=1}^{p} n_i = n$$
$$\forall i \le p, (m_i, (n_i, \alpha_i)) \in \mathrm{app}\}$$

For all $n \in !U, M \in \mathcal{P}(!U)$, let

$$\varphi(n, M) = \{m \in !U \mid \exists[\alpha_1, ..., \alpha_p] \in M, m_1, ..., m_p, n_1, ..., n_p \in !U :$$
$$m = \sum m_i \wedge n = \sum n_i \wedge \forall i \leq p, (m_i, (n_i, \alpha_i)) \in \text{app}\}$$

In particular we observe that for $\alpha \in U$, $\varphi(n, \{[\alpha]\}) = \{m \mid (m, (n, \alpha)) \in \text{app}\}$. We note $\varphi(n, \alpha) = \varphi(n, \{[\alpha]\})$.

Then

$$(RS)^! = \{(\rho{+}\sigma, \sum_{i=1}^{p}[\alpha_i]) \mid \exists m, n \in !U : (\rho, m) \in R^! \wedge (\sigma, n) \in S^! \wedge m \in \varphi(n, \{\sum_{i=1}^{p}[\alpha_i]\})\}$$

and at last

$$RST = \{(\rho{+}\sigma{+}\tau, \alpha) \mid \exists m', n' \in !U : (\rho{+}\sigma, m') \in (RS)^! \wedge (\tau, n') \in T^! \wedge (m', (n', \alpha)) \in \text{app}\}$$
$$= \{(\rho{+}\sigma{+}\tau, \alpha) \mid \exists m, n, m', n' \in !U : (\rho, m) \in R^! \wedge (\sigma, n) \in S^! \wedge (\tau, n') \in T^!$$
$$m' \in \varphi(n', \alpha) \wedge m \in \varphi(n, \{m'\})\}$$
$$RST = \{(\rho + \sigma + \tau, \alpha) \mid \exists m, n, n' \in !U : (\rho, m) \in R^! \wedge (\sigma, n) \in S^! \wedge (\tau, n') \in T^!$$
$$m \in \varphi(n, \varphi(n', \alpha))\}$$

We have come to a simple expression of RST. We observe here a phenomenon due to the fact that we do not make any hypothesis on the maps app and lam of our reflexive object. Most often app and lam are assumed to be linear, that is there is a function $\tilde{\varphi} : U \Rightarrow U \to U$ such that app $= \{(\tilde{\varphi}(m, \alpha), (m, \alpha)) \mid (m, \alpha) \in U \Rightarrow U\}$ and lam $= \{([(m, \alpha)], \tilde{\varphi}(m, \alpha))\}$. Then

$$RST = \{(\rho{+}\sigma{+}\tau, \alpha) \mid \exists n, n' \in !U : (\rho, \tilde{\varphi}(n, \tilde{\varphi}(n', \alpha))) \in R \wedge (\sigma, n) \in S^! \wedge (\tau, n') \in T^!\}$$

The purpose of the notation φ is to "hide" the non-linearity of app.

Central elements. Now let E be a central element. First we show that E is similar to a projection, in that:

Lemma 5.2. *For all $R, S \in A$, $ERS \subset R \cup S$.*

We prove this lemma using only the first equality of central elements: for all $R \in A$, $ERR = R$.

Let $\rho_0 \in !U^{\text{Var}}$, $m_0, n_0, n'_0 \in !U$ and $\alpha \in U$ be such that $(\rho_0, m_0) \in E^!$ and $m_0 \in \varphi(n_0, \varphi(n'_0, \alpha))$. The reason we consider such elements is that, from the expression of the "if then else" given above, we know that for any $\rho \in !U^{\text{Var}}$, $R \in A$, if $(\rho, n_0 + n'_0) \in R^!$ then $(\rho_0 + \rho, \alpha_0) \in ERR$. Since $ERR = R$, if $(\rho, n_0 + n'_0) \in R^!$ then $(\rho_0 + \rho, \alpha_0) \in R$. Note that we do not know yet if this is an equivalence, since n_0 and n'_0 are chosen independantly of R.

If $R_1 = \{[]\} \times supp(n_0+n_0')$, of course $([], n_0+n_0') \in R_1^!$, so $(\rho_0, \alpha_0) \in R_1$. But then this means by the definition of R_1 that $\rho_0 = []$, and that $\alpha_0 \in supp(n_0+n_0')$.

Now about the cardinality of n_0+n_0', let $u \in U^{\mathrm{Var}}$ and $R_2 = \{[u]\} \times U$. Then for $\rho \in !U^{\mathrm{Var}}$, $(\rho, n_0+n_0') \in R_2^!$ if and only if $\rho = [u^{\#(n_0+n_0')}]$. But it means that $([u^{\#(n_0+n_0')}], \alpha_0) \in R_2$, so $\#(n_0+n_0') = 1$.

From these results, we deduce that $(n_0, n_0') = ([\alpha], [])$ or $(n_0, n_0') = ([], [\alpha])$.

We define two sets corresponding to these two cases:

$$E_1 = \{\alpha \in U \mid \exists m \in \varphi([\alpha], \varphi([], \alpha)) : ([], m) \in E^!\}$$

$$E_0 = \{\alpha \in U \mid \exists m \in \varphi([], \varphi([\alpha], \alpha)) : ([], m) \in E^!\}$$

Then for all $\rho \in !U^{\mathrm{Var}}$, $\alpha \in U$, $R, S \in A$,

$$\begin{aligned}
(\rho,\alpha) \in ERS &\Leftrightarrow \rho = \rho_1+\rho_2+\rho_3 \wedge \exists m, n, n' \in !U : (\rho_1, m) \in E^! \wedge m \in \varphi(n, \varphi(n', \alpha)) \\
&\qquad (\rho_2, n) \in R^! \wedge (\rho_3, n') \in S^! \\
&\Leftrightarrow (\exists m \in !U : ([], m) \in E^! \wedge (\rho, [\alpha]) \in R^! \wedge m \in \varphi([\alpha], \varphi([], \alpha))) \\
&\quad \vee (\exists m \in !U : ([], m) \in E^! \wedge (\rho, [\alpha]) \in S^! \wedge m \in \varphi([], \varphi([\alpha], \alpha))) \\
(\rho,\alpha) \in ERS &\Leftrightarrow (\alpha \in E_1 \wedge (\rho,\alpha) \in R) \vee (\alpha \in E_0 \wedge (\rho,\alpha) \in S)
\end{aligned}$$

This proves the lemma, and more specifically

$$ERS = (\{(\rho,\alpha) \mid \alpha \in E_1\} \cap R) \cup (\{(\rho,\alpha) \mid \alpha \in E_0\} \cap S)$$

The sets E_1 and E_0. To prove that E behaves like 1 or 0 we have only left to prove $(E_1, E_0) = (U, \emptyset)$ or $(E_1, E_0) = (\emptyset, U)$, from which follows $\forall R, S \in A, ERS = R$ or $\forall R, S \in A, ERS = S$.

From the expression obtained in the previous proof we see that

$$E\nabla\nabla = \{(\rho,\alpha) \mid \alpha \in E_1\} \cup \{(\rho,\alpha) \mid \alpha \in E_0\}$$

But E being central, $E\nabla\nabla = \nabla$, so for all $\alpha \in U$, $\alpha \in E_1 \vee \alpha \in E_0$

We also deduce from this expression that

$$E(E\emptyset\nabla)\emptyset = E\{(\rho,\alpha) \mid \alpha \in E_0\}\emptyset$$

$$= \{(\rho,\alpha) \mid \alpha \in E_1\} \cap \{(\rho,\alpha) \mid \alpha \in E_0\}$$

But as E is central, $E(E\emptyset R)\emptyset = E\emptyset\emptyset = \emptyset$. Thus $E_1 \cap E_0 = \emptyset$.

Lastly we prove that if there is an element $\alpha \in E_1$, then $E_1 = U$. Let $\beta \in U$, we use the fact that for all $R_1, R_2, S_1, S_2 \in A$, $E(R_1R_2)(S_1S_2) = (ER_1S_1)(ER_2S_2)$ to prove $\beta \in E_1$.

Since $\mathrm{app} \circ_! \mathrm{lam} = \mathrm{Id}_{U \Leftrightarrow U}$ there exists $m = [\gamma_1, ..., \gamma_p] \in !U$ such that $([([\beta], \alpha)], m) \in \mathrm{lam}^!$ and $(m, ([\beta], \alpha)) \in \mathrm{app}$. Let $x \in \mathrm{Var}$, we observe that:

- $([(x, \gamma_1), ..., (x, \gamma_p)], m) \in (x^{\mathcal{A}})^!$
- $([], [\beta]) \in (\{([], \beta)\})^!$

so by definition of the application, $([(x, \gamma_1), ..., (x, \gamma_p)], \alpha) \in x^{\mathcal{A}}(\{([], \beta)\})$.

Then since $\alpha \in E_1$, we have

$$\begin{aligned} ([(x, \gamma_1), ..., (x, \gamma_p)], \alpha) \in \{(\rho, \alpha) \mid \alpha \in E_1\} \cap (x^{\mathcal{A}}(\{([], \beta)\})) \\ = E(x^{\mathcal{A}}(\{([], \beta)\}))(\emptyset\emptyset) \\ = (Ex^{\mathcal{A}}\emptyset)(E(\{([], \beta)\})\emptyset) \end{aligned}$$

Observe that $E(\{([], b)\})\emptyset = \{(\rho, \alpha) \mid \alpha \in E_1\} \cap \{([], \beta)\}$ is empty if $\beta \notin E_1$. In that case, $([(x, \gamma_1), ..., (x, \gamma_p)], \alpha) \in (Ex^{\mathcal{A}}\emptyset)\emptyset$, so by definition of the application there exists $n \in !U$ such that $([(x, \gamma_1), ..., (x, \gamma_p)], n) \in (Ex^{\mathcal{A}}\emptyset)^!$ and $(n, ([], \alpha)) \in \mathrm{app}$.

Since $(Ex^{\mathcal{A}}\emptyset) \subset x^{\mathcal{A}}$, we know that $([(x, \gamma_1), ..., (x, \gamma_p)], n) \in x^{\mathcal{A}}$, and the only possibility is $n = m = [\gamma_1, ..., \gamma_p]$.

But then we would have $([([\beta], \alpha)], m) \in \mathrm{lam}^!$ and $(m, ([], \alpha)) \in \mathrm{app}$, so $([([\beta], \alpha)], ([], \alpha)) \in \mathrm{app} \circ_! \mathrm{lam}$. As $\mathrm{app} \circ_! \mathrm{lam} = \mathrm{Id}_{U \Leftrightarrow U}$, this is impossible. So $\beta \in E_1$.

If $E_1 \neq \emptyset$ then $E_1 = U$.

Conclusion. According to the last property of central elements,

$$E = E10 = \{(\rho, \alpha) \mid \alpha \in E_1 \wedge (\rho, \alpha) \in 1\} \cap \{(\rho, \alpha) \mid \alpha \in E_0 \wedge (\rho, \alpha) \in 0\}$$

Since $(E_1, E_0) = (U, \emptyset)$ or $(E_1, E_0) = (\emptyset, U)$, this expression implies $E = 1$ or $E = 0$. $\mathcal{A}$ has only two central elements, thus it is undecomposable.

Conclusion

The goal of this internship was first for me to understand how algebraic and categorical models of the λ-calculus work, as I was familiar with neither algebras nor categories. For that purpose I studied some particular categories, as well as results about the structures of different kinds of algebras besides LAA. I also saw some transformation between algebras and categories to better understand how the structures are related, and how they differ.

The second objective, finding a result similar to Salibra's about categories, has not been reached. I do not know of any structural property that could be used to prove that a class of algebraic models is incomplete, that would match the decomposability in algebraic models. Yet by trying to find one I made a deep study of the particular category $\mathbf{Rel}_!$. While this category is often used, all the models build in it were linear, so the result shown here without making any hypothesis on the linearity of the models is brand new.

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