RECOGNIZING WELL-PARENTHEZED EXPRESSIONS
IN THE STREAMING MODEL∗
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Abstract. Motivated by a concrete problem and with the goal of understanding the relationship between the complexity of streaming algorithms and the computational complexity of formal languages, we investigate the problem $\text{Dyck}(s)$ of checking matching parentheses, with $s$ different types of parentheses. We present a one-pass randomized streaming algorithm for $\text{Dyck}(2)$ with space of $O(\sqrt{n \log n})$ bits, time per letter polylog$(n)$, and one-sided error. We prove that this one-pass algorithm is optimal, up to a log$n$ factor, even when two-sided error is allowed. Surprisingly, the space requirement shrinks drastically if we have access to the input stream in reverse. We present a two-pass randomized streaming algorithm for $\text{Dyck}(2)$ with space of $O((\log n)^2)$, time polylog$(n)$ and one-sided error, where the second pass is in the reverse direction. Both algorithms can be extended to $\text{Dyck}(s)$ since this problem is reducible to $\text{Dyck}(2)$ for a suitable notion of reduction in the streaming model. Except for an extra $O(\sqrt{\log s})$ multiplicative overhead in the space required in the one-pass algorithm, the resource requirements are of the same order. For the lower bound, we exhibit hard instances $\text{Ascension}(m)$ of $\text{Dyck}(2)$ with length in $\Theta(mn)$. We embed these in what we call a “one-pass” communication problem with $2m$-players, where $m \in \tilde{O}(n)$. To establish the hardness of $\text{Ascension}(m)$, we follow the “information cost” approach, but with a few twists. We prove a direct sum result that reduces $\text{Ascension}(m)$ to a two-player protocol for $\text{Mountain}$, which is in fact a variant of $\text{Index}$, a fundamental problem in communication complexity. We finish the argument with a new information cost lower bound for $\text{Mountain}$.

Key words. streaming algorithms, communication complexity, information cost, Dyck languages, well-parenthesized expressions

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1. Introduction. The area of streaming algorithms has experienced tremendous growth in many applications since the late 1990s. Streaming algorithms sequentially scan the whole input piece by piece in one pass, or in a small number of passes (i.e., they do not have random access to the input), while using sublinear memory space, ideally polylogarithmic in the size of the input. The design of streaming algorithms is motivated by the explosion in the size of the data that algorithms are called upon to process in everyday real-time applications. Examples of such applications occur in bioinformatics for genome decoding, in Web databases for the search of documents, or in network monitoring. The analysis of Internet traffic [2], in which traffic logs are

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queried, was one of the first applications of this kind of algorithm. Although these would have ramifications for massive data such as DNA sequences and large XML files, few studies have been made in the context of formal languages. For instance, in the context of databases, properties decidable by streaming algorithms have been studied [39, 38], but only in the restricted case of deterministic and constant memory space algorithms.

Motivated by a concrete problem and with the goal of understanding the relationship between the complexity of streaming algorithms and the computational complexity of formal languages, we investigate the problem \( \text{DYCK}(s) \) of checking matching parentheses, with \( s \) different types of parentheses. Regular languages are by definition decidable by deterministic streaming algorithms with constant space. The \( \text{DYCK} \) languages are some of the simplest context-free languages and yet already are powerful. These languages play a central role in the theory of context-free languages, since every context-free language \( L \) can be mapped to a subset of \( \text{DYCK}(s) \) [16], for some \( s \). In addition to its theoretical importance, the problem of checking matching parentheses is encountered frequently in database applications, for instance, in verifying that an XML file is well-formed.

The problem of deciding membership in \( \text{DYCK}(s) \) has already been addressed in the massive data setting, more precisely through property testing algorithms. An \( \varepsilon \)-property tester [10, 11, 22] for a language \( L \) accepts all strings of \( L \) and rejects all strings which are \( \varepsilon \)-far from strings in \( L \) with respect to the normalized Hamming distance. For every fixed \( \varepsilon > 0 \), \( \text{DYCK}(1) \) is \( \varepsilon \)-testable in constant time [1], whereas for \( s > 1 \), \( \text{DYCK}(s) \) is \( \varepsilon \)-testable in time \( O(n^{2/3}) \), with a lower bound of \( \Omega(n^{1/11}) \) [36]. Feigenbaum et al. [20] have compared property testers and streaming algorithms. Property testers are constrained to read only small portions of the input due to expectation of small processing time. In contrast, streaming algorithms have the advantage of access to the entire string, albeit not in a random access fashion.

With random access to the input, context-free languages are known to be recognizable in space \( O((\log n)^2) \) [23]. In the special case of \( \text{DYCK}(s) \), logarithmic space is sufficient, as we may run through all possible levels of nesting, and check parentheses at the same level. This scheme does not seem to translate easily to streaming algorithms, even with a small number of passes over the input.

In the streaming model, \( \text{DYCK}(1) \) has a one-pass streaming algorithm with logarithmic space, using a height counter. Using the linear lower bound for two-way deterministic communication protocols for \( \text{EQUALITY} \), we can deduce that \( \text{DYCK}(2) \) requires space \( \Omega(n/T) \) for deterministic streaming algorithms with \( T \) passes. In particular, \( \text{DYCK}(2) \) requires linear space for deterministic one-pass streaming algorithms. A relaxation of \( \text{DYCK}(s) \) is \( \text{IDENTITY}(s) \) in the free group with \( s \) generators, where local simplifications \( \pi a = \epsilon \) are allowed in addition to \( a \pi = \epsilon \), for every type of parenthesis \( (a, \pi) \). There is a logarithmic space algorithm for recognizing the language \( \text{IDENTITY}(s) \) [32] that can easily be massaged into a one-pass streaming algorithm with polylogarithmic space. Again, this algorithm does not extend to \( \text{DYCK}(s) \).

We show that \( \text{DYCK}(s) \) is reducible to \( \text{DYCK}(2) \), for a suitable notion of reduction in the streaming model, with a \( \log s \) factor expansion in the input length. First, we present a one-pass randomized streaming algorithm for \( \text{DYCK}(2) \).

**Theorem 3.9.** Let \( c > 0 \) be any constant. There is a one-pass randomized streaming algorithm that checks if a stream of length \( n \) belongs to \( \text{DYCK}(2) \) and uses space \( O(\sqrt{n \log n}) \) and time \( \text{polylog}(n) \) per letter. If the stream belongs to \( \text{DYCK}(2) \) then the algorithm accepts it with certainty; otherwise it rejects it with probability at least \( 1 - n^{-c} \).
If the length of the stream $x$ is not known in advance, we may use standard techniques to extend the algorithm. Namely, we use an estimate for the length that is scaled geometrically as needed. The extended algorithm continues to accept streams in $\text{Dyck}(2)$ with certainty. The error probability of the resulting algorithm when $x \not\in \text{Dyck}(2)$ is guaranteed to be smaller than a prespecified constant $\delta > 0$.

If we had no space constraints, deciding $\text{Dyck}(2)$ would be very simple: when we encounter an upstep ($a$ or $b$), we push it on a stack; when we encounter a downstep ($\bar{a}$ or $\bar{b}$), we pop the top item from the stack and check that they match. However, the stack may grow to linear size in this process. To avoid this growth, the basic strategy of our algorithm is to use a linear hash function to periodically (every $\sqrt{n/\log n}$ letters) compress stack information. As long as we compress sequences of only upsteps or only downsteps, all at different heights, we are able to detect mismatches with high probability. The algorithm has one-sided error; it accepts words that belong to the language with certainty. Although it is simple, we show that this appealing algorithm is nearly optimal in its space usage, even when two-sided error is allowed.

**Corollary 4.7.** Every one-pass randomized streaming algorithm for $\text{Dyck}(2)$ with (two-sided) error $O(1/n \log n)$ on inputs of length $n$ uses $\Omega(\sqrt{n \log n})$ space.

In the preliminary version of this article [33], we conjectured that a similar lower bound continues to hold if we read the stream several times, but always in the same direction. This conjecture has since been confirmed; we elaborate on this in section 5. Surprisingly, the situation is drastically different if we can read the data stream in reverse. We present a second algorithm, a randomized two-pass streaming algorithm for $\text{Dyck}(2)$ with polylogarithmic space and time, where the second pass is in the reverse direction.

**Theorem 1.** Let $c > 0$ be a constant. There is a bidirectional two-pass randomized streaming algorithm for $\text{Dyck}(2)$ with space $O((\log n)^2)$ and time polylog($n$) for inputs of length $n$. If the input belongs to $\text{Dyck}(2)$, then the algorithm accepts it with certainty; otherwise it rejects it with probability at least $1 - n^{-c}$.

The above algorithm may be extended to streams of unknown length in the same manner as the unidirectional one. The rejection probability for inputs not in $\text{Dyck}(2)$ is then only guaranteed to be at least $1 - \delta$ for any prespecified constant $\delta$, whereas inputs in $\text{Dyck}(2)$ are accepted with certainty.

The bidirectional algorithm uses a hierarchical decomposition of the stream into blocks; whenever the algorithm reaches the end of a block, it compresses the information about subwords from within the block. This compression is what reduces the stack size from $\Theta(\sqrt{n \log n})$ down to $O(\log n)$ but prevents us from checking that certain matching pairs of parentheses are well-formed. However, given the profile of the word (i.e., the sequence of heights), we can pinpoint exactly the matching pairs that do not get checked. As it turns out, a pair that does not get checked when scanning the input left to right is necessarily checked when scanning in the reverse direction. Like the one-pass algorithm, this algorithm has only a one-sided error and always accepts words that belong to the language. We note that it is straightforward to extend the algorithms so that they recognize the language of substrings (which are subwords of consecutive letters) of $\text{Dyck}(2)$.

As mentioned above, we also investigate the lower bound on the space required for any one-pass randomized streaming algorithm. Such a lower bound is by nature hard to prove because of the connection of the problem with $\text{Identity}(2)$. Moreover, proving a nontrivial lower bound based on two-party communication complexity is hopeless; the related communication problem automatically reduces to $\text{Equality}$
after local checks and simplifications by both players, leading to only an $\Omega(\log n)$ lower bound. Instead, we build hard instances $\text{Ascension}(m)$ of $\text{Dyck}(2)$ with length in $\Theta(mn)$, which we embed in a “one-pass” communication problem with $2m$ players, where $m \in \tilde{\Theta}(n)$. The constraint is that the length of each message in the protocol be less than $\sigma$, a function of $n$. Our main lower bound result (Theorem 4.6) is that such a protocol requires $\sigma \in \Omega(n)$, which proves that our one-pass algorithm is optimal for probability of error of order $1/n \log n$ and within an $O(\log n)$ factor of optimal for constant error (Corollary 4.7).

To establish the hardness of $\text{Ascension}(m)$, we follow the “information cost” approach taken in [15, 37, 8, 28, 26], among other works before and since. The technique comes with a few twists in our case. We prove a direct sum result that captures the relationship of $2m$-player problem $\text{Ascension}(m)$ to solving $m$ instances of an intermediate problem $\text{Mountain}$, which involves only two players. $\text{Mountain}$ is a variant of $\text{Index}$, a fundamental problem in communication complexity. This variant has been studied in the one-way communication model as “serial encoding” [3, 35] and in later works on streaming and sketching as an “augmented index” (see, e.g., [29, 19]).

We adapt the notion of information cost to suit the nature of both streaming algorithms and our problem. The idea is to focus on the information about a part of the input contained in a part of the protocol transcript, given the remaining inputs. Using this notion of information cost, we prove the direct sum result (Lemma 4.5). A remarkable device here, originally developed by Bar-Yossef et al. [8], is the use of an “easy” distribution for the information cost for protocols that are correct with high probability in the worst case. The use of an easy distribution “collapses” $\text{Ascension}(m)$ to an instance of $\text{Mountain}$, which may be planted in any one of the $m$ coordinates. Finally, we prove a new information cost lower bound for $\text{Mountain}$ using a medley of combinatorial and information-theoretic means.

In protocols for $\text{Ascension}(m)$ we allow access to only public coins by all players, whereas in protocols for $\text{Mountain}$ we allow one of the players, Bob, access also to private coins (while Alice, the other player, may access only public coins). This mixture of public and private coins for $\text{Mountain}$ arises from a balancing act between the direct sum result and our lower bound for $\text{Mountain}$ (Theorem 4.4). Namely, we prove the lower bound for $\text{Mountain}$ when Alice uses only public coins, whereas the direct sum holds, with our definition of information cost, only when Bob has access to additional private coins. The mixing of public and private coins in the analysis of information cost has also been observed and similarly tackled in earlier works (see, e.g., [14]).

We note that as a bonus, our lower bound (Theorem 4.6) provides a $\tilde{\Omega}(\sqrt{n})$ lower bound for the problem of checking priority queues in the one-pass streaming model, solving an open problem posed by Chu, Kannan, and McGregor [17].

2. Definitions and preliminaries.

**Definition 2.1 (Dyck).** Let $s$ be a positive integer. Then $\text{Dyck}(s)$ denotes the language over alphabet $\Sigma = \{a_1, \overline{a}_1, \ldots, a_s, \overline{a}_s\}$ defined recursively by

$$\text{Dyck}(s) = \epsilon + \sum_{i \leq s} a_i \cdot \text{Dyck}(s) \cdot \overline{a}_i \cdot \text{Dyck}(s).$$

We also denote by $\text{Dyck}(s)$ the problem of deciding whether a word $w \in \Sigma^*$ is in the language $\text{Dyck}(s)$. 


In streaming algorithms, a pass on an input \( x \in \Sigma^n \) means that \( x \) is presented as an input stream \( x_1, x_2, \ldots, x_n \), which arrives sequentially, i.e., letter by letter in this order. For simplicity, we assume throughout this article that the length \( n \) of the input is always given to the algorithm in advance. Nonetheless, all our algorithms can be adapted using standard techniques to the case in which \( n \) is unknown until the end of a pass. See [34] for an introduction to streaming algorithms.

**Definition 2.2** (streaming algorithm). Fix an alphabet \( \Sigma \). A \( k \)-pass deterministic (resp., randomized) streaming algorithm \( A \) with space \( s(n) \) and time \( t(n) \) is a deterministic (resp., randomized) algorithm such that for every input stream \( x \in \Sigma^n \),

1. \( A \) performs \( k \) sequential passes on \( x \);
2. \( A \) maintains a memory space of size \( s(n) \) bits while reading \( x \);
3. \( A \) has running time at most \( t(n) \) per letter \( x_i \);
4. \( A \) has preprocessing and postprocessing time \( t(n) \).

We say that \( A \) is bidirectional if it is allowed to access the input in the reverse order, after reaching the end of the input. Then the parameter \( k \) is the total number of passes in either direction.

**Definition 2.3** (streaming reduction). Fix two alphabets \( \Sigma_1 \) and \( \Sigma_2 \). A problem \( P_1 \) is \( f(n) \)-streaming reducible to a problem \( P_2 \) with space \( s(n) \) and time \( t(n) \) if for every input \( x \in \Sigma_1^n \), there exists \( y = y_1 y_2 \ldots y_n \), with \( y_i \in \Sigma_2^{f(n)} \), such that

1. \( y_i \) can be computed deterministically from \( x_i \) using space \( s(n) \) and time \( t(n) \);
2. from a solution of \( P_2 \) with input \( y \), a solution on \( P_1 \) with input \( x \) can be computed with space \( s(n) \) and time \( t(n) \).

The following is immediate.

**Proposition 2.4.** Let \( P_1 \) be \( f(n) \)-streaming reducible to a problem \( P_2 \) with space \( s_0(n) \) and time \( t_0(n) \). Let \( A \) be a \( k \)-pass streaming algorithm for \( P_2 \) with space \( s(n) \) and time \( t(n) \). Then there is a \( k \)-pass streaming algorithm for \( P_1 \) with space \( s(n \times f(n)) + s_0(n) \) and time \( t(n \times f(n)) + t_0(n) \) with the same properties as \( A \) (deterministic/randomized, unidirectional/bidirectional).

Moreover, we need only study \( \text{DYCK}(s) \) with \( s = 2 \).

**Proposition 2.5.** \( \text{DYCK}(s) \) is \( \lfloor \log s \rfloor \)-streaming reducible to \( \text{DYCK}(2) \) with space and time \( O(\log s) \).

**Proof.** We encode a parenthesis \( a_i \) by a word of length \( l = \lceil \log s \rceil \) with only parentheses of type \( b, c \). We let \( f(a_i) \) be the binary expansion of \( i \) over \( l \) bits where 0 is replaced by \( b \) and 1 by \( c \). Then \( f(\pi_i) \) is defined similarly, except that we write the binary expansion of \( i \) in the opposite order and replace 0 by \( b \) and 1 by \( c \). Then \( x_1 \ldots x_n \) is in \( \text{DYCK}(s) \) if and only \( f(x_1) \ldots f(x_n) \) is in \( \text{DYCK}(2) \).

Since the parameter \( s \) is a constant independent of the length of the input stream, the above reduction can be implemented with constant space and time. For example, in parsing XML files, given an upstep (start-tag) \( <w> \) (resp., a downstep (end-tag) \(</w>\), where \( w \) is an ASCII string denoting the type of parenthesis (tag), we can generate the above encoding of \( w \) into \( b, c \) (resp., into \( \overline{b}, \overline{c} \)), while reading \( w \) as a stream itself, i.e., character by character.

By Proposition 2.5, it is enough to design streaming algorithms for \( \text{DYCK}(2) \). That is the objective of the next section.

**3. Algorithms.** From now on we consider \( \text{DYCK}(2) \) where the input is a stream of \( n \) letters \( x_1 x_2 \ldots x_n \) in the alphabet \( \Sigma = \{a, \pi, b, \overline{b}\} \). We first introduce a few definitions. An upstep is a letter \( a \) or \( b \), and a downstep is a letter \( \pi \) or \( \overline{b} \). For integers \( i \leq j \), we denote by \( [i, j] \) the set of integers \( \{i, i+1, \ldots, j\} \) and by \( x[i, j] \) the subword \( x_i x_{i+1} \ldots x_j \). We also use the notation \( x[i] \) for \( x_i \) when we also consider sequences
of words. For ease of notation, we identify an increasing sequence \(i_1 < i_2 < \cdots < i_m\) of indices with the corresponding subword \(x_{i_1}x_{i_2}\ldots x_{i_m}\) of \(x\). We also use this correspondence in reverse when the indices of the subword are clear from the context.

The number of occurrences of the letter \(p\) in a word \(x\) is denoted by \(|x|_p\). In the absence of any subscript, \(|x|\) denotes the length of the word.

**Definition 3.1** (height, matching pair, well-formed). Let \(x \in \Sigma^n\). The height of \(x\) is \(\text{height}(x) = |x_a| + |x_b| - |x|\sigma - |x|\bbar\). For \(1 \leq i < j \leq n\), \((i, j)\) is a matching pair for \(x\) if \(\text{height}(x[1, i-1]) = \text{height}(x[1, j])\) and \(\text{height}(x[1, k]) > \text{height}(x[1, i-1])\) for all \(k \in \{i, \ldots, j - 1\}\). The height of a matching pair \((i, j)\) is \(\text{height}(x[1, i-1])\).

A matching pair \((i, j)\) for \(x\) is well-formed if \((x[i], x[j]) = (a, \sigma)\) or \((b, \bbar)\) and is ill-formed otherwise.

It follows that any index \(i\) forms a matching pair with at most one other index and that a matching pair consists of an upstep and a downstep. These definitions are extended to subsets \(I \subseteq [1, n]\) of indices of letters of \(x\). For instance, we say that \(I\) is a matching set for \(x\) if \(I\) is the union of \(\{i, j\}\) over the matching pairs \((i, j)\) for \(x\).

Observe that when \(i < j\) we have the following equivalence: \((i, j)\) is a matching pair for \(x\) if and only if \(\{i, i+1, i+2, \ldots, j\}\) is a matching set for \(x\).

Define a partial order \(\prec\) between words such that \(u \prec v\) if and only if \(u\) is obtained from \(v\) by removing zero or more of its matching pairs. This order is well defined, and in particular transitive, since matching pairs of \(u\) are still matching pairs of \(v\), up to reindexing. (This may be proved by a straightforward inductive argument.)

**Proposition 3.2.** Let \(u, v\) be words such that \(u \prec v\) and \(u = u_1u_2\ldots u_m = v_1v_2\ldots v_m\) is obtained by removing the matching set \([1, n]\backslash\{i_1, i_2, \ldots, i_m\}\) from \(v\). If \((j, k) \in [1, m]^2\) is a matching pair for \(u\), then \((i_j, i_k)\) is a matching pair for \(v\).

To prove correctness of our algorithms, we use the following characterization of \(\text{DYCK}(2)\).

**Proposition 3.3.** Let \(x \in \Sigma^n\). Then

1. \([1, n]\) is a (possibly ill-formed) matching set for \(x\) if and only if \(\text{height}(x) = 0\) and the height of every prefix of \(x\) is nonnegative;
2. \([1, n]\) is a well-formed matching set for \(x\) if and only if \(x \in \text{DYCK}(2)\).

**Proof.** The proof is by induction on the length \(n\) of \(x\). The first part may be established by a straightforward induction on \(n\). We prove the second part. For \(n = 0\) the result is true since the empty word is in \(\text{DYCK}(2)\) and \(\emptyset\) is a well-formed set.

Let \(x \in \text{DYCK}(2)\) of length \(n\). By Definition 2.1, there exist \(y, z \in \text{DYCK}(2)\) such that \(x = cy\sigma z\), where \(c \in \{a, b\}\). Then \((1, 2 + |y|)\) is a well-formed pair. By the inductive hypothesis, \([1, |y|]\) and \([1, |z|]\) are well-formed matching sets for \(y\) and \(z\), respectively. All together this gives the well-formed set \([1, n]\) for \(x\), after appropriate reordering.

Conversely, assume that \([1, n]\) is a well-formed set for \(x\). Let \(j_1\) be such that \((1, j_1)\) is a (well-formed) matching pair for \(x\). We prove that every matching pair \((i, j)\) for \(x\) satisfies \(1 < i, j < j_1\) or \(j_1 < i, j \leq n\). Thus the matching set \([1, n]\) is partitioned into \([1, j_1]\), \([2, j_1 - 1]\) (which is a translation of the matching set for \(x[2, j_1 - 1]\)) and \([j_1 + 1, n]\) (which is a translation of the matching set for \(x[j_1 + 1, n]\)). By the inductive hypothesis, \(x[2, j_1 - 1], x[j_1 + 1, n] \in \text{DYCK}(2)\), and the statement follows.

We return to the property of matching pairs \((i, j)\) described above. Assume, for a contradiction, that there is a matching pair \((i, j)\) such that \(i < j < j_1 < j\). Then, by Definition 3.1, \(\text{height}(x[1, j_1]) > \text{height}(x[1, i - 1])\) and also \(\text{height}(x[1, j_1]) = \text{height}(x[1, 0]) = 0\). Thus \(\text{height}(x[1, i - 1]) < 0\), a contradiction to the first part of the proposition. \(\square\)
Observe that checking that \([1, n]\) is a (possibly ill-formed) matching set, or equivalently that \(\text{height}(x) = 0\) and the height of every prefix of \(x\) is nonnegative, can be done deterministically within one pass over stream \(x\), using \(\log n\) memory. Our algorithms do not explicitly check this but nonetheless ensure this property when accepting \(x\). During the computation our algorithms implicitly keep track of the height of the word read so far. They reject when the height of any prefix is negative, so for ease of exposition, we assume that the height of the stream is always nonnegative.

Let \(p\) be a prime number such that \(n^{1+\gamma} \leq p < 2n^{1+\gamma}\) for some fixed constant \(\gamma > 0\). The algorithm uses a random function \(\text{hash}(\cdot)\) that maps subwords \(v\) of \(x\) to integers in \([0, p-1]\), as follows: \(\text{hash}(x_i, x_{i+1}, \ldots, x_m) = \sum_i \text{hash}(x_i)\) with

\[
\text{hash}(x_i) = \begin{cases} 
\alpha \cdot \text{height}(x[1,i-1]) \pmod{p} & \text{if } x_i = a, \\
-\alpha \cdot \text{height}(x[1,i]) \pmod{p} & \text{if } x_i = \bar{a}, \\
0 & \text{otherwise},
\end{cases}
\]

where \(\alpha\) is a uniformly random integer in \([0, p-1]\). Note that the computation of \(\text{hash}(v)\) depends not just on \(v\) but also on the height of its letters within \(x\).

Given \(x\) and \(v\), the value of \(\text{hash}(v)\) is a polynomial in \(\alpha\) of degree bounded by the maximum height of a prefix of \(x\), which is at most \(n\). A polynomial of degree \(d\) over \(\mathbb{F}_p\) has at most \(d\) roots. Therefore, if the polynomial corresponding to \(\text{hash}(v)\) is not identically zero, for a uniformly random \(\alpha\), the probability that \(\text{hash}(v) = 0\) is at most \(n/p < n^{-\gamma}\). In particular, we have the following.

**Proposition 3.4.** Let \(x \in \Sigma^n\) be such that every prefix of \(x\) has nonnegative height, and let \(v = x_{i_1}, x_{i_2}, \ldots\) be a subword of \(x\). If \(v \in \text{DYCK}(2)\), then \(\text{hash}(v) = 0\) for all \(\alpha\). Moreover, if there is a height \(d\) at which \(v\) has a single ill-formed pair (and possibly other ill-formed matching pairs at heights \(\neq d\)), then \(\text{hash}(v) \neq 0\) with probability at least \(1 - n^{-\gamma}\), for a uniformly random integer \(\alpha \in [0, p-1]\).

**Proof.** If \(v \in \text{DYCK}(2)\), then by Proposition 3.3 the set \([1, n]\) is well-formed, and then each well-formed matching pair \((i, j)\) at height \(d\) contributes

\[
\begin{cases} 
\alpha^d - \alpha^d = 0 & \text{if } (x_i, x_j) = (a, \bar{a}); \\
0 - 0 = 0 & \text{if } (x_i, x_j) = (b, \bar{b}).
\end{cases}
\]

Therefore, we get \(\text{hash}(v) = 0\).

Now, assume there is a height \(d\) at which \(v\) has a single ill-formed pair. Since every prefix of \(x\) has nonnegative height, the value \(\text{hash}(v)\) is a polynomial \(q(z)\) evaluated at \(z = \alpha\). Every well-formed pair at height \(d\) cancels, and so the coefficient of \(z^d\) in \(q\) is \(+1\) if \((x_i, x_j) = (a, \bar{b})\) and \(-1\) if \((x_i, x_j) = (b, \bar{a})\). Thus \(q\) is not identically zero. The claim follows from the uniformly random choice of \(\alpha\).

For any letter \(x_i\), we may compute \(\text{hash}(x_i)\) in time \(\text{polylog } n\) and space \(O(\log n)\). Moreover, for any word \(v\) the value of \(\text{hash}(v)\) can be maintained with \(O(\log n)\) space.

### 3.1. The one-pass algorithm.

The algorithm is easiest to understand if \(x = uv\), where \(u\) has only upsteps and \(v\) has only downsteps, in equal numbers. To check whether \(uv \in \text{DYCK}(2)\), the naive algorithm would grow a stack of size \(n/2\). Here is a simple alternative. We read the input in blocks of length \(q\). For simplicity, assume that \(n\) is divisible by \(2q\). While the algorithm is reading letters of \(u\), the stack stores the values of \(\text{hash}(x[iq+1, (i+1)q])\), one stack item for each \(i \in \{0, \ldots, n/2q - 1\}\), and notes that \(\text{height}(x[iq+1, (i+1)q]) = q\). While the algorithm is reading letters
of $v$, it adds $hash(x[jq + 1, (j + 1)q])$ to $hash(x[iq + 1, (i + 1)q])$ for $j = n/q - i - 1$ and checks whether their sum is 0. The input $x$ is ill-formed if any of the sums is nonzero.

Our algorithm is a generalization of this stack compression idea, and the block length is chosen to be $q = \lceil \sqrt{n \log n} \rceil$ to minimize the space used.

Algorithm 1 attempts to collect a sequence of $\ell = \lceil \sqrt{n \log n} \rceil$ upsteps while doing obvious checks. Using a straightforward stack-based algorithm, any upstep followed by a downstep is checked for well-formedness, and once checked, the pair is discarded. The stack, called $S_{temp}$ in the algorithm, allows us to apply this check for every matching pair that is encountered before reaching the limit of $\ell$ upsteps. When the stack $S_{temp}$ collects a sequence $v$ of $\ell$ upsteps, the algorithm hashes $v$ to $hash(v)$ and empties $S_{temp}$. The hash value is pushed to a second stack $S$. The stack $S$ encodes the subword given by the letters seen so far that remain to be checked. Each item of $S$ of the form $(h, \ell)$ encodes a subword $v$ of the stream $x$, in the sense that $h = hash(v)$ and $\ell = height(v)$. The algorithm accesses $S$ to look up information about the blocks previously read.

To process a downstep $y$, the algorithm either checks for a match in $S_{temp}$ or incorporates it into the topmost stack item of $S$. More precisely for the second case, given a downstep $y$ and given $(h, \ell) = (hash(v), height(v))$, it computes $hash(vy) = h + hash(y)$ and $height(vy) = \ell - 1$, thus encoding $vy$ without explicit knowledge of $v$. Note that this relies on the linearity of the hash function. When the encoded subword $v$ has height 0, to test whether it is well-formed, the algorithm checks whether $hash(v) = 0$. If this test succeeds, the entry of the stack encoding $v$ is removed. An example execution of the algorithm is presented in Figure 1.

For the analysis, we start with the following invariants of Algorithm 1.

**Proposition 3.5.** Let $(h, \ell)$ be an item of $S$ that encodes a subword $v$. Then $v = v_1v_2$, where $v_1$ has only upsteps, $v_2$ has only downsteps, $\ell = |v_1| - |v_2|$, and $\ell > 0$.

The proof is by a straightforward induction on the number of operations on $S$ and is omitted.

Fig. 1. Example of execution of Algorithm 1. Here there are eight blocks, and they are shown after the internal simplifications have already been done. The dotted vertical lines mark times at which the stack changes size, either starting a new stack item (for example, at time $t_0$) or discarding a stack item (for example, at time $t_4$). Note that blocks and stack items are staggered: the first item incorporates the first block and the downsteps of the second block, the second item incorporates the upsteps of the second block and the downsteps of the third block, etc. The bullets mark times when the algorithm checks and discards an item if the hash value is 0. The horizontal lines go from the time when a stack item is created to the time when it is checked and discarded. For example, at time $t_5$ the algorithm checks and discards an item $(h_m, \ell_m)$ such that $h_m$ incorporates the upsteps marked in bold on the figure, namely, $x(t_5, t_6)$, and incorporates the downsteps marked in bold on the figure, namely, $x(t_2, t_3)$, $x(t_4, t_5)$, and $x(t_6, t_7)$.
Let \( v \) the stream. Initially, \( k \) an analogous statement holds after \( x \) processing \( \preceq v \) in advance). We have \( x \) \( k \) modified such that they encode \( e \). Therefore the propositions still hold just before processing \( x_{k+1} \).

**Algorithm 1.** One-pass algorithm (when the length \( n \) of the stream is known in advance).

1: \( S_{\text{temp}} \leftarrow \) empty stack of upsteps; \( S \leftarrow \) empty stack of items \((h, \ell)\)
2: \( (h_{\text{temp}}, \ell_{\text{temp}}) \leftarrow (0, 0) \) \{This pair encodes the subword contained in \( S_{\text{temp}} \).\}
3: Compute a prime \( p \) such that \( n^{1+\gamma} \leq p < 2n^{1+\gamma} \); Pick a uniformly random \( \alpha \in [0, p - 1] \)
   \{The pair \((p, \alpha)\) are used in the function hash; \( \gamma > 0 \) is a constant of our choice.\}
4: while stream is not empty do
5:   read next letter \( y \) from stream
6:   if \( y \) is an upstep then
7:     push \( y \) on \( S_{\text{temp}} \)
8:     update \((h_{\text{temp}}, \ell_{\text{temp}})\) with \( y \): \( h_{\text{temp}} \leftarrow (h_{\text{temp}} + \text{hash}(y) \mod p)\); \( \ell_{\text{temp}} \leftarrow \ell_{\text{temp}} + 1 \)
9:     if \( S_{\text{temp}} \) has size \( \lceil \sqrt{n \log n} \rceil \) then
10:    push \((h_{\text{temp}}, \ell_{\text{temp}})\) on to \( S \) and reset \( S_{\text{temp}} \) to empty; \((h_{\text{temp}}, \ell_{\text{temp}}) \leftarrow (0, 0)\)
11:   end if
12: else \{\( y \) is a downstep\}
13:   if \( S_{\text{temp}} \) is not empty then
14:     pop \( z \) from \( S_{\text{temp}} \)
15:     check that \( zy \) is well-formed: \( zy \in \{a\bar{a}, b\bar{b}\} \) (if not, \textbf{reject} “mismatch”)
16:     update \((h_{\text{temp}}, \ell_{\text{temp}})\) for removal of \( z \): \( h_{\text{temp}} \leftarrow (h_{\text{temp}} - \text{hash}(z) \mod p)\);
17:     \( \ell_{\text{temp}} \leftarrow \ell_{\text{temp}} - 1 \)
18:   else \{\( S_{\text{temp}} \) is empty\}
19:     pop \((h, \ell)\) from \( S \) (if empty, \textbf{reject} “extra closing parenthesis”)
20:     update \((h, \ell)\) with \( y \): \( h \leftarrow (h + \text{hash}(y) \mod p)\); \( \ell \leftarrow \ell - 1 \)
21:     if \( \ell = 0 \) then check that \( h = 0 \) (if not, \textbf{reject} “mismatch”)
22:     else push \((h, \ell)\) on \( S \)
23:   end if
24: end if
25: end while
26: if \( S \) and \( S_{\text{temp}} \) are not both empty then \textbf{reject} “missing closing parenthesis”
27: \textbf{accept}
Suppose now that $x_k$ is a downstep. We assume that $(S,S_{\text{temp}})$ are not both empty, otherwise the algorithm rejects at line 18 before processing $x_{k+1}$. We analyze the processing of $x_k$ in order to complete the induction step.

First, if $v_{\text{temp}} \neq \epsilon$, then the last letter of $v$ is the last letter of $v_{\text{temp}}$, which is an upstep. Therefore $(v_{[v]}, x_{k+1})$ is a matching pair for $vx_k$, and $v_{[v]} | v_{[v]} | v_{[v]} - 1 < vx_k < x_{[1,k]}$. If the algorithm does not reject at this point, the last upstep of $v_{\text{temp}}$ is deleted at line 14, and $(S,S_{\text{temp}})$ encodes $v_{[v]} | v_{[v]} - 1$. So the statement holds in this case.

Second, if $v_{\text{temp}} = \epsilon$ and height$(v_{\text{m}}) = \ell$ from Proposition 3.5. Therefore $\ell > 1$, and $(h, \ell)$ is updated in order to encode $v_{\text{m}}x_k$, and then pushed back to $S$. Now $(S,S_{\text{temp}})$ encodes $vx_k$ which satisfies $vx_k < x_{[1,k]}$. So the statement holds in this case as well.

The last case is when $v_{\text{temp}} = \epsilon$ and height$(v_{\text{m}}) = 0$. By Proposition 3.5, the item $(h, \ell)$ popped from $S$ encodes $v_{\text{m}}$ and satisfies height$(v_{\text{m}}) = \ell$. If the algorithm does not reject after processing $x_k$, this item is deleted from $S$ and $(S,S_{\text{temp}})$ now encodes $v_1v_2 \ldots v_{m-1}$. Recall that $x_k$ is a downstep. From Proposition 3.5, the subword $v_{\text{m}}x_k$ is a sequence of upsteps followed by the same number of downsteps. Therefore $\epsilon < v_{\text{m}}x_k$. Since $v_{\text{m}}x_k$ is also the suffix of $vx_k$, we get that it is a matching set for $vx_k$, that is, $v_1v_2 \ldots v_{m-1} < vx_k < x_{[1,k]}$, and the statement holds.

Lemma 3.7. Algorithm 1 satisfies the following properties:  
1. At line 15, the pair $(z,y)$ is a matching pair for $x$.  
2. At line 20, if $\ell = 0$, then $(h,0)$ encodes a subword $v$ which is a matching set for $x$.

Proof. For both properties, let $y$ be the letter $x_j$ that the algorithm is currently processing. Let $(S, S_{\text{temp}})$ be the stacks just before processing $x_j$, and let $v$ be the subword encoded by $(S, S_{\text{temp}})$. Since we consider properties at line 15 and line 20, $x_j$ is necessarily a downstep.

We start with the first property. Therefore stack $S_{\text{temp}}$ is not empty before processing $x_j$ and the upstep $z$ is on its top. Therefore $v$ ends with $z$, and $(z,x_j)$ is a matching pair for $vx_j$. By Proposition 3.6, subword $v$ satisfies $v < x_{[1,j]}$, so $vx_j < x_{[1,j]}$. By Proposition 3.2, $(z,x_j)$ is also a matching pair for $x_{[1,j]}$, and for $x$.

For the second property, $S_{\text{temp}} = \emptyset$ and $S \neq \emptyset$. Let $v_{\text{m}}$ be the subword encoded by the topmost element of $S$. Then $(h,0)$ encodes $v_{\text{m}}x_j$. Moreover, by Proposition 3.5, $v_{\text{m}}x_j$ is a sequence of upsteps followed by the same number of downsteps. Therefore $v_{\text{m}}x_j$ is a matching set for $vx_j$. By Proposition 3.6, subword $v$ satisfies $v < x_{[1,j-1]}$, so $vx_j < x_{[1,j]}$. By Proposition 3.2, $v_{\text{m}}x_j$ corresponds to a matching set for $x_{[1,j]}$, and therefore for $x$.

Lemma 3.8. Let $(i,j)$ be a matching pair for $x$. Then just before processing $x_j$, the stacks $S$ and $S_{\text{temp}}$ of Algorithm 1 satisfy one of the following properties, if the algorithm has not already rejected $x$:  
1. $S_{\text{temp}}$ is not empty and has $x_i$ on top.  
2. $S_{\text{temp}}$ is empty but not $S$, and the topmost item of $S$ encodes a subword containing $x_i$.

Proof. Fix some matching pair $(i,j)$. Let $(S, S_{\text{temp}})$ be the stacks of the algorithm just before processing $x_j$. By Proposition 3.6, the subword encoded by $(S, S_{\text{temp}})$ contains $x_i$. Therefore the stacks are not both empty.

If $S_{\text{temp}} \neq \emptyset$ just before processing $x_j$, then, by Lemma 3.7, $x_j$ matches, possibly as an ill-formed pair, the topmost element of $S_{\text{temp}}$. Since any index (in our case, $j$) may form a matching pair with at most one other index (in our case, $i$), the second property is satisfied.
Assume now that $S_{\text{temp}} = \emptyset$ and $S \neq \emptyset$. All upsteps in the stream, including $x_i$, were first pushed onto $S_{\text{temp}}$, unless the algorithm rejected before reading them. Since $S_{\text{temp}}$ is now empty, the upstep $x_i$ was later popped. By Lemma 3.7, $x_i$ was not popped from $S_{\text{temp}}$ at line 15. (Otherwise $i$ would match another index $k \neq j$, which is impossible.) Therefore $x_i$ was encoded into a stack element and pushed on to $S$ at line 10. By Lemma 3.7 it follows that this stack element was not popped from $S$ at line 20 and therefore is still in $S$ just before $x_j$ is read.

It remains to be proved that the stack element containing $x_i$ is at the top of $S$. Let $v_1v_2 \ldots v_m$ be the subwords encoded by $S$. By Propositions 3.5 and 3.6, we have $v_1v_2 \ldots v_m x_j \prec x[1,j]$, and $\text{height}(v_k) > 0$, for $k = 1, 2, \ldots, m$. Since $(x_i, x_j)$ is a matching pair for both $v_1v_2 \ldots v_m x_j$ and $x[1,j]$, we have that $x_i$ is in $v_m$.

We conclude with the correctness of our algorithm.

**Theorem 3.9.** Algorithm 1 is a one-pass randomized streaming algorithm for Dyck(2) with space $O(\sqrt{n \log n})$ and time polylog($n$). If the stream belongs to Dyck(2), then the algorithm accepts it with certainty; otherwise it rejects it with probability at least $1 - n^{-c}$, where $c > 0$ is a constant.

**Proof.** The stack elements of $S_{\text{temp}}$ and $S$ take space $O(1)$ and $O(\log n)$ bits, respectively. Stack $S_{\text{temp}}$ has size bounded by $[\sqrt{n \log n}]$ and therefore uses space $O(\sqrt{n \log n})$. A new element is pushed on to $S$ only when $S_{\text{temp}}$ is full (has size $[\sqrt{n \log n}]$, after which $S_{\text{temp}}$ is emptied. Therefore the algorithm processes at least $[\sqrt{n \log n}]$ letters between each increase of the size of $S$, bounding the number of stack elements of $S$ by $n/\sqrt{n \log n}$. Hence $S$ also uses space $O(\sqrt{n \log n})$.

Using well-known results in algorithmic number theory [7, sections 8.2 and 9.7], the prime $p$ used for the hash function may be computed probabilistically in time $\text{polylog}(n)$. The probability that the procedure returns a prime is at least $1 - n^{-\gamma}$ for a constant $\gamma > 0$ of our choice. The processing time of any letter in the stream is dominated by the computation of the hash function, specifically by the modular exponentiation. Since the modular exponentiation involves $(\log n)$-bit integers, the time taken is $\text{polylog}(n)$.

With probability $n^{-\gamma}$, the number returned by the prime number generation procedure may be composite. We analyze the algorithm assuming that the number is prime, and then consider the case when it is composite.

To prove correctness, we first argue that the algorithm rejects when $[1,n]$ is not a matching set for $x$. We prove this property by contraposition. Assume that the algorithm accepts $x$. Then the stacks $S, S_{\text{temp}}$ are both empty after processing $x$ and therefore encode the empty word. By Proposition 3.6, we have that $\emptyset \prec [1,n]$, i.e., $[1,n]$ is a matching set for $x$.

We assume in the rest of the proof that $[1,n]$ is a matching set for $x$. By Proposition 3.6, $S$ and $S_{\text{temp}}$ are never both empty while processing a downstep. The proposition also implies that if the algorithm processes the full stream, then it accepts. Therefore, the algorithm only rejects after processing a downstep at either line 15 or line 20. Let $x_j$ be any downstep, and let $i$ be the unique integer such that $(i,j)$ is a matching pair. We prove that the algorithm does not reject after processing $x_j$ if $x \in \text{Dyck}(2)$, whereas it rejects with high probability if $(i, j)$ is ill-formed.

Consider first the case $x \in \text{Dyck}(2)$. By Lemma 3.7, the tests at lines 15 and 20 check whether a matching pair or set of $x$ is well-formed. Since $x$ is well-formed, those matching sets are all well-formed. Therefore the tests always succeed (thanks to Proposition 3.4 for the test at line 20).
We consider the remaining case, where \( (i, j) \) is ill-formed and the algorithm has not already rejected before processing \( x_j \). Consider the stacks \( S \) and \( S_{\text{temp}} \) just before \( x_j \) is processed. They are not both empty since \([1, n]\) is a matching set. By Lemma 3.8, the topmost element of \( S_{\text{temp}} \) is \( x_i \) when \( S_{\text{temp}} \neq \emptyset \), and the topmost element of \( S \) encodes a subword containing \( x_i \) when \( S_{\text{temp}} = \emptyset \). In the first case, the algorithm checks the well-formedness of \((x_i, x_j)\) at line 15 and rejects with probability 1. In the second case, the algorithm updates the stack element so that it contains both \( x_i \) and \( x_j \) at line 19. This element either is checked at line 20 or is pushed back to the stack at line 21. In the latter case, the algorithm either rejects while processing subsequent letters or eventually checks this stack element at line 20. We consider the first time (if any) that this stack element is checked. Recall that the stack element encodes a subword \( v \) that contains \( x_i \) and \( x_j \) and that \( v \) is a matching set by Lemma 3.7. The crucial observation is that, by Proposition 3.5, \((i, j)\) is the only ill-formed matching pair in \( v \) at the corresponding height. Therefore Proposition 3.4 implies that the probability that the algorithm rejects is at least \( 1 - \frac{n}{2} \).

We point out that the algorithm continues to accept streams \( x \in \text{Dyck}(2) \) with certainty even if the modulus used in the hash function is composite. When the stream \( x \notin \text{Dyck}(2) \), the union bound tells us that the probability that the algorithm does not reject is at most \( 2^{n-\gamma} \).

### 3.2. The bidirectional algorithm

The second algorithm uses a (virtual) hierarchical decomposition of the stream \( x \) into nested blocks of \( 2^i \) letters for \( i \leq k = \lceil \log n \rceil \) (see Figure 2). We define an \( i \)-block to be any substring of the form \( x[(q - 1)2^i + 1, q2^i] \) for \( 1 \leq q \leq n/2^i \). We may omit the parameter \( i \) when referring to an \( i \)-block if its precise value is not important. The algorithm maintains a decomposition of the prefix \( x[1, j] \) read so far into \( m \leq \lceil \log j \rceil \) contiguous blocks of decreasing sizes. The decomposition is given by the binary decomposition of \( j \). Let \( 0 \leq i_1 < \cdots < i_m \leq k \) be such that \( j = \sum_{i=1}^{m} 2^{i_i} \). Then \( x[1, j] \) is partitioned from left to right into adjacent blocks of decreasing lengths \( 2^{i_m}, 2^{i_{m-1}}, \ldots, 2^{i_1} \). We call such a decomposition the binary partition of \( x[1, j] \) and call the block of size \( 2^{i_k} \) the last block of the binary partition. We extend the definition and notation related to blocks to intervals \([1, j]\) as well. The binary partition and the last block of an interval \([1, j]\) play an important role in the bidirectional algorithm (see line 13 of Algorithm 3).

We assume that \( n = 2^k \), for some \( k \geq 1 \). Thanks to this assumption, the algorithm uses the same hierarchical decomposition whether we read the stream from left to right.
or from right to left. The assumption is without loss of generality, as we can append to $x$ the word $(\bar{a}a)^i$ for a suitable $i \geq 1$. This is required only if $|x|$ is even; otherwise $x \notin \text{DYCK}(2)$. At the end of the first pass, we use $O(\log n)$ bits of memory to store the number of letters added. Algorithm 2, the bidirectional algorithm, simply runs Algorithm 3 twice, once reading the stream in the forward direction, and a second time in reverse. The algorithm accepts if there is no rejection during either pass. During the right-to-left pass, letters $\bar{a}, b$ are interpreted as $a, b$, respectively (and vice versa).

**Algorithm 2.** Bidirectional algorithm (when the length $n$ of the stream is a power of 2, and is known in advance).

```
Compute a prime $p$ such that $n^{1+\gamma} \leq p < 2n^{1+\gamma}$; Pick a uniformly random $\alpha \in [0, p-1]$

{The pair $(p, \alpha)$ are used in the function hash; $\gamma > 0$ is a constant of our choice.}
Run Algorithm 3, reading the stream from left to right
Run Algorithm 3, reading the stream from right to left
{While reading the stream right to left, $\bar{a}, b$ are interpreted as $a, b$, respectively (and vice-versa)}
accept
```

Algorithm 3 continuously maintains the binary partition of the prefix $x[1, j]$ of the stream that has been read so far. We use a stack data structure to encode the entire prefix $x[1, j]$. Each stack item is now of the form $(h, \ell, f)$ and encodes a subword $v$ of $x$, in the sense that $h = \text{hash}(v)$, $\ell = \text{height}(v)$, and $f$ is the position in $x$ of the first letter of $v$. An item remains in the stack while $\ell > 0$.

The main difference between Algorithm 3 and Algorithm 1 is that whenever the algorithm reaches the end of a block, it “compresses” without checking the stack items encoding subwords from within the block. This compression is what reduces the stack size from $\sqrt{n}/\log n$ down to $O(\log n)$, but now Proposition 3.5 no longer holds for this stack; since hash is commutative, we may lose information. For example, compressing $\text{hash}(ba\bar{a})$ with $\text{hash}(\bar{b}b\bar{a}\bar{a})$ gives $\text{hash}(ba\bar{a}b\bar{a}\bar{a})$, which is equal to $\text{hash}(ba\bar{a}b\bar{a}\bar{a})$: one word is in $\text{DYCK}(2)$, the other one is not, but after compressing we can no longer distinguish between them. In processing the ill-formed word $ba\bar{a}b\bar{a}\bar{a}$ from left to right, the algorithm compresses the first four letters to $\text{hash}(ba\bar{a})$ and consequently does not detect ill-formedness. The crux of the analysis is that such information loss does not occur both when reading the stream from left to right and when reading it from right to left (see Figure 3). Every matching pair is checked in at least one of the two passes. In the example above, in processing the word $ba\bar{a}b\bar{a}\bar{a}$ from right to left (with upsteps interpreted as downsteps and vice versa), a mismatch is detected when the seventh letter is read.

For the analysis of Algorithm 3, we first derive the following invariant that is weaker than Proposition 3.5. The proof follows from induction and is omitted.

**Proposition 3.10.** Let $(h, \ell, f)$ be an item of $S$ encoding a subword $v$. Then $\ell = \text{height}(v) > 0$, and every prefix of $v$ has positive height.

We can adapt Lemmas 3.7 and 3.8 to our new algorithm. The proofs are straightforward and are omitted.

**Lemma 3.11.** At line 10 of Algorithm 3, if $\ell = 0$, then $(h, 0, f)$ encodes a subword $v$ that is a matching set for $x$. 

Fig. 3. Asymmetry of the two passes. The bold lines represent matching pairs between the two \((i-1)\)-blocks \(B, B'\) within the same \(i\)-block \(B_i\). In this example, these pairs are checked during the left-to-right pass, since the minimum height \(m\) within the left \((i-1)\)-block \(B\) is larger than the minimum height \(m'\) with the right \((i-1)\)-block \(B'\) (during the right-to-left pass, they are compressed without any checks when \(B_i\) is processed).

ALGORITHM 3. ONE PASS OF THE BIDIRECTIONAL ALGORITHM.

1: \(S \leftarrow \) empty stack of items \((h, \ell, f)\)
2: \(j \leftarrow 0\) \{This records the length of the stream read so far\}
3: \(\text{while} \) stream is not empty \(\text{do}\)
4: \(\text{read next letter } y\), and set \(j \leftarrow j + 1\)
5: \(\text{if } y\ \text{is an upstep} \text{ then}\)
6: \(\text{push the item } (\text{hash}(y), 1, j) \text{ on to } S\) \{This encodes the letter \(y\}\)
7: \(\text{else} \{y\ \text{is a downstep}\}\)
8: \(\text{pop } (h, \ell, f) \text{ from } S\) \{if empty, reject: “extra closing parenthesis”\}
9: \(\text{update } (h, \ell, f) \text{ with } h \leftarrow (h + \text{hash}(y) \mod p); \ell \leftarrow \ell - 1\)
10: \(\text{if } \ell = 0 \text{ then check that } h = 0 \text{ (if not, reject: “mismatch”)}\)
11: \(\text{else push } (h, \ell, f) \text{ on } S\)
12: \(\text{end if}\)
13: \(\text{while the top 2 elements of } S \text{ both start in the last block of the binary partition of } [1, j] \text{ do}\)
14: \(\text{combine them into one element: pop } (h_2, \ell_2, f_2); \text{ pop } (h_1, \ell_1, f_1); \text{ push } (h_1 + h_2, \ell_1 + \ell_2, f_1)\)
15: \(\text{end while}\)
16: \(\text{end while}\)
17: \(\text{if } S \text{ is not empty then reject: “missing closing parenthesis”}\)

**Lemma 3.12.** Let \((j, j')\) be a matching pair for \(x\). Then, either Algorithm 3 rejects before processing \(x_{j'}\), or the stack \(S\) just before processing \(x_{j'}\) is not empty and its topmost item encodes a subword containing \(x_j\).

We now state a simple observation from the definition of matching pairs. Recall from the convention introduced before Definition 3.1 that we identify a subword \(x_{i_1} x_{i_2} \ldots x_{i_m}\) of \(x\) with the set of indices \(\{i_1, i_2, \ldots, i_m\}\) corresponding to it.

**Proposition 3.13.** Let \(v = uu'\) be a subword of \(x\), and let \(d \geq 0\). Then \(u \times u'\) has at most one matching pair at height \(d\). In other words, in \(v\) there exists at most one matching pair \((j, j')\) at height \(d\) such that \(j \in u\) and \(j' \in u'\).

**Proof.** By contradiction, assume that \((i, i')\) and \((j, j')\) are two matching pairs in \(u \times u'\) at height \(d\). For simplicity suppose that \(i < j\). From the definition of matching
pair for \((i, i')\), we get that \(\text{height}(x[1,k]) > \text{height}(x[1,i-1])\) for all \(i \leq k < i'\). Since \((i, i')\) and \((j, j')\) are both at height \(d\), indices \(i, j\) satisfy \(\text{height}(x[1,i-1]) = \text{height}(x[1,j-1]) = d\). Therefore \(j > i'\), leading to \(j \not\in u\), which contradicts that \((j, j')\) is in \(u \times u'\). \(\square\)

We conclude with the correctness of our algorithm.

**Theorem 3.14.** Let \(c > 0\). Algorithm 2 is a bidirectional two-pass randomized streaming algorithm for \(\text{Dyck}(2)\) with space \(O((\log n)²)\) and time \(\text{polylog}(n)\). If the input belongs to \(\text{Dyck}(2)\), then the algorithm accepts it with certainty; otherwise it rejects it with probability at least \(1 - n^{-c}\).

**Proof.** As before, we use well-known results in algorithmic number theory [7, sections 8.2 and 9.7] to compute the prime \(p\) for the hash function. This computation is probabilistic, takes time \(\text{polylog}(n)\), and takes space at most \(O(\log^2 n)\). With probability \(n^{-\gamma}\), for a constant \(\gamma > 0\) of our choice, the number returned may be composite. We first analyze the algorithm assuming the number is prime and discuss the composite case later.

Each stack element takes space \(O(\log n)\) and the stack has size at most \(2k = 2 \log n\), hence space \(O(\log^2 n)\). The processing time is dominated by the computation of the hash function and the compression of stack elements. Each letter read generates at most one new stack item, after which Algorithm 3 may combine the elements on top of the stack (at most \(\log n\) times). The net time is therefore \(\text{polylog}(n)\) per letter.

To analyze the algorithm, observe by induction, and Proposition 3.3, that the algorithm rejects in either direction with probability 1 if \([1, n]\) is not a matching set. A matching set may be ill-formed, and in the rest of the proof we focus on proving that the algorithm detects this with high probability.

The above observation implies that we may assume that \([1, n]\) is a matching set. In particular, it implies that \(S\) is never empty while processing a downstep. Moreover, if the algorithm processes the full stream in one direction, then the stack is empty at the end and Algorithm 3 does not reject.

By Lemma 3.11, each check at line 10 consists of verifying that a matching set is well-formed. Therefore the algorithm always accepts whenever \(x \in \text{Dyck}(2)\).

Consider now the case when \(x\) has an ill-formed matching pair. Let \(i\) be a minimum number such that some \(i\)-block \(B_i\) contains an ill-formed matching pair \((j, j')\). By minimality, \(x_j\) and \(x_{j'}\) are in different \((i-1)\)-blocks \(B\) and \(B'\). Let \(m\) be the minimum, over upsteps \(x_i\) of \(B\), of \(\text{height}(x[1,l-1])\). Let \(m'\) be the minimum, over downsteps \(x_i\) of \(B'\), of \(\text{height}(x[1,l])\) (see Figure 3). Up to swapping left-to-right and right-to-left directions, we may assume that \(m \geq m'\).

Assume that the algorithm does not reject before processing \(x_{j'}\). The stack \(S\) is empty since \([1, n]\) is a matching set. Then, by Lemma 3.12, the topmost element of \(S\) encodes a subword containing \(x_j\). Moreover, since all compressions in \(B\) involve items with the first letter in \(B\), the first letter \(f\) of that word is in \(B\) and hence starts at height \(\geq m\). Since \(m \geq m'\), the letter \(f'\) matching \(f\) is in \(B_i\), and so from Proposition 3.10 by the end of reading \(B'\) that item is discarded. Let \((h, 0, f)\) be that discarded item, encoding a subword \(v\) containing both \(x_j\) and \(x_{j'}\).

Since the first letter \(f\) of \(v\) is in \(B\), all the letters of \(v\) are in \(B \cup B'\). Recall that \(v\) is a matching set, and, by Proposition 3.13, its matching pairs in \(B \times B'\) are all at different heights. So, at the height \(d\) of pair \((j, j')\), \(v\) only contains \((j, j')\), which is ill-formed, plus possibly some matching pairs coming from \(B \times B\) or from \(B' \times B'\), pairs that are all well-formed by minimality of \(i\). Altogether, at height \(d\) the word \(v\) has exactly one ill-formed matching pair, so by Proposition 3.4, the probability that \(v\)
passes the hash test of Algorithm 3 is at most \( n^{-\gamma} \) for a uniformly random choice of \( \alpha \). So the algorithm is correct with probability \( 1 - n^{-\gamma} \).

The algorithm continues to accept streams \( x \in \text{DYCK}(2) \) with certainty even if the modulus used in the hash function is composite. When the stream \( x \not\in \text{DYCK}(2) \), the union bound tells us that the probability that the algorithm does not reject is at most \( 2n^{-\gamma} \). \( \square \)

4. Lower bounds. In this section, we prove a space lower bound for \( \text{DYCK}(2) \). We start with a family of hard instances that we embed in a communication problem \( \text{ASCENSION}(m) \). A streaming algorithm that uses space \( \sigma \) (a function of \( m,n \)) implies a multiparty communication protocol for \( \text{ASCENSION}(m) \) with \( 2m \) players, in which every message has length \( \sigma \). We then appeal to a direct sum argument to derive a two-party communication protocol for \( \text{MOUNTAIN} \) with “low” information cost. Finally, we show that such a protocol is impossible, unless \( \sigma \in \Omega(n) \).

4.1. Reduction from Dyck(2), and an overview. We define the family of hard instances for \( \text{DYCK}(2) \) as follows. For any word \( z \in \{a,b\}^n \), let \( \mathcal{T} \) be the minimal matching word associated with \( z \) (so that \( z\mathcal{T} \) is well-formed). For positive integers \( m,n \), consider the following instances of length in \( \Theta(mn) \):

\[

w = x_1\mathcal{T}_1y_1 x_2\mathcal{T}_2y_2 \cdots x_m\mathcal{T}_my_m\mathcal{T}_m \cdots \mathcal{T}_2 \mathcal{T}_1,
\]

where for every \( i \), \( x_i \in \{a,b\}^n \), \( y_i = x_i[n-k_i+2,n] \) for some \( k_i \in \{1,2,\ldots,n\} \), and \( c_i \in \{\text{a,b}\} \). The word \( w \) is in \( \text{DYCK}(2) \) if and only if, for every \( i \), \( c_i = x_i[n-k_i+1] \).

Intuitively, for \( m = n/\log n \) recognizing \( w \) is difficult with space \( o(n) \). After reading \( x_i \), the streaming algorithm does not have enough space to store information about the bit at unknown index \( (n-k_i+1) \). When it reads \( c_i \) it is therefore unable to decide whether \( c_i = x_i[n-k_i+1] \). Moreover, after reading \( \mathcal{T}_m \) it does not have enough space to store information about all indices \( k_1,k_2,\ldots,k_m \). When it reads \( \mathcal{T}_m \cdots \mathcal{T}_2 \mathcal{T}_1 \) it therefore misses out on its second chance to check whether \( c_i = x_i[n-k_i+1] \) for every \( i \). The formal proof contains several subtleties and is executed in the language of communication complexity.

We define a communication problem \( \text{ASCENSION}(m) \) (see Figure 4) associated with the hard instances described above. For convenience, we replace suffixes by prefixes and identify \( a \) with 0 and \( b \) with 1. Formally, in the problem \( \text{ASCENSION}(m) \) there are \( 2m \) players \( A_1,A_2,\ldots,A_m \) and \( B_1,B_2,\ldots,B_m \). Player \( A_i \) has \( x_i \in \{0,1\}^n \), \( B_i \) has \( k_i \in [n] \), a bit \( c_i \), and the prefix \( x_i[1,k_i-1] \) of \( x_i \). Let \( x = (x_1,x_2,\ldots,x_m) \), \( k = (k_1,k_2,\ldots,k_m) \), and \( c = (c_1,c_2,\ldots,c_m) \). The goal is to compute \( f_m(x,k,c) = \bigvee_{i=1}^m f(x_i,k_i,c_i) = \bigvee_{i=1}^m (x_i[k_i] \oplus c_i) \), which is 0 if \( x_i[k_i] = c_i \) for all \( i \) and 1 otherwise.

Motivated by the streaming model, we require each message to have length at most \( \sigma \) bits, where the parameter \( \sigma \) is a function of \( m \) and \( n \) and corresponds to the space used in the streaming algorithm. We also require the communication between the \( 2m \) participants in a one-pass protocol to be in the following order:

- **Round 1:**
  - For \( i \) from 1 to \( m-1 \), player \( A_i \) sends message \( M_{A_i} \) to \( B_i \), then \( B_i \) sends message \( M_{B_i} \) to \( A_{i+1} \);
  - \( A_m \) sends message \( M_{A_m} \) to \( B_m \);

- **Round 2:**
  - \( B_m \) sends message \( M_{B_m} \) to \( A_m \);
  - For \( i \) from \( m \) down to 2, \( A_i \) sends message \( M_{A_i}' \) to \( A_{i-1} \);
  - \( A_1 \) computes the output.
Problem Ascension \((m)\). The figure presents the \(m\)-fold nesting of streams of the form depicted in Figure 5. The stream is divided between \(2m\) players. There are \(m\) potential mismatches, the \(i\)th one caused by the letter \(c_i\) in \(B_i\)'s input. The word is well-formed if and only if \(c_i = x_i\) for all \(i\).

A streaming algorithm for Dyck(2) with space \(\sigma\) implies a communication protocol for Ascension \((m)\) as described above. So a lower bound on \(\sigma\) follows from a lower bound on the communication complexity of Ascension \((m)\).

To establish the hardness of solving Ascension \((m)\), we prove a direct sum result that captures its relationship to solving \(m\) instances of a “primitive” problem Mountain. In the problem Mountain (see Figure 5), Alice has an \(n\)-bit string \(x \in \{0,1\}^n\), and Bob has an integer \(k \in [n]\), a bit \(c\), and the prefix \(x[1,k-1]\) of \(x\). The goal is to compute the Boolean function \(f(x,k,c) = (x[k] \oplus c)\) which is 0 if \(x[k] = c\) and 1 otherwise. In a one-pass protocol for Mountain, the communication occurs in the following order: Alice sends a message \(M_A\) to Bob, Bob sends a message \(M_B\) to Alice, then Alice outputs \(f(x,k,c)\).

As mentioned in section 1, we follow the “information cost” approach, a method that has been particularly successful in recent works on direct sum results. The method comes in a variety of flavors, each crafted to suit the application at hand. We describe the approach as adapted for Ascension \((m)\). Information cost is often defined in terms of the entire input and the full transcript of the protocol. We enforce the nature of both streaming algorithms and our problem by restricting our attention to only one message \(M_{B_m}\) from the transcript. We also split the input into two parts and measure the information in the message \(M_{B_m}\) about one part \((k,c)\), conditioned on the other part \(x\). In our case, the conditioning corresponds to information that is...
in the hands of the subsequent players. The closest such measures of which we are aware were considered in [28, 9].

The direct sum result is proven using the superadditivity of mutual information for inputs \((k_1, c_i)\) picked independently from a carefully chosen distribution. In the defining information cost, we measure mutual information with respect to a distribution on which the \textsc{Mountain} function is the constant 0, even though we consider protocols for the problem that are correct with high probability in the worst case (or, equivalently, when the inputs are chosen from a “hard” distribution). The use of this easy distribution collapses the function \textsc{Ascension}(m) to an instance of \textsc{Mountain} in any chosen coordinate. We massage this technique into a form that is better suited to the streaming model and to proving lower bounds for the primitive function \textsc{Mountain}.

We finish by giving a combinatorial argument that protocols computing \textsc{Mountain} in the worst case necessarily reveal a lot of information even when its inputs are chosen according to the easy distribution. Privacy loss, a measure similar to information cost, has been studied previously in protocols for \textsc{Index} (see, e.g., [27, 25] and the references therein). Although this communication problem is closely related to \textsc{Mountain}, prior works study \textsc{Index} under hard distributions and do not seem to extend directly to our case.

4.2. Information cost. We now implement the program laid out above. We use standard notions from information theory such as Shannon entropy \(H(A)\), mutual information \(I(A : B)\), and their conditional variants \(H(A|C), I(A : B|C)\), respectively (where \(A, B, C\) are jointly distributed random variables). For a primer on these notions and their properties, we refer the reader to the text [18].

We measure the information cost of a one-pass public coin randomized protocol \(P\) for \textsc{Ascension}(\(m\)) (of the form described in the previous section), with respect to some distribution \(\nu\) by \(IC_{\nu}(P) = I(K, C : M_{\text{asc}}|X, R)\), where \((X, K, C)\) are inputs drawn from \(\nu\), and \(R\) denotes the public coins of \(P\). From this we define the information cost of the problem \textsc{Ascension}(\(m\)) itself with respect to a distribution \(\nu\) and error parameter \(\delta\) as follows: \(IC_{\nu}^\text{pub}(\textsc{Ascension}(m), \delta) = \min IC_{\nu}(P)\), where the minimum is over one-pass public coin randomized protocols \(P\) for the problem, with worst-case error at most \(\delta\). Note that the information cost implicitly depends on \(\sigma\), the length of each message.
For the problem Mountain we play a subtle game between public and private coins. We consider protocols in which Alice has access only to public coins $R$, whereas Bob additionally has access to some independent private coins $R_B$. We define $IC_{\nu}(P) = I(K, C : M_B | X, R)$, where $R$ denotes only the public coins of $P$. Further, we define $IC_{\nu}^{\text{mix}}(\text{Mountain}, \delta) = \min \{ IC_{\nu}(P) \}$, where $P$ ranges over “mixed” public and private coin randomized protocols with worst-case error at most $\delta$ where Alice and Bob share public coins, and only Bob has access to extra private coins.

We also make use of a related measure of complexity for Mountain when $P$ ranges over protocols where Alice’s message is deterministic, and Bob has access to private coins $R_B$: $\text{DIC}_{\nu}^{\text{mix}}(\text{Mountain}, \mu, \delta) = \min \{ IC_{\nu}(P) \}$, i.e., the minimum information cost with respect to $\nu$, where $P$ ranges over protocols for Mountain, in which Alice’s message $M_A$ is deterministic given her input $X$, while Bob may use his private coins $R_B$ to generate his message. Further, the distributional error of $P$ is at most $\delta$ when the inputs are chosen according to $\mu$. Note that in general, and certainly in our application, $\nu$ and $\mu$ may be different, meaning that we measure the information cost of the protocol with respect to some distribution $\nu$, while we measure its error under a potentially different distribution $\mu$. For later use, we recall that the distributional error under $\mu$ is $E_{X,K,C} \sim \mu \left( \Pr(P \text{ fails on } (X, K, C)) \right)$, where the probability is over the private coins $R_B$ of Bob.

We begin by relating the information cost for protocols in which Alice is deterministic to that of mixed randomized protocols. A similar argument for eliminating public randomness is seen in [14, Lemma 3.3].

Lemma 4.1.

$$\text{DIC}_{\nu}^{\text{mix}}(\text{Mountain}, \mu, 2\delta) \leq 2 \times \text{IC}_{\nu}^{\text{mix}}(\text{Mountain}, \delta).$$

Proof. Consider a randomized protocol $P$ for Mountain with worst-case error at most $\delta$ such that $IC_{\nu}^{\text{mix}}(\text{Mountain}, \delta) = IC_{\nu}(P)$. We further assume that Alice and Bob have uniformly distributed public coins $R$, and only Bob has extra private coins $R_B$. Then

$$IC_{\nu}^{\text{mix}}(\text{Mountain}, \delta) = \mathbb{E}_r \left( I(K, C : M_{B_m} | X, R = r) \right).$$

Since $P$ has worst-case error at most $\delta$, it has distributional error at most $\delta$ under $\mu$:

$$\mathbb{E}_r \left( \mathbb{E}_{(X,K,C) \sim \mu} \left( \Pr(P \text{ fails on } (X, K, C)|R = r) \right) \right) \leq \delta.$$

Therefore, by the Markov inequality, there is a set $\mathcal{R}$ with $\Pr(R \in \mathcal{R}) \geq \frac{1}{2}$ such that

$$\forall r \in \mathcal{R}, \mathbb{E}_{(X,K,C) \sim \mu} \left( \Pr(P \text{ fails on } (X, K, C)|R = r) \right) \leq 2\delta.$$

Now consider the information cost of $P$ under the distribution $\nu$ over inputs. Let $U(\mathcal{R})$ denote the uniform distribution on $\mathcal{R}$. We have

$$\mathbb{E}_{r \sim U(\mathcal{R})} \left( I(k, c : M_{B_m} | X, R = r) \right) \leq 2 \times IC_{\nu}^{\text{mix}}(\text{Mountain}, \delta),$$

since the event $\mathcal{R}$ has probability at least $1/2$. Therefore, there exists an $r \in \mathcal{R}$ such that $I(K, C : M_{B_m} | X, R = r) \leq 2 \times IC_{\nu}^{\text{mix}}(\text{Mountain}, \delta)$. Let $P_r$ be the protocol obtained by fixing the public coins used in $P$ to $r$. Then Alice’s message $M_A$ is deterministic. By definition of $\mathcal{R}$, the protocol $P_r$ has distributional error at most $2\delta$ under $\mu$, and $IC_{\nu}(P_r) \leq 2 \times IC_{\nu}^{\text{mix}}(\text{Mountain}, \delta)$.  \qed
4.3. Information cost of MOUNTAIN. As explained before, and formally proved in the next section, the information cost approach entails showing that the MOUNTAIN problem is “hard” even when we restrict our attention to an easy distribution. We prove such a result here.

Let $\mu$ be the distribution over inputs $(x, k, c)$ in which $X$ is a uniformly random $n$-bit string, $K$ is a uniformly random integer in $[n]$, and $C$ is a uniformly random bit. This is a hard distribution for MOUNTAIN (as is implicit in [35, 4]). We consider the information cost of MOUNTAIN under the distribution $\mu_0$ obtained by conditioning $\mu$ on the event that the function value is 0: $\mu_0(x, k, c) = \mu(x, k, c) X[K] = C$.

**Lemma 4.2.** If $\sigma \leq n/100$, then

$$\text{DIC}_{\mu_0}^\text{mix}(\text{MOUNTAIN}, \mu, 1/16n^2) \in \Omega(\log n).$$

**Proof.** Let $P$ be a randomized protocol for MOUNTAIN, where Alice’s message $M_A$ is deterministic, with distributional error at most $1/16n^2$ under the distribution $\mu$, such that $|M_A| \leq n/100$. We prove that $IC_{\mu_0}(P) \in \Omega(\log n)$. In the following, all expressions involving mutual information and entropy are with respect to the distribution $\mu_0$.

By the Markov inequality, there are at least $2^{n-1}$ strings $u$ on which $P$ fails with error at most $1/8n^2$ on average on input $(u, K, C)$, where $(K, C)$ are uniformly distributed. Let $S \subseteq \{0, 1\}^n$ of size at least $2^{n-1}$ be the set of such strings $u$. When $u \in S$, the protocol $P$ has error probability at most $1/4n$ on input $(u, k, c)$ for every $(k, c)$.

Let $\alpha$ be a possible message $M_A$ from Alice to Bob when her inputs range in $S$, and let $S_\alpha = \{u \in S : M_A(u) = \alpha\}$. For every string $v \in S_\alpha$, we bound from below the mutual information of $K$ and $M_B$, the randomized message that Bob sends back to Alice. For this we construct a set $J_v \subseteq [n]$ such that the message distributions $M_k = M_B(\alpha, v[1, k-1], k, v[k])$ for $k \in J_v$ are pairwise well-separated in $\ell_1$ distance. This is in turn established by exhibiting, for each $k \in J_v$, a string $u_k \in S_\alpha$ such that $u_k[1, k-1] = v[1, k-1]$ and $u_k[k] \neq v[k]$. The details follow.

Associate with $S_\alpha$ its dictionary $T$, a 2-rank tree (a tree with either one or two children at any internal node), all of whose nodes except the root are labeled by bits; the root has an empty label. Each string $v$ in $S_\alpha$ is in one-to-one correspondence with a top-down path $\pi$ in $T$ from the root to one of its leaves, where the label of the $(\nu+1)$th node in $\pi$ is $v[\nu]$. We identify $v \in S_\alpha$ with the path $\pi$ in $T$ and refer to this path as $v$.

The tree $T$ has $|S_\alpha|$ leaves, each at depth $n$. For a fixed $v \in S_\alpha$, let $J_v$ be the set of integers $k$ such that the $(k+1)$th node in $\pi$ has out-degree $2$. By construction, for every $k \in J_v$, there exists another string, say, $u_k \in S_\alpha$, such that $u_k[1, k-1] = v[1, k-1]$ and $u_k[k] \neq v[k]$. Set $c_k = v[k]$ for every $k \in [n]$. Then the message distributions satisfy $M_B(\alpha, v[1, k-1], k, v[k]) = M_B(\alpha, u_k[1, k-1], k, c_k)$ for all $k \in J_v$. Let $M_k = M_B(\alpha, v[1, k-1], k, c_k)$. Let $k, k' \in J_v$ be distinct indices such that $k \neq k'$. As $u_{k'}[1, k'-1] = v[1, k'-1]$, the message distribution $M_B(\alpha, u_{k'}[1, k'-1], k, c_{k'})$ on input $(u_{k'}, k, c_{k'})$ equals $M_k$, and also $M_B(\alpha, u_{k'}[1, k'-1], k', c_{k'})$ on input $(u_{k'}, k', c_{k'})$ equals $M_k$. However, $u_{k'}[k] = v[k] = c_k$, so the function evaluates to 0 on input $(u_{k'}, k, c_k)$, and $u_{k'}[k'] \neq v[k'] = c_{k'}$, so the function value is 1 on $(u_{k'}, k', c_{k'})$. The protocol $P$ computes its outputs from $M_k, u_{k'}$ and $M_{k'}, u_{k'}$, respectively, on these instances and errs with probability at most $1/4n$.

We use the above property of the distributions $\{M_k\}$ to bound from below the mutual information of $K$ and the message $M_B$, given $v$. 
Proposition 4.3.

\[ I(K : M_B | X = v) \geq \left( \frac{4|J_v| - n}{4n} \right) \log n - 2. \]

We prove this below.

Next, we observe from the properties of 2-rank trees that the number of strings \( v \in S_\alpha \) for which \(|J_v| = l\) is at most \(2^l\). The number of \( v \) for which \(|J_v| \leq l - 2\) is therefore at most \(2^{l-1}\). Now fix \( l = \log |S_\alpha|\), and note that the proportion of \( v \in S_\alpha \) with \(|J_v| \geq l - 1\) is at least 1/2. Therefore \( \mathbb{E}_{v \sim U(S_\alpha)} |J_v| \geq \frac{l - 1}{2} \).

We now concentrate on the messages \( \alpha \) such that \( \Pr_{X \sim \text{uniform}}(M_A(X) = \alpha | X \in S) \geq 2^{-n/10} \). Then \( l = \log |S_\alpha| \geq n - 1 - n/10 = 0.9n - 1 \), and by Proposition 4.3 for \( n \geq 3\),

\[
\mathbb{E}_{v \sim U(S_\alpha)} (I(K, C : M_B | X = V)) \geq \frac{1}{n} \mathbb{E}_{v \sim U(S_\alpha)} |J_v| - \frac{1}{4} \log n - 2 \\
\geq \frac{l - 1}{2n} - \frac{1}{4} \log n - 2 \\
\geq \frac{0.9n - 2}{2n} - \frac{1}{4} \log n - 2 \\
\geq \frac{10}{n} \log n - 2.
\]

Consider the set \( A \) of messages \( \alpha \) which have probability at most \( 2^{-n/10} \) given \( X \in S \). These messages occur with probability at most \( 2^{n/100}2^{-n/10} = 2^{-9n/10} \), which is negligible. Therefore we conclude that \( I(K, C : M_B | X) \in \Omega(\log n) \).

Proof of Proposition 4.3. Fix a string \( v \) and the corresponding set of indices \( J_v \). Suppose we are given as input a distribution \( M = M_k \) for some \( k \in J_v \). We recover \( k \) using the following procedure II:

1. For each \( k' \in J_v \), simulate Alice’s computation of the output in the protocol \( P \) by setting \( M_B = M \), the distribution given as input to \( P \), and \( X = u_{k'} \).
2. Let \( (D_{k'})_{k' \in J_v} \) be the sequence of outputs Alice generates from the above simulation. Output the largest \( k' \) for which \( D_{k'} = 1 \). This is our guess for \( k \).

On input \( M_k \), the procedure II above generates \( D_k = 1 \), and \( D_{k'} = 0 \) for \( k' > k \), each with probability at least \( 1 - 1/4n \) for any fixed \( k' \geq k \). Therefore, the procedure outputs \( k \) with probability at least \( 3/4 \).

We now argue that the entropy of \( K \) is significantly reduced when given \( M_B, X = v \), under the distribution \( \mu_0 \) (i.e., when \( c_k = v[k] \)). This is equivalent to saying that the mutual information of \( k \) and \( M_B \) is high. When the inputs are picked according to the distribution \( \mu_0 \), we have

\[
I(K, C : M_B | X = v) = H(K | X = v) - H(K | M_B, X = v) \\
= \log n - H(K | M_B, X = v).
\]

We bound from above the conditional entropy \( H(K | M_B, X = v) \). We first separate the values of \( k \not\in J_v \) as follows. Let \( p = |J_v|/n \), and define the Boolean random variable \( L \) as 1 if and only if \( K \in J_v \). We have
By the Fano inequality [18, Theorem 2.11.1, p. 39], we have

\[ H(K|M_B, X = v) \]
\[ = H(K|L|M_B, X = v) \]
\[ = H(L|M_B, X = v) + H(K|M_B, X = v, L) \]
\[ = H(p) + (1 - p)H(K|M_B, X = v, K \notin J_v) \]
\[ + pH(K|M_B, X = v, K \in J_v) \]
\[ \leq 1 + (1 - p)\log n + H(K|M_B, X = v, K \in J_v) \]
\[ \leq 1 + (1 - p)\log n + H(K|K_{II}, X = v, k \in J_v), \]

where \( K_{II} \) is the random variable output by the procedure \( \Pi \). The second equality follows from the chain rule for entropy [18, Theorem 2.2.1, p. 16], and the final step follows from the data processing inequality [18, Theorem 2.8.1, p. 32]. For any fixed \( k \in J_v \), given \( M_k \) the procedure \( \Pi \) computes \( K_{II} = k \) with probability at least 3/4. By the Fano inequality [18, Theorem 2.11.1, p. 39], we have

\[ H(K|K_{II}, X = v, K \in J_v) \leq H \left( \frac{1}{4} \right) + \frac{1}{4} \log(|J_v| - 1) \]
\[ \leq 1 + \frac{1}{4} \log n. \]

By combining Lemmas 4.1 and 4.2 we get the next theorem.

**Theorem 4.4.**

\[ I_{\mu_0}^{\text{mix}}(\text{Mountain}, 1/32n^2) \in \Omega(\log n). \]

**4.4. Reduction from Ascension to Mountain.** We now study the information cost of \( \text{Ascension}(m) \) for the distribution \( \mu_0^n \) over \((\{0,1\}^n \times [n] \times \{0,1\})^m\) for the inputs \( x = (x_1, x_2, \ldots, x_m) \), \( k = (k_1, k_2, \ldots, k_m) \), and \( c = (c_1, c_2, \ldots, c_m) \). We state a direct sum property that relates this cost to that of one instance of \( \text{Mountain} \), and then conclude.

**Lemma 4.5.**

\[ I_{\mu_0}^{\text{pub}}(\text{Ascension}(m), \delta) \geq m \times I_{\mu_0}^{\text{mix}}(\text{Mountain}, \delta). \]

**Proof.** Let \( P \) be a public coin randomized protocol for \( \text{Ascension}(m) \) with worst-case error \( \delta \) such that \( I_{\mu_0}^{\text{pub}}(P) = I_{\mu_0}^{\text{pub}}(\text{Ascension}(m), \delta) \).

From \( P \), we construct the following protocol \( P'_j \) for \( \text{Mountain} \), where \( j \in [n] \). Let \((x, k, c)\) be the input for \( \text{Mountain} \).

1. Alice sets \( A_j \)'s input \( x_j \) to her input \( x \).
2. Bob sets \( B_j \)'s input \((k_j, x_j[1, k_j - 1], c_j)\) to his input \((k, x[1, k - 1], c)\).
3. Alice and Bob generate, using public coins, \((X_i, K_i, C_i)\) according to \( \mu_0 \), independently for all \( i < j \), and \( X_i \) uniformly independently for \( i > j \).
4. Bob generates \( (K_i) \) uniformly independently for \( i > j \), but using his private coins. Then Bob sets \( C_i = X_i[K_i] \) for \( i > j \) (so that \( X_i, K_i, C_i \) are distributed according to \( \mu_0 \), independently for all \( i > j \)).
5. Alice and Bob run the protocol \( P \) by simulating the players \((A_i, B_i)\) as follows:
   (a) Alice runs \( P \) until she generates the message \( M_{A_i} \) from player \( A_j \). She sends this message to Bob.
   (b) Bob continues running \( P \) until he generates the message \( M_{B_m} \) from player \( B_m \). He sends this message to Alice.
(c) Alice completes the rest of the protocol $P$ until the end and produces as output for $P'_j$ the output of player $A_1$ in $P$.

By definition of the distribution $\mu_0$, we have $f(X_i, K_i, C_i) = 0$ for all $i \neq j$. So $f_m(X, K, C) = f(x, k, c)$, and each protocol $P'_j$ computes the function $f$, i.e., solves \textsc{Mountain}, with worst-case error $\delta$.

We prove that $\text{IC}_{\mu_0}^{\text{pub}}(P) = \sum_j \text{IC}_{\mu_0}(P'_j)$, which implies the result, since only Bob uses private coins in $P'_j$.

Let $R$ denote the public coins used in the protocol $P$. By applying the chain rule [18, Theorem 2.5.2, p. 22] to $\text{IC}_{\mu_0}^{\text{pub}}(P)$, we get

$$\text{IC}_{\mu_0}^{\text{pub}}(P) = \text{I}(K, C : {X_B}, R)$$
$$= \sum_j \text{I}(K_j, C_j : {X_B}, {X}, K_1, \ldots, K_{j-1}, C_{j-1}, R).$$

Let $R_j = (R_i, (X_i)_{i \neq j}, (K_i, C_i)_{i < j})$. These are all the public random coins used in the protocol $P'_j$, and any further random coins $(K_i, C_i)_{i > j}$ are used only by Bob. Since for all $j$

$$\text{IC}_{\mu_0}(P'_j) = \text{I}(K_j, C_j : {X_B}, R_j),$$

which is the same as

$$\text{I}(K_j, C_j : {X_B}, X, K_1, \ldots, K_{j-1}, C_{j-1}, R),$$

the direct sum result follows.

We can now conclude a lower bound for \textsc{Ascension}$(m)$.

**Theorem 4.6.** Let $P$ be a public coin randomized protocol for \textsc{Ascension}$(n/\log n)$ with worst-case error probability $1/32n^2$; then $\sigma \in \Omega(n)$.

**Proof.** Let $m = n/\log n$ and $\delta = 1/32n^2$, and let $P$ be a public coin randomized protocol for \textsc{Ascension}$(m)$ with worst-case error probability $\delta$. $\text{IC}_{\mu_0}^{\text{pub}}(P)$ is at most $\sigma$, and by definition $\text{IC}_{\mu_0}^{\text{pub}}(\textsc{Ascension}(m), \delta)$ is less than or equal to $\text{IC}_{\mu_0}^{\text{pub}}(P)$. By Lemma 4.5, we have $\text{IC}_{\mu_0}^{\text{pub}}(\textsc{Ascension}(m), \delta) \geq m \times \text{IC}_{\mu_0}^{\text{mix}}(\textsc{Mountain}, \delta)$. By Theorem 4.4, we get $\text{IC}_{\mu_0}^{\text{mix}}(\textsc{Mountain}, \delta) \in \Omega(\log n)$. Combining yields $\sigma \in \Omega(m \log n) \in \Omega(n)$.

**Corollary 4.7.** Every one-pass randomized streaming algorithm for \textsc{Dyck}(2) with (two-sided) error $O(1/n' \log n')$ uses $\Omega(\sqrt{n'} \log n')$ space, where $n'$ is the input length.

**Proof.** Assume we have a one-pass randomized streaming algorithm for \textsc{Dyck}(2) with (two-sided) error $O(1/n' \log n')$ that uses space $\sigma$, where $n'$ is the input length. Then, by the discussion at the beginning of section 4, there is a public coin randomized protocol for \textsc{Ascension}(n/\log n) with $n \in \Theta(\sqrt{n'} \log n')$ and with worst-case error probability $1/32n^2$. By Theorem 4.6, the messages have length $\Omega(n)$, and therefore the streaming algorithm has space $\Omega(n) = \Omega(\sqrt{n'} \log n')$.

**5. Concluding remarks.** Existing computing infrastructure typically supports unidirectional streams. A question that naturally arises from our work is whether we can achieve the performance of the bidirectional algorithm in Theorem 3.14 by making multiple passes in the same direction. Two sets of authors, Chakrabarti et al. [12, 13] and Jain and Nayak [24], independently and concurrently proved that allowing a larger constant number of passes in the same direction does not help. More precisely,
they showed that for any $T \geq 1$, any unidirectional randomized $T$-pass streaming algorithm that recognizes length $n$ instances of Dyck(2) with a constant probability of error uses space $\Omega(\sqrt{n}/T)$. The lower bound in both works goes via an extension of the reduction from Ascension($m$) to Mountain (cf. section 4.4). When specialized to one-pass algorithms, the above gives us a bound that is a factor of $\sqrt{\log n}$ better than the one in Corollary 4.7 for constant error probability. However, it falls short of optimal (by the same factor) for polynomially small error.

A number of later works have explored applications of the fingerprinting technique in streaming algorithms, the relationship of formal language theory to streaming, or the advantage of bidirectional streams over unidirectional ones. Chakrabarti et al. [12, 13] use fingerprinting in a one-pass streaming algorithm for checking priority queues. They also observe that the algorithms in this article extend to checking stacks, queues, and double-ended queues. Konrad and Magniez [30] show a qualitatively similar dichotomy between one-pass and bidirectional two-pass algorithms as in this article for validating XML documents. In addition, they present an algorithm when access to external memory is available. The multipass lower bound for Dyck(2) described above [12, 24] extends to the problem of checking priority queues. François and Magniez [21] prove a lower bound of $\Omega(\sqrt{n}/T)$ for this problem even in the presence of timestamps, with $T$ passing in the same direction. They complement this with a polylogarithmic space bidirectional algorithm, thus providing another example of a language for which bidirectional streams are exponentially more powerful than unidirectional ones. Krebs, Limaye, and Srinivasan [31] give streaming algorithms for nearly well-parenthesized “one-turn” expressions, and Babu, Limaye, and Varma [6] (see also [5]) study the streaming complexity of subclasses of context-free languages. We expect the ideas in this article to have further such ramifications.

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