Proving Lower bounds using Information Theory

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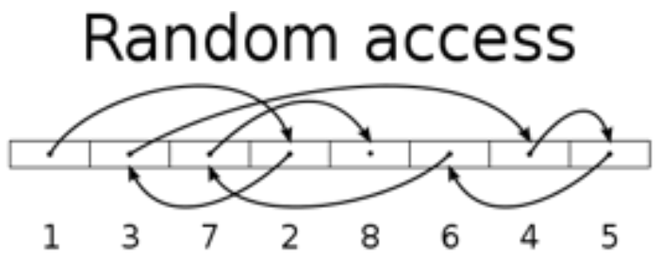
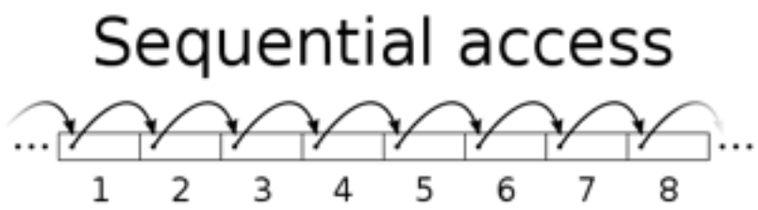
•Massive data sets

- Examples:
  - Network monitoring
  - Genome decoding
  - Web databases
- Impossible to store full input in memory
  - Random access model is not realistic



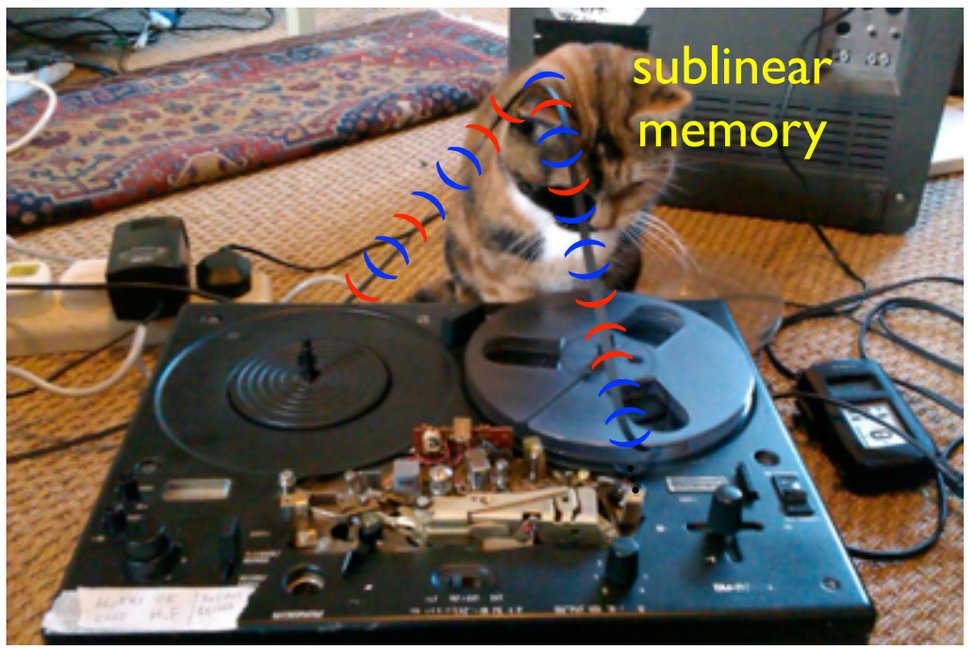
•Data streams

- Sequential access to input in **one** pass



- Prove lower bounds for streaming algorithms

1. Check if all elements are pairwise distinct
2. Decide if there is an element with frequency > 50%
3. Check if a document is well-parenthesized

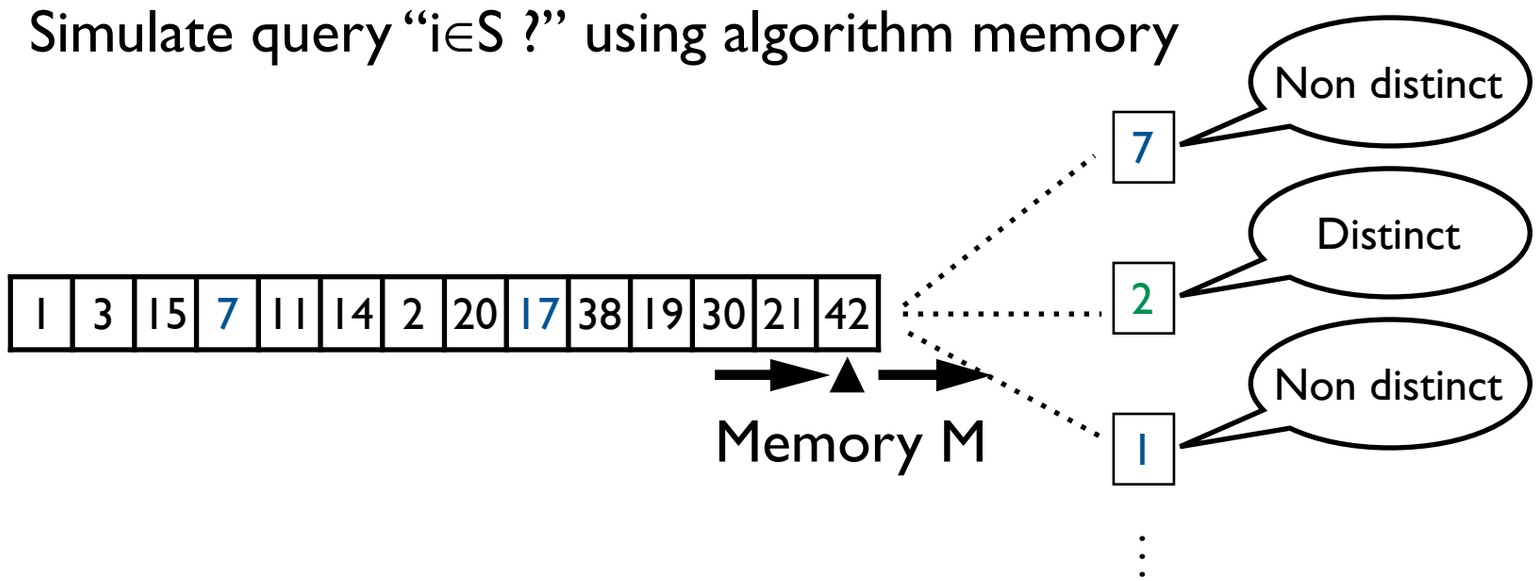


- Theorem
  - For 1-pass randomized streaming algorithms

    Problems 1 & 2 require memory |input|

    Problem 3 requires memory $\sqrt{|\text{input}|}$

- Adversary argument
  - Run streaming algorithm on a long sequence S of pairwise distinct integers
  - Simulate query "i∈S ?" using algorithm memory



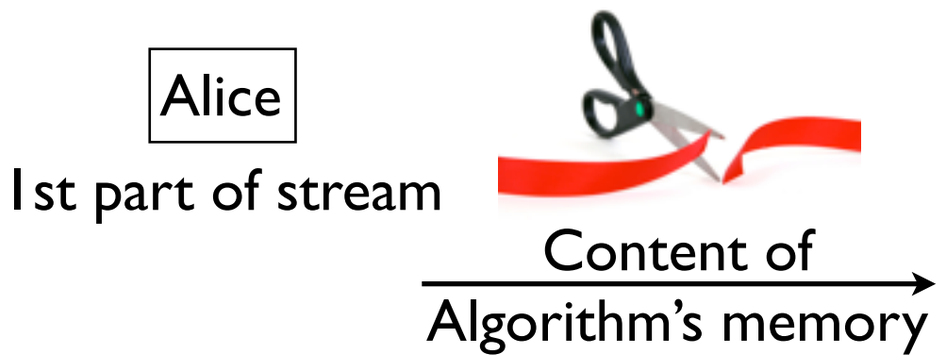
  - Therefore memory should be of same size than S
- Generalization
  - Randomized algorithms
  - Other problems
    - Decide if there is an element with frequency > 50%
    - Check if a document is well-parenthesized

- From algorithm to protocol
  - Build a communication protocol

    Cut the stream in 2 or more pieces, one for each player

    Simulate algorithm on distributed stream

    Message size ↔ Memory space



- From protocol to information theory
  - Lower bound the information in the message of some player

    Correctness of protocol

    → Leakage of information by some player

    → There is a large message

Entropy H(X)

•Definition

- Setting

  Random variables (distributed according to fixed distributions)

  Might be dependent

- Intuition

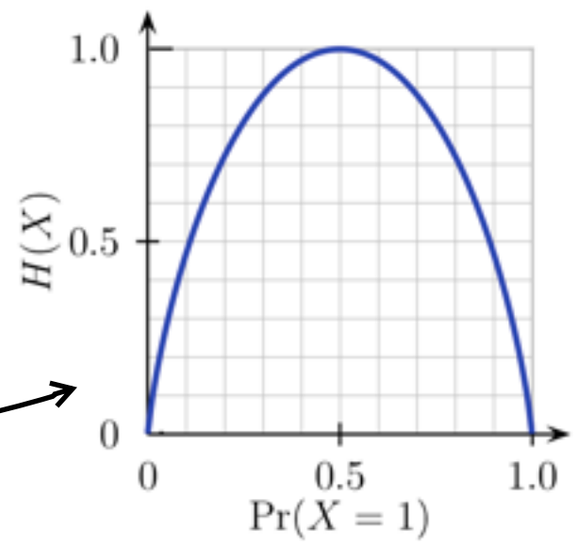
  H(X) = Amount of randomness in X

- Definition (from coding theory)

  H(X) = Average length of the best encoding for X

•Examples

- Uniform distribution on n bits: H(X) = n
- Dirac distribution: H(X) = 0
- Bernoulli distribution



•First principles

- $H(X) \leq H(XY) \leq H(X) + H(Y)$
- $H(X) \leq n$ when X has n bits

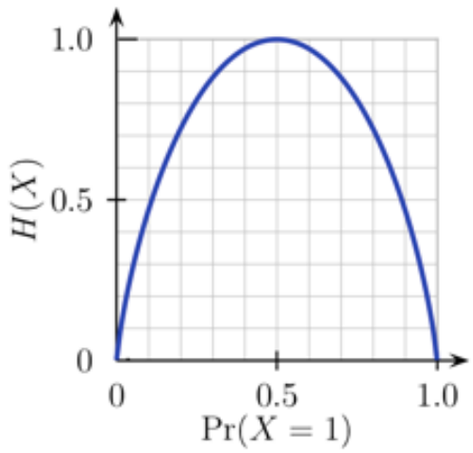
Conditional entropy H(X|Y)

- Definition
  - Intuition

    H(X|Y) = Amount of randomness in X while Y is known
  - Definition

    H(X|Y) = Average length of the best encoding for X, given Y

    $H(X|Y) = E_y\, H(X|_{Y=y})$ where y is distributed according to Y
- Chain rule
  - $H(XY) = H(Y) + H(X|Y)$ $\quad H(X_1X_2...X_n) = \sum_k H(X_k|X_1...X_{k-1})$
- First principles
  - H(X|Y) = 0 when X=Y
  - H(X|Y) = H(X) when X&Y are independent
  - $H(X|Y) \leq H(X)$
- Fano's inequality
  - $H(X|Y) \leq H(\varepsilon)$ when $Pr(X \neq Y) \leq \varepsilon$

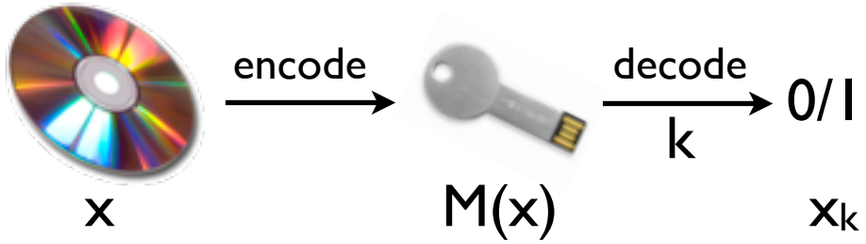


Mutual information I(X:Y)

- Definition
  - Intuition

    I(X:Y) = Amount of information that Y carries on X
  - Definition

    $I(X:Y) = H(X) - H(X|Y)$
- Examples
  - $I(X:Y) = H(X)$ when X=Y
  - $I(X:Y) = 0$ when X&Y are independent
- First principles
  - $I(X:Y) \leq H(X)$
  - $I(X:Y) = H(X) + H(Y) - H(XY) = I(Y:X)$
  - $I(X:YZ) \leq I(X:Y)$
  - $I(X_1X_2...X_n:Y) = \sum_k I(X_k:Y \mid X_1...X_{k-1})$
- Warning
  - $I(X:Y|Z) \geq I(X:Y)$ when X&Z are independent
  - $I(X:Y|Z) \leq I(X:Y)$ when X=Z



•Problem

- Input: n-bit string x
- Goal: Encode x into M(x) such that

Any bit of x can be decoded from M(x) with probability ≥ 1-ε

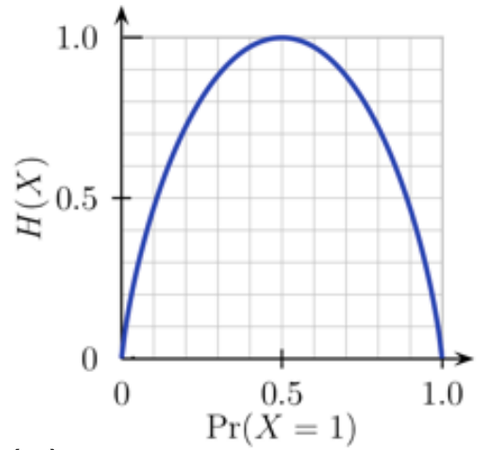
For all x,k: $Pr_{Enc\&Dec}(Decoding(M(x),k) = x_k) \geq 1-\varepsilon$

→ Worst-case error

While minimizing the size of M(x) (# of bits)

•Theorem

- At least (1-H(ε))n of bits are needed



•Proof

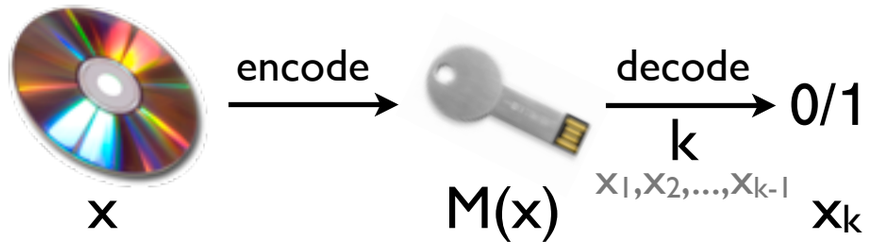
- X: uniform distribution for x, M: resp. distribution for M(x)
- I(M:X) ≤ H(M) ≤ size of the encoding
- I(X:M) ≥ (1-H(ε))n

$I(X:M) = \sum_k I(X_k:M|X_1...X_{k-1}) \geq \sum_k I(X_k:M)$ chain rule & (X_k) independent

$H(X_k) = 1$ and $H(X_k|M) \leq H(\varepsilon)$ Fano's inequality

- $n(1-H(\varepsilon)) \leq I(X:M) = I(M:X) \leq$ size of the encoding

Random Access Code with distributed error



•Extension

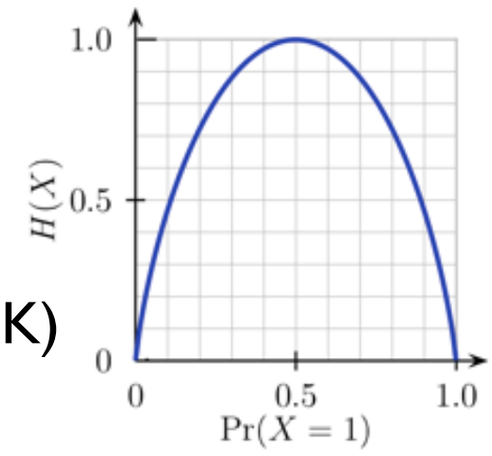
- Random input: uniform n-bit string x
- Goal: Encode x into M(x) such that

$$\Pr_{x,k,\text{Enc\&Dec}}(\text{Decoding}(M(x),k) = x_k) \geq 1-\varepsilon$$

→ Distributed error (over x)

•Theorem

- At least (1-H(ε))n of bits are needed



•Proof

- $I(X:M) = \sum_k I(X_k:M|X_1...X_{k-1}) \leq \sum_k I(X_k:M) = n\, I(X_K:M|K)$

  where K is uniformly distributed in [n]

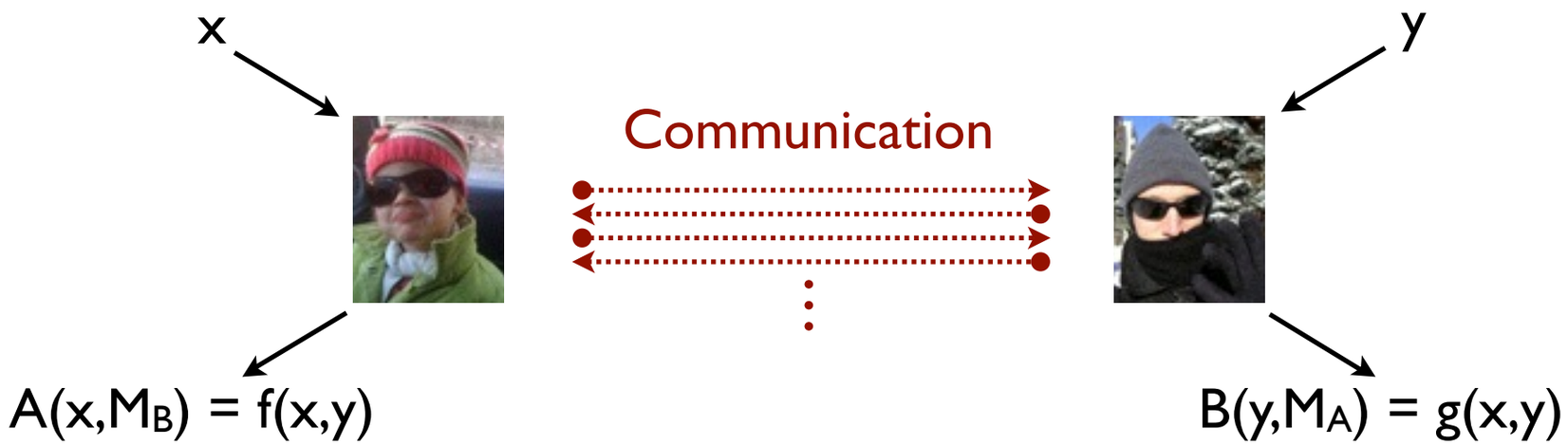
•Remarks

- Same lower bounds if $x_1x_2...x_{k-1}$ are given to the decoder
- For distributed error, best encoding/decoding is deterministic
- Yao's principle

  Encoding size of (RAC with worst-case error)

  ≥ Encoding size of (det-RAC with distributed error)

- 2-player game
  - Alice & Bob wants to solve a common task
    based on their respective inputs
    while minimizing # of exchanged bits



- Powerful tool
  - For stressing limitations of **various** computational models
    Complexity theory (Circuits)
    VLSI design
    Turing machines
    Streaming algorithms

•Problem

- One-way communication

  Only Alice sends a message M_A

- Goal

  Bob wants to learn one bit of x

•Theorem

- M_A has size at least (1-H(ε))n when error probability ≤ ε

•Preuve

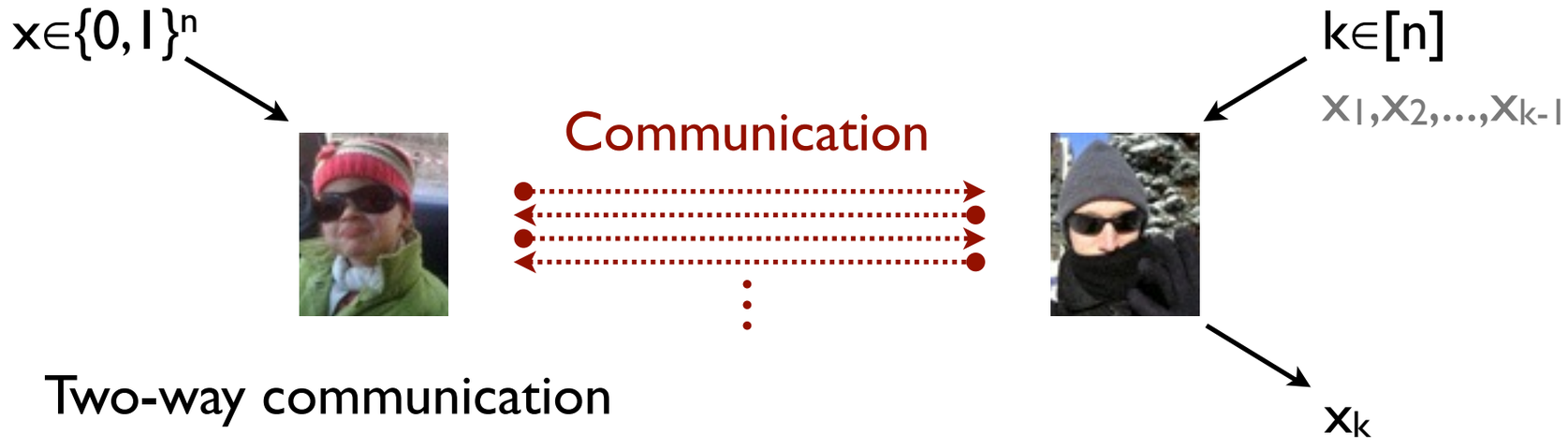
- M_A is a random access code for x

  $I(X:M_A|K) \geq (1-H(\varepsilon))n$

  when X,K are uniformly distributed

INDEX with interaction

- Problem



  - Two-way communication

- Theorem
  - If M_B has size b then M_A has size $\Omega(n/2^b)$ (when error probability ≤ 1/3)

- Proof by round elimination
  - Special case: one round Alice→Bob→Alice, and Alice outputs x_i
  - Alice guesses M_B by flipping b uniform random bits

    Error probability: $\varepsilon = 1-\Omega(1/2^b) \rightarrow H(\varepsilon)= H(1-\varepsilon)= \Omega(1/2^b)$

    One-way protocol → M_A has size $\geq (1-H(\varepsilon))n = \Omega(n/2^b)$

- Theorem
  - If $I(K:M_B|X) \leq b$ then $I(X:M_A|K)=\Omega(n/2^b)$ (when error probability ≤ 1/3)

•Problem

- Two-way communication

•Theorem

- If M_B has size $t \times b$ then M_A has size $\Omega(t \times n/2^b)$ (when error probability $\leq 1/3$)

•Proof

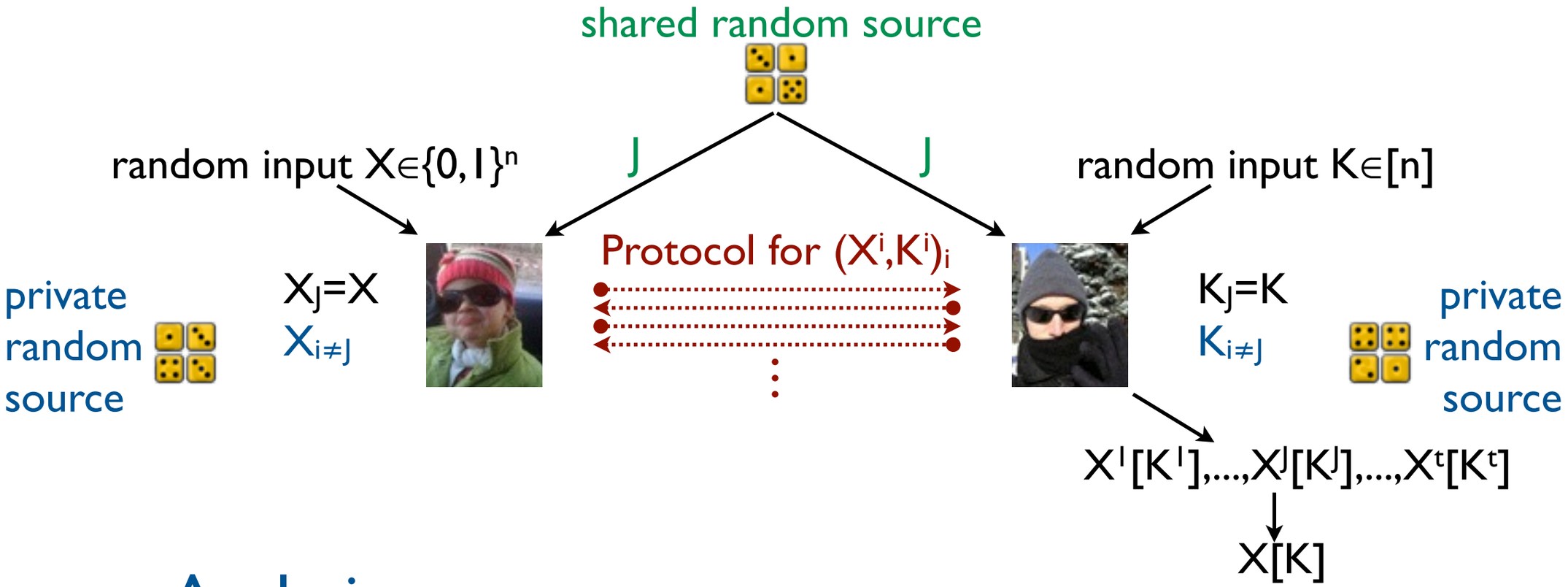
- Using the protocol, A&B can build a protocol for INDEX such that

  $I(X:M_A|K) \leq |M_A|/t$ and $I(K:M_B|X) \leq |M_B|/t \leq b$

- Single instance lower bound

  $\rightarrow |M_A|/t = \Omega(n/2^b)$

- Reduction



- Analysis
  - $|M_A| \geq I((X^i)_i:M_A|(K^i)_i) = \sum_j I(X^j:M_A|X^1...X^{j-1}(K^i)_i)$

    $\geq \sum_j I(X^j:M_A|K^j) = t \times I(X^J:M_A|K^J,J) = t \times I(X:M_A|K,J)$

    where J is the shared randomness
  - $|M_B| \geq t \times I(K:M_B|X,J)$ similarly

•Streaming algorithm for problem P

- Read a stream w of length n
- Solve P on w
- Complexity

  # of passes (1, constant, log)

  memory space (polylog, sublinear)

  processing time per symbol (polylog)

sublinear space
polylog time

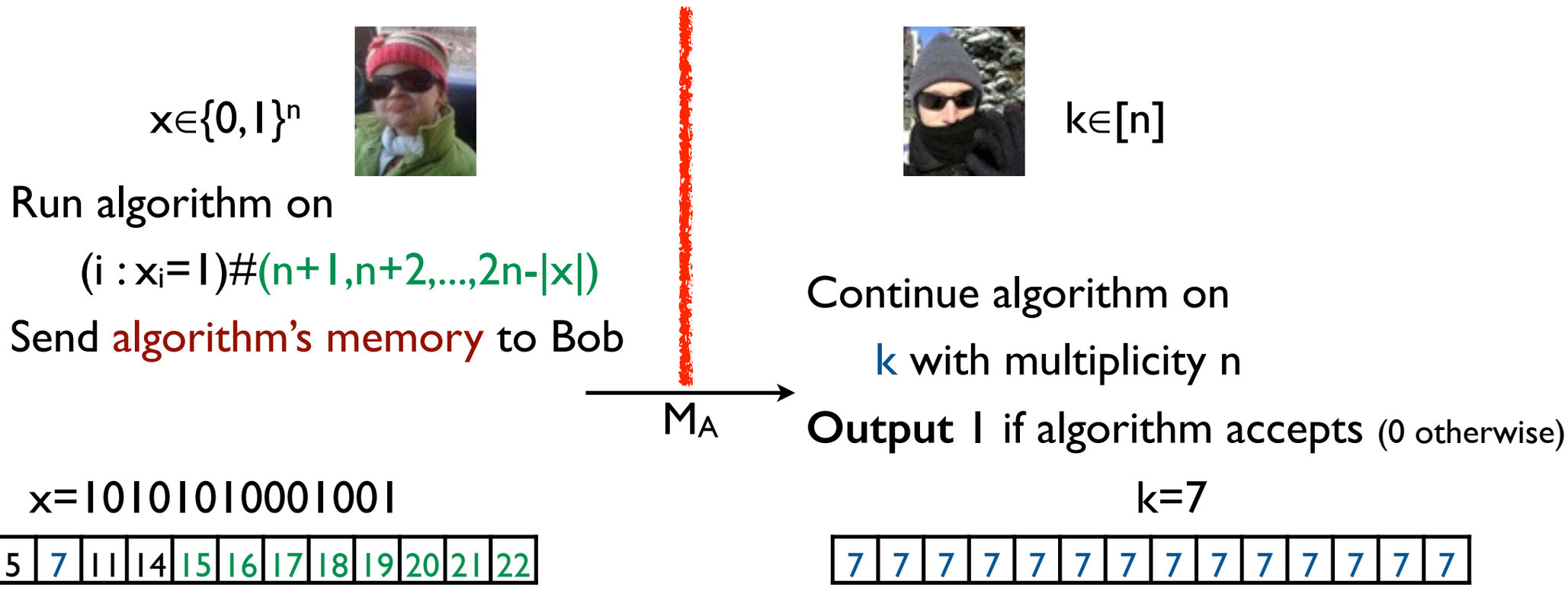
•1-pass deterministic algorithms

- Generalization of automata
- Recognize Regular languages with constant space!

Frequent elements

- Problem
  - Input stream of 2n integers in [2n]
  - Decide if there is an element with frequency > 50%
- Theorem
  - Any 1-pass randomized streaming algorithm requires memory $\Omega(n)$

    (There exists a 2-pass deterministic algorithm with memory $O(\log n)$)
- Proof
  - Use the algorithm to construct a one-way protocol for INDEX



Frequent elements with 2 passes

- 1-pass algorithm
  - c ← 0
  - While stream non empty

    a ← next item

    If c = 0 then v ← a

    If v = a then c ← c + 1

    else c ← c - 1
- Fact
  - If u has frequency > 50%, then v=u

- 2-pass algorithm
  - Run previous algorithm on 1st pass
  - On 2nd pass check that v has frequency > 50%

Dyck languages

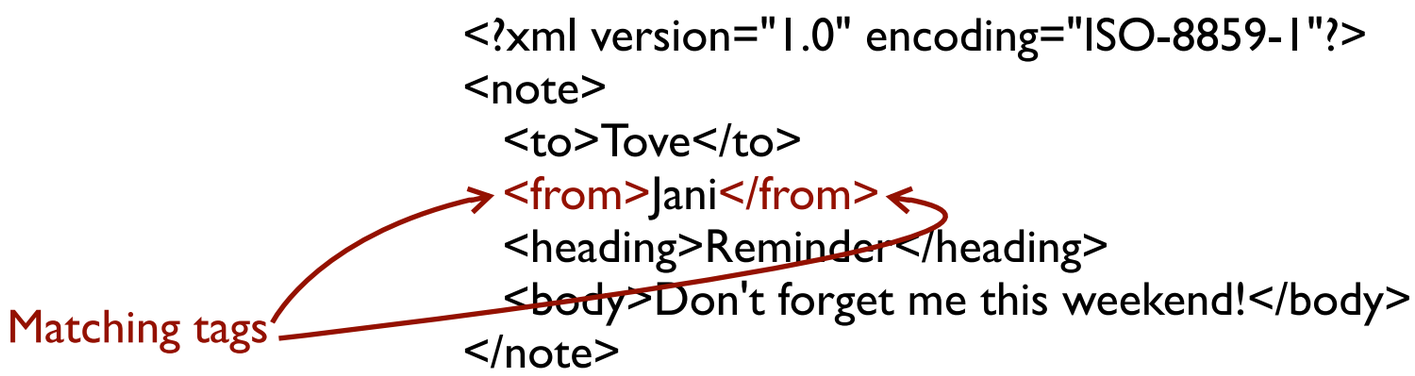
- **Dyck(s)**
  - Well-formed expressions over s types of parentheses
  - Example:

    ((( )) ( )) : **well**-formed ((( )) ( )) : **ill**-formed

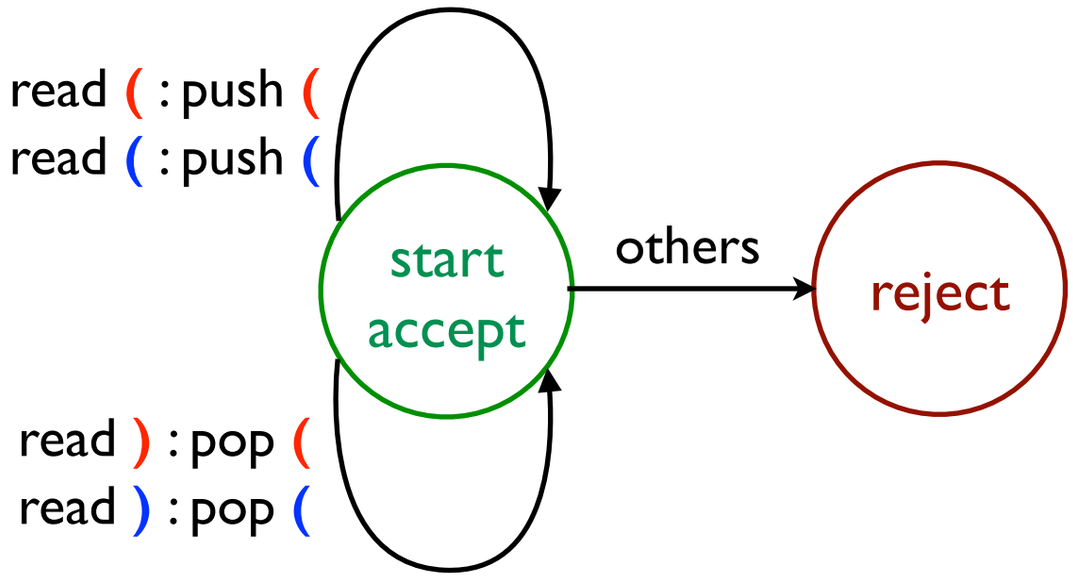
Walther von Dyck

- **Database application**
  - Well-formedness of large XML file

    Check if document is well-parenthesized (opening tags are correctly closed)



•Stack automaton

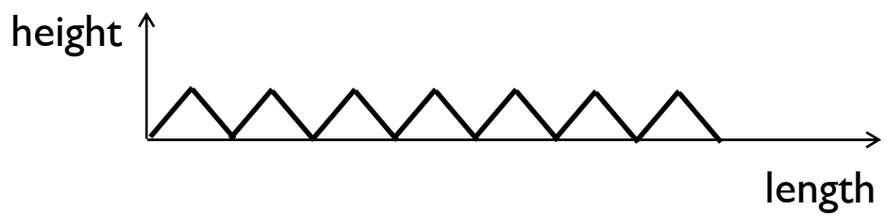


•Simple algorithm

- One-pass algorithm for Dyck(2) with space max height

•Observation

- Small height ⇒ small space



Results for Dyck(s)

n: input size

Fact

- Dyck(1): 1-pass deterministic algorithm with space log n
  - Dyck(2): p-passes deterministic algorithms need space Ω(n/p)
- Dyck(s) is (streaming-)reducible to Dyck(2)
- From now s=2 and alphabet is (,), (,)

•Results for Dyck(2)

- **One pass**

  Randomized algorithm with space √(n log n) and one-sided error 1/poly(n)

  Matching lower bound

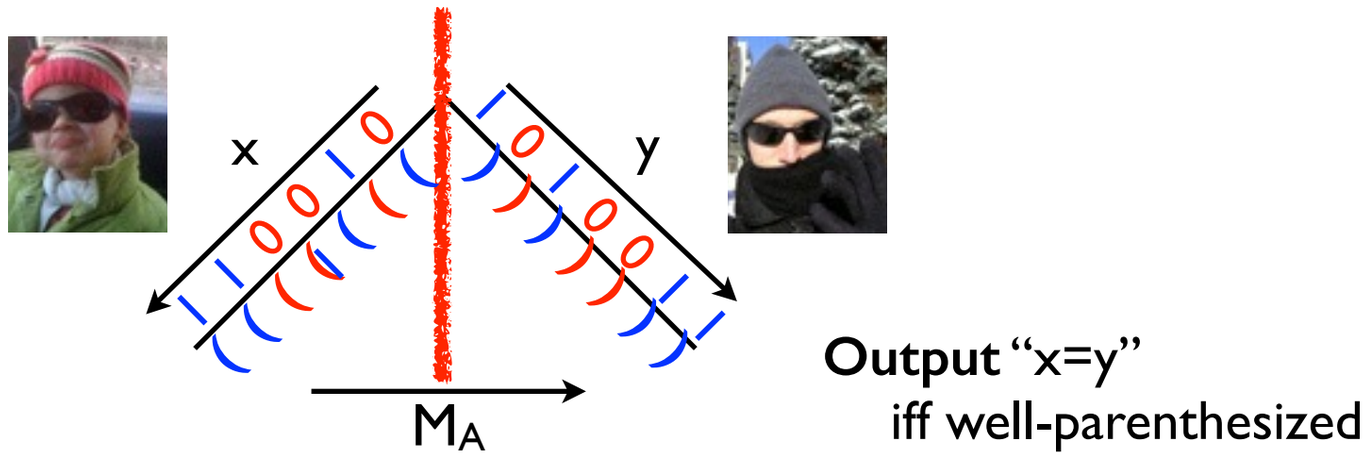
- **More passes in same direction do not help**

  Requires space Ω(√n / p) for p passes

- **Two passes: Forward and backward**

  Randomized algorithm with space $(\log n)^2$ and one-sided error 1/poly(n)

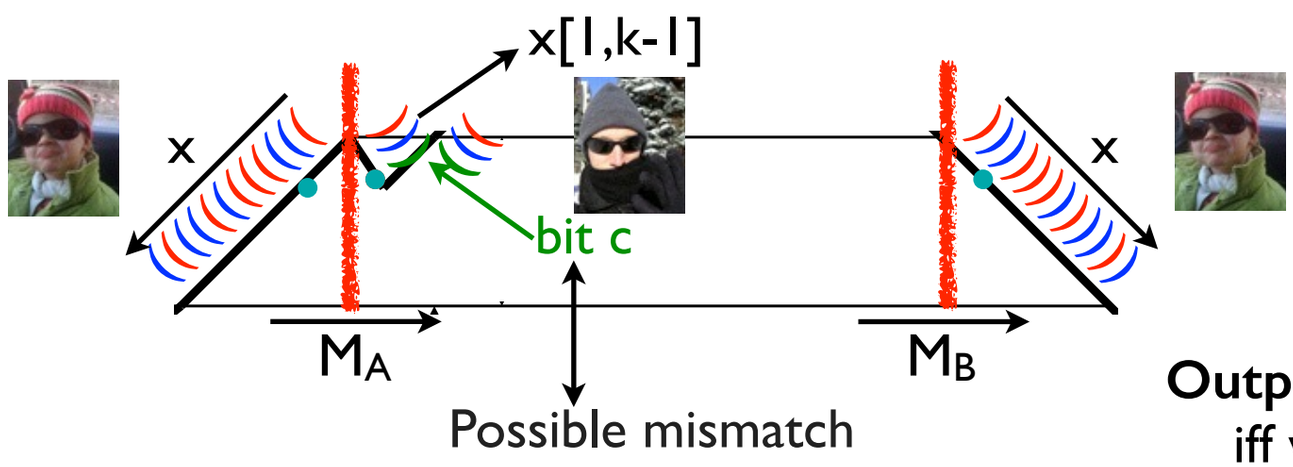
- **Theorem**
  - Any 1-pass deterministic streaming algorithm requires memory $\Omega(n)$
- **Proof**
  - Idea: Use the algorithm to construct a one-way protocol for EQUALITY



- **Lemma**
  - A deterministic protocol for EQUALITY requires a message of linear size
- **Fact**
  - There is a randomized one-way protocol for EQUALITY
    with a message of logarithmic size

Randomized lower bound for Dyck(2): 1st attempt

- Idea
  - Use the algorithm to construct a one-way protocol for Augmented-INDEX



Output "x[k]=c"
iff well-parenthesized

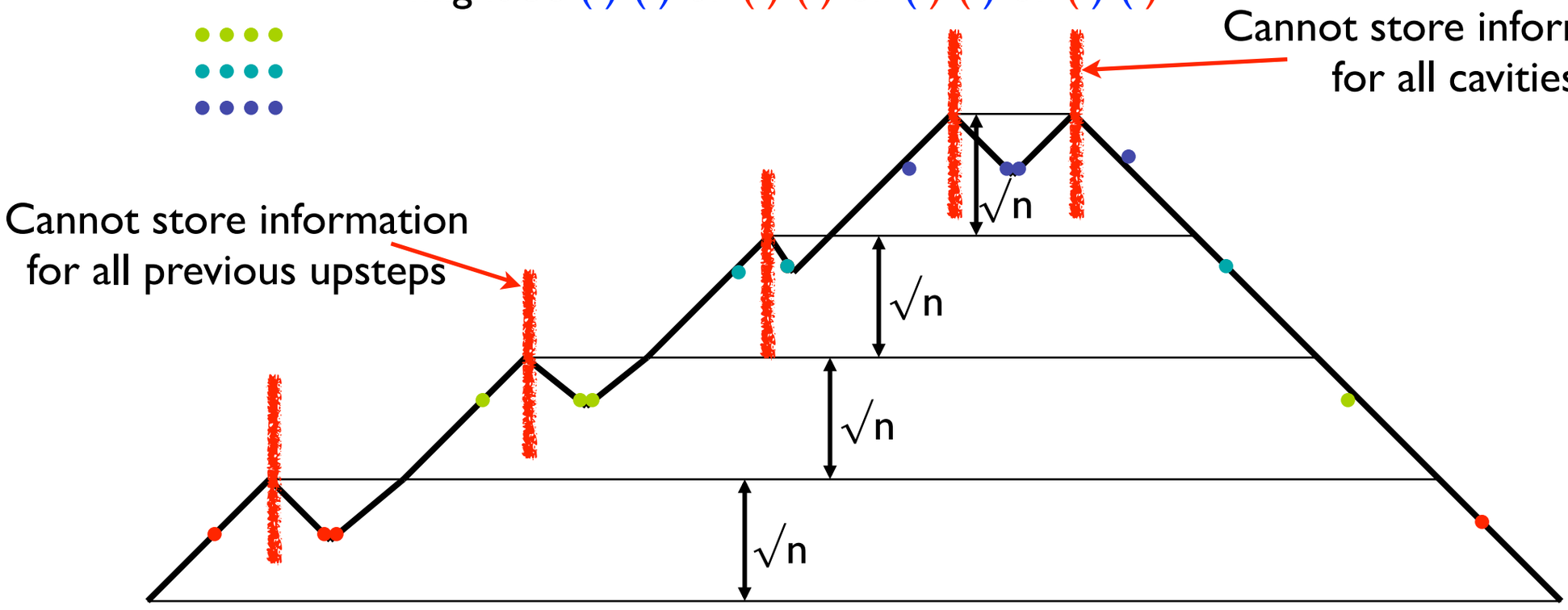
- Lemma
  - Any 1-pass streaming algorithm for Dyck(2) (for input size n)
    induces a two-way protocol for INDEX s.t.
    $|M_A|, |M_B| \leq$ memory space (of algorithm)
- Issue
  - M_B is too large

Hard instances for Dyck(2)

•Hard instance

- String of size Θ(n): $\sqrt{n}$ interleaved slices of size Θ($\sqrt{n}$)
- Maximize the stack size of the 1-pass algorithm
- Forbid any possible simplifications on the stack

might be ( ) ( ) or ( ) ( ) or ( ) ( ) or ( ) ( )

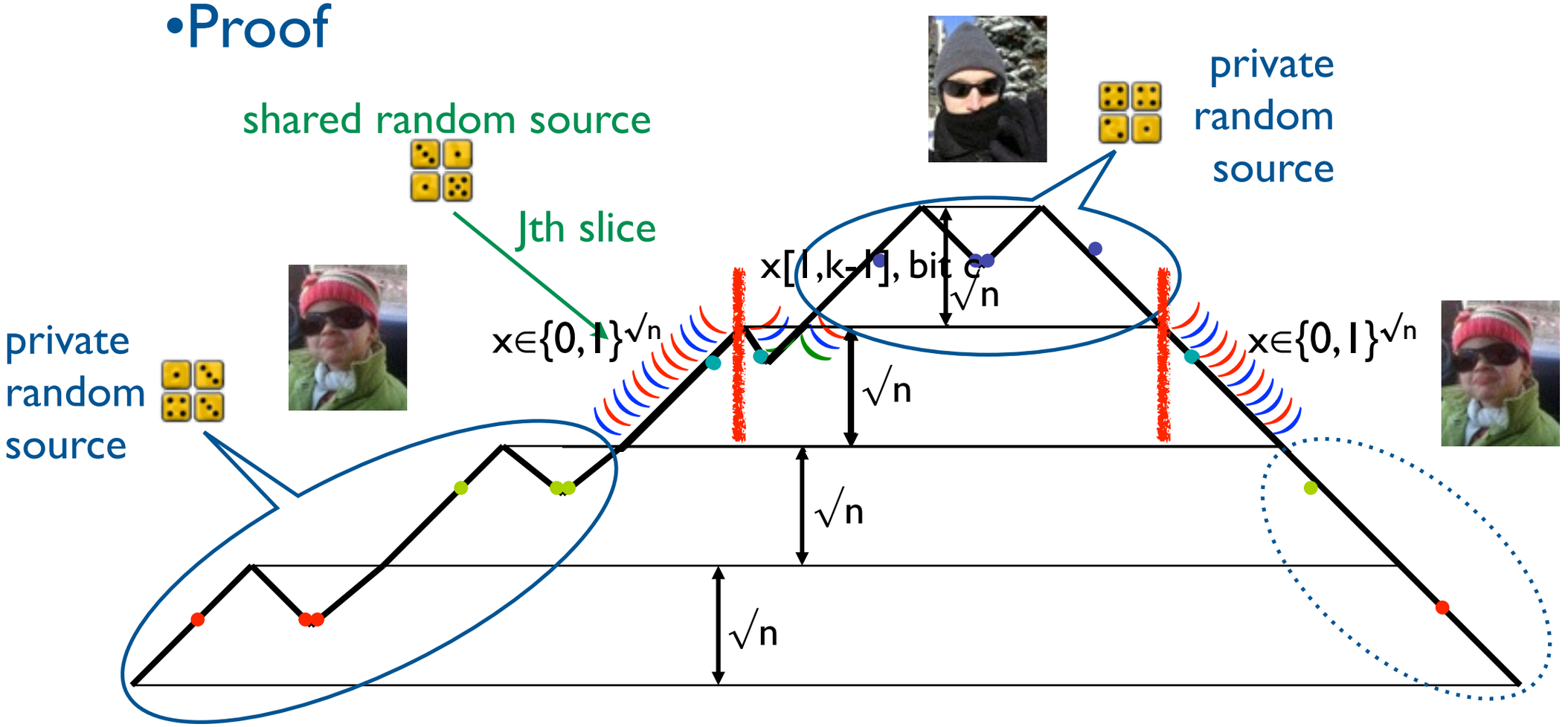


- Theorem
  - Any streaming algorithm for Dyck(2) (for input size n)

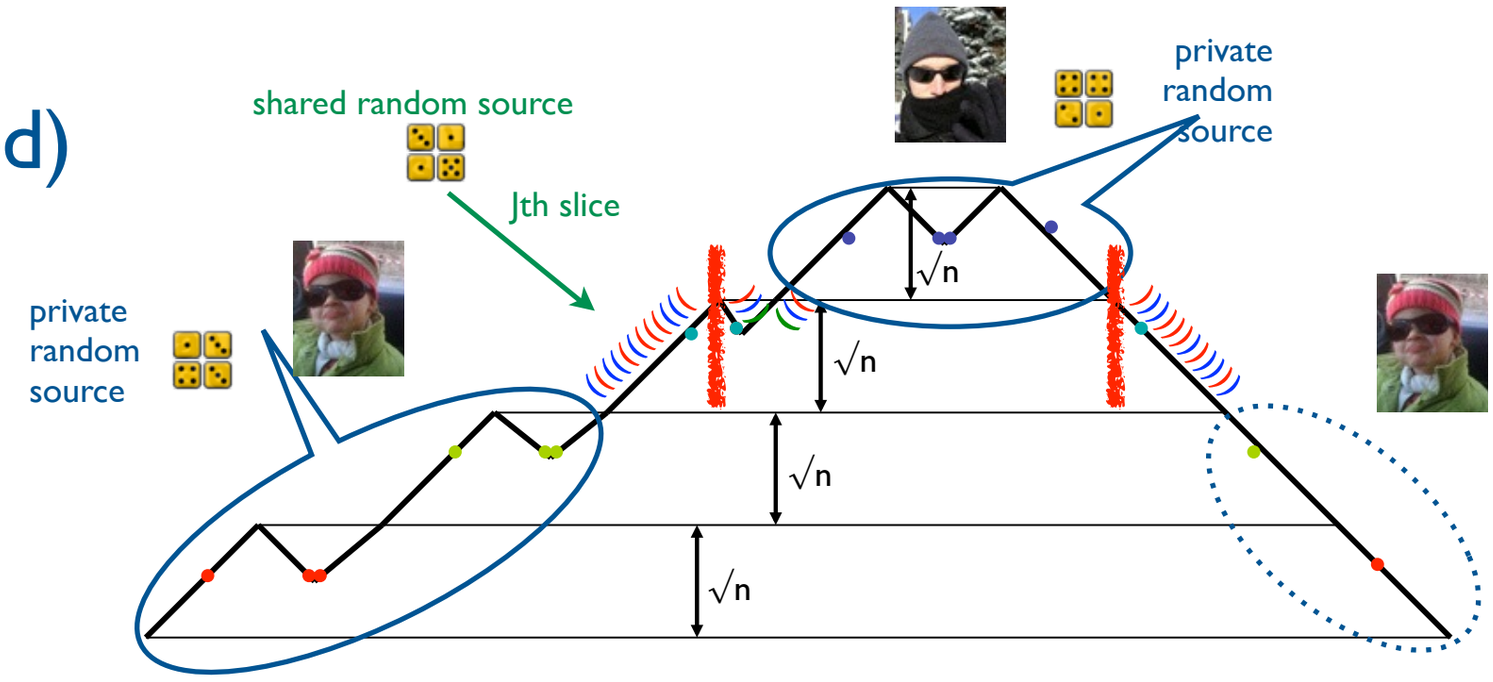
    induces a protocol for Augmented-INDEX (for input size $\sqrt{n}$) s.t.

    $|M_A|, |M_B| \leq$ memory space (of algorithm)

    $I(K:M_B|X) \leq$ (memory space) / $\sqrt{n}$
- Proof



•Proof (continued)



- Requirement:

  FULL expression is well-parenthized iff SLICE is well-parenthesized

  → No error for other slices

  → Collapsing distribution (for Aug-INDEX): uniform (x,k) with c=x[k]

  - $|M_A|, |M_B| \leq$ memory space (of algorithm)
- On the collapsing distribution

  $|M_B| \geq I((K^i)_i : M_B | (X^i)_i) = \sum_j I(K^j : M_B | K^1 \ldots K^{j-1} (X^i)_i)$

  $\geq \sum_j I(K^j : M_B | X^j) = t \times I(K^J : M_B | X^J, J) = t \times I(K : M_B | X, J)$

Issue

- Need to extend INDEX lower bound for collapsing distribution

Randomized lower bound for Dyck(2)

- **Theorem**
  - Every streaming algorithm for Dyck(2) (for input size n)

    induces a protocol for Augmented-INDEX (for input size $\sqrt{n}$) s.t.

    $|M_A|, |M_B| \leq$ memory space (of algorithm)

    $I(K:M_B|X) \leq$ (memory space) / $\sqrt{n}$

    for the collapsing distribution

- **Lemma**
  - Any protocol for Augmented-INDEX satisfies

    If $I(K:M_B|X) \leq b$ then $I(X:M_A|K)=\Omega(|X|/2^b)$ (when error probability $\leq 1/3$)

    for the collapsing distribution

- **Corollary**
  - Every streaming algorithm for Dyck(2) (for input size n)

    requires memory space $\Omega(\sqrt{n})$

References (selection)

•Index

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