Introduction to Quantum Computing

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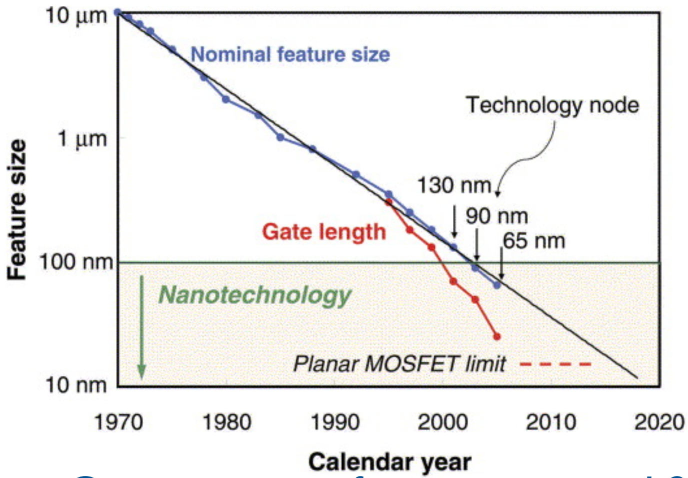
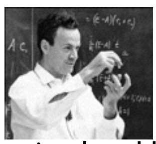
INF554 - lectures 8 & 9

End of Moore's Law?



"No exponential is forever. Your job is to delay forever.", Andrew Gordon Moore Feb. 2003.

Quantum interferences around 2020...



- Current approach: avoid them
- Quantum computing: get benefit of them!

  Feynman'81: "Can quantum systems be probabilistically simulated by a classical computer? [...] the answer is certainly, No!"

  Deutsch'85: Universal quantum Turing machine

The superiority of Quantum Computing

Cryptography

- Secrete Key Distribution Protocol [Bennett, Brassard'84]
  - Implementation: ~100 km

Information Theory

- EPR Paradox [Einstein, Podolsky, Rosen'35]
  - Realization: 1982 [Orsay]
- Teleportation [Bennett, Brassard, Crépeau, Jozsa, Peres, Wootters'93]
  - Realization: 1997 [Innsbruck]

Algorithms

- Polynomial algorithm for Period Finding [Simon, Shor'94]
  - ⇒ Factorization, Discrete Logarithm
- Quadratic speedup for Database Search [Grover'96]
- Quantum computer?
  - 1995: 2-qubit [ENS], 2000: 5-qubit [IBM], 2006: 12-qubit [Waterloo

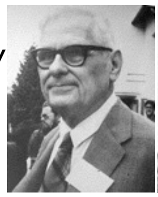
Quantum proofs for classical theorems

- http://arxiv.org/abs/0910.3376 [Drucker, de Wolf'09]

Computing?

Formal concepts

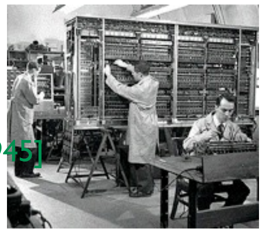
- Model of computation
  - What is a machine, a program?
  - Mathematical model of a computer?
- Hardness of a problem
  - Calculable / Non-calculable
  - Easy / Hard
  - [Turing 1936]: Turing machine, calculability, universality



Church-Turing theses

- Weak version
  - Any *reasonable* model of computation can be simulated on a Turing machine
    - reasonable: physically realizable
    - Turing machine ≈ today computer
- Strong version
  - Any reasonable model of computation can be *efficiently* simulated on a *probabilistic* Turing machine
    - efficiently: using same amount of resources (time and space)

Computers?



Classical computing

- Turing machine, calculability, universality [Turing 1936]
- Proposition: EDVAC (Electronic Discrete VAriable Computer) [von Neumann 1945]
- First computer: Mark I [Robinson-Tootill-Williams 1949]

Quantum computing

- Idea: simulation of quantum systems [Feynman 1982]
- Turing machine, calculability, universality [Deutsch 1985, 1989][Bernstein-Vazirani 1993], circuits [Yao 1993], cellular automata, finite automata...
- Technology: 2-qubit [1995], 5-qubit [2000], 12-qubit [2006]

Validity of Church-Turing theses

- Weak version is still valid
  - Calculability: quantum and classical computation have same power
- Strong version *could* be violated
  - Complexity: evidences that quantum computers can be exponentially faster than classical computers

In this talk

1 qubit

- Definition
- Quantum key distribution

2 qubits

- Definition
- EPR Paradox and applications

Algorithms

- Toward factorization
  - Quantum Fourier transform
  - Applications
- Generalization
- Grover algorithm

Conclusion

Qubit state

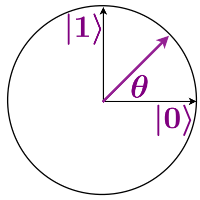
Logical bit

- Deterministic element: $b \in \{0,1\}$

Probabilistic bit

- Probabilistic distribution: $d = \begin{pmatrix} p \\ q \end{pmatrix}$ $p, q \in [0,1]$, $p + q = 1$

Quantum bit (qubit)



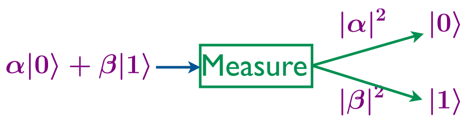
- State: 2-dimensional unit vector

$$|\psi\rangle = \cos\theta|0\rangle + \sin\theta|1\rangle$$

general case (complex amplitudes):

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha|0\rangle + \beta|1\rangle, \quad |\alpha|^2 + |\beta|^2 = 1$$

- Measure: randomized orthogonal projection



Qubit evolution

Logical bit

- Function: $f : \{0,1\} \longrightarrow \{0,1\}, \quad b \mapsto f(b)$

Probabilistic bit

- Stochastic matrix:

$$P = \begin{pmatrix} p & p' \\ q & q' \end{pmatrix}, \quad d \mapsto d' = Pd$$

Quantum bit

- Evolution: unitary transformation $G \in \mathcal{U}(2)$ ($\Rightarrow$ reversible)

Definition: $G \in \mathbb{C}^{2\times 2}$ s.t. $G^*G = \mathrm{Id}$

$$|\psi\rangle \dashrightarrow \boxed{G} \dashrightarrow |\psi'\rangle = G|\psi\rangle$$

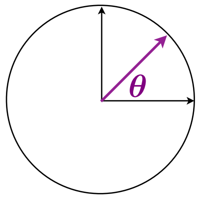
$$|\psi'\rangle = G|\psi\rangle \dashrightarrow \boxed{G^*} \dashrightarrow |\psi\rangle$$

Polarization of photons

State

- Polarization: 2-dimensional vector

  $|\theta\rangle = \cos\theta|\rightarrow\rangle + \sin\theta|\uparrow\rangle$

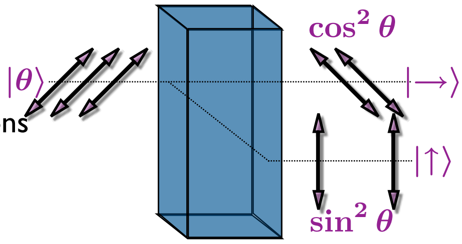


Measure

- Calcite crystal

  separates horizontal and vertical polarizations

  STOP A measure modifies the system

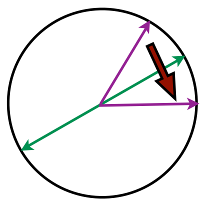


Transformation

- Well known transformation: half-wave blade

  orthogonal symmetry around its axis

- Any rotations (possibly with complex angles)



Examples of transformations

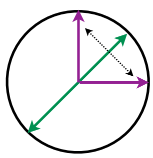
Reversible classical transformation

- Identity

  $|b\rangle$ - - - - - Id - - - - - $|b\rangle$

- Negation

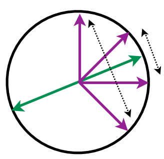
  $|b\rangle$ - - - - - NOT - - - - - $|1-b\rangle$



Hadamard transformation

- Definition: half-wave blade at 22,5° $H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$

  $|b\rangle$ - - - - - $\boxed{H}$ - - - - - $\frac{1}{\sqrt{2}}(|0\rangle + (-1)^b|1\rangle)$



- Properties: quantum coin flipping

  $|0\rangle \rightarrow \boxed{H} \rightarrow \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \rightarrow \boxed{\text{Measure}} \begin{cases} \xrightarrow{\frac{1}{2}} |0\rangle \\ \xrightarrow{\frac{1}{2}} |1\rangle \end{cases}$

  $|b\rangle \rightarrow \boxed{H} \rightarrow \boxed{H} \rightarrow \boxed{\text{Measure}} \longrightarrow |b\rangle$

  STOP Measure does not commute!

Quantum key distribution

Problem

- Setting

  No prior shared secret information between Alice and Bob

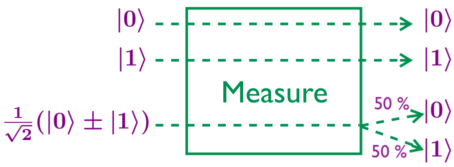
  Authenticated classical channel
- Goal: Get a private key between Alice and Bob

Classical results

- **Impossible**, since all the information is in the canal
- However, one **can** (using randomized techniques):

  Amplify the privacy of an imperfect private key by shortening it

Incertitude in the measure



Impossibility of cloning

- Impossibility of duplicating an unknown state
- Proof based on the linearity of quantum transformations

Main idea of quantum key distribution

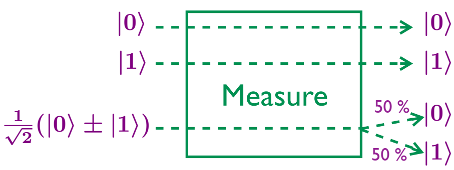
Primitive

- Alice choses 2 random bits *a,c*
- Alice creates and sends to Bob qubit $H^c|a\rangle$
- Bob gets qubit from $|\psi\rangle$ Alice
- Bob choses 1 random bit *d*
- Bob measures $H^d|\psi\rangle$ and gets bit *b*

*H*²=Id

Facts

- c=d → b=a with **probability 1**
- c≠d → b=a with **probability 1/2**



Reconciliation

- Alice & Bob exchange their value *c,d*

Remarks

- If *c=d*, Alice & Bob know *a=b without revealing a,b*
  - "*without revealing*" can be formalized...

The protocol BB84 [Bennett-Brassard 84]

Protocol: quantum part

Key:	0	1	1	0	0	1	0	1	1	1	0
Encoding:		*H*			*H*			*H*	*H*		*H*
Qubit:	↔	↘	↕	↔	↗	↕	↔	↘	↘	↕	↗

Decoding:	*H*		*H*			*H*		*H*			*H*
Qubit:	↗	↘	↘	↔	↗	↘	↔	↕	↘	↕	↔
Key:	1	1	0	0	1	1	0	1	0	1	0

Protocol: classical part

- Reconciliation: Alice and Bob publicly announce their coding choices
  A&B only keep key bits with same choices (prob. 1/2)
  If no third party observes communication, then A&B get same key
- Security: A&B check few key bits at random positions
- Secret amplification using with few other more key bits

Conclusion

- Key generation without any prior shared secret information but using an authenticated classical channel
- Small initial private key → large private key

Preliminaries: Tensor product

Vector spaces

- V,W: vector spaces
- $V\otimes W$ is the free vector space Span ($v\otimes w : v\in V, w\in W$)
  with equivalence relations

$$(v_1+v_2)\otimes w = v_1\otimes w + v_2\otimes w$$

$$v\otimes (w_1+w_2) = v\otimes w_1 + v\otimes w_2$$

$$(c\cdot v)\otimes w = v\otimes(c\cdot w) = c\cdot(v\otimes w)$$

Linear maps

- $S: V\to X,\quad T: W\to Y$: linear maps
- $S\otimes T : V\otimes W\to X\otimes Y$ is the linear map satisfying

$$S\otimes T\ (v\otimes w) = S(v)\otimes T(w)$$

(and extended by linearity)

Applications

- Joint probability distributions on spaces V,W

$$\mathcal{D}(V\times W) = \mathcal{D}(V)\otimes\mathcal{D}(W) \neq \mathcal{D}(V)\times\mathcal{D}(W)$$ (: product distributions)

Definition

- $|\psi\rangle \in \mathbb{C}^{\{0,1\}^n}$ such that $\| |\psi\rangle \| = 1$

$$|\psi\rangle = \sum_{x\in\{0,1\}^n} \alpha_x |x\rangle \qquad \text{with} \quad \sum_{x\in\{0,1\}^n} |\alpha_x|^2 = 1$$

STOP $\mathbb{C}^{\{0,1\}^2} = \mathbb{C}^{\{0,1\}} \otimes \mathbb{C}^{\{0,1\}} \neq \mathbb{C}^{\{0,1\}} \times \mathbb{C}^{\{0,1\}}$

Examples: $\frac{|00\rangle+|01\rangle}{\sqrt{2}} = |0\rangle \otimes \frac{|0\rangle+|1\rangle}{\sqrt{2}}$

$\frac{|00\rangle+|11\rangle}{\sqrt{2}} \neq |\psi_1\rangle \otimes |\psi_2\rangle$

Unitary transformations: $G \in \mathcal{U}(2^n)$

$G \in \mathbb{C}^{2^n \times 2^n}$ s.t. $G^* G = \mathrm{Id}$

$$|\psi\rangle \dashrightarrow \boxed{G} \dashrightarrow |\psi'\rangle = G|\psi\rangle$$

Measure

$$\sum_{x\in\{0,1\}^n} \alpha_x |x\rangle \dashrightarrow \boxed{\text{Measure}} \xrightarrow{|\alpha_x|^2} |x\rangle$$

Definition

$\text{c–NOT}|0b\rangle = |0b\rangle$

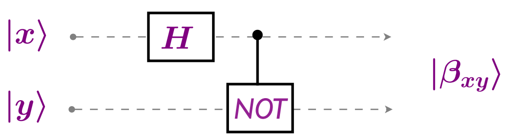
$\text{c–NOT}|1b\rangle = |1\rangle|(1-b)\rangle$

$\text{c–NOT}|ab\rangle = |a\rangle|a \oplus b\rangle$

$$\text{c–NOT} = \begin{pmatrix} 1000 \\ 0100 \\ 0001 \\ 0010 \end{pmatrix}$$

Representation

Bell basis change



$$|\beta_{00}\rangle = \tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$
$$|\beta_{01}\rangle = \tfrac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$
$$|\beta_{10}\rangle = \tfrac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$
$$|\beta_{11}\rangle = \tfrac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

Measure of first qubit

- Projectors $P_0 = |00\rangle\langle 00| + |01\rangle\langle 01| = |0\rangle\langle 0| \otimes \mathbf{I}_2$

$$P_1 = |10\rangle\langle 10| + |11\rangle\langle 11| = |1\rangle\langle 1| \otimes \mathbf{I}_2$$

$$P_0 \overset{\perp}{\oplus} P_1 = Id$$

- Measure of first qubit

$$|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle \dashrightarrow \boxed{\text{Measure 1}}$$

$$\xrightarrow{||P_0|\psi\rangle||^2 = a^2+b^2} \frac{1}{||P_0|\psi\rangle||} P_0|\psi\rangle = |0\rangle \frac{a|0\rangle + b|1\rangle}{\sqrt{a^2+b^2}}$$

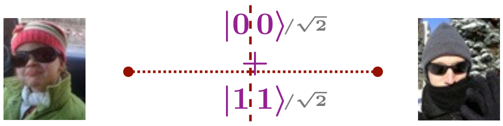
$$\xrightarrow{||P_1|\psi\rangle||^2 = c^2+d^2} \frac{1}{||P_1|\psi\rangle||} P_1|\psi\rangle = |1\rangle \frac{c|0\rangle + d|1\rangle}{\sqrt{c^2+d^2}}$$

Interpretation

- Partial measure project to a subspace compatible with the observation
  - Probability = square norm of the projection
  - Outcome = renormalization of the projection

Protocol

- Assume Alice & Bob shares an EPR state: $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$
  - Alice has the first qubit, and Bob the second one



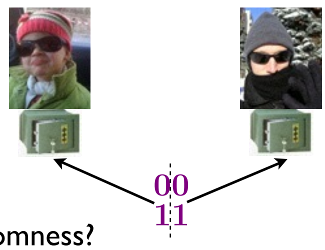
- Alice & Bob observe their qubit and respectively get bit *a,b*

Fact

- *a*=*b* with probability 1
- *a* (resp. *b*) is a uniform random bit

Classical analogue?

- Shared randomness model:
  - Alice and Bob has access to shared random bits
  - → Non product distribution:
    - 00 with prob. 1/2 and 11 with prob. 1/2
- Can we simulate quantum physic using shared randomness?



Bell-CHSH inequality as a classical game

Game

- Alain and Bob share some initial information but cannot communicate
- Alain receives a random bit x, Bob y
- Alain returns a bit a, Bob b

- Goal: maximize $p = \Pr_{x,y}(a \oplus b = x \wedge y)$

$\wedge$	0	1
0	0	0
1	0	1

$\oplus$	0	1
0	0	1
1	1	0

Classically: CHSH inequality [1969]

- Best deterministic strategy: $a = b = 0 \implies p = \frac{3}{4}$
- Theorem: the best probabilistic strategy is not better than the best deterministic strategy

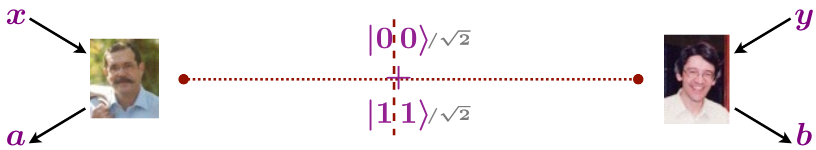
Bell-CHSH inequality as a quantum game

Reminder

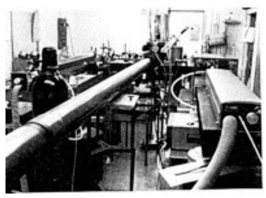
- Goal: maximize $p = \Pr_{x,y}(a \oplus b = x \wedge y)$

Quantumly

- Alain and Bob share an EPR state



- Bob performs a rotation of angle $\frac{\pi}{8}$
- If $x = 1$, Alain performs a rotation of angle $\frac{\pi}{4}$
- If $y = 1$, Bob performs a rotation of angle $-\frac{\pi}{4}$
- Alain et Bob observe their qubit and send their respective outcomes
- Theorem: $p = \cos^2(\frac{\pi}{8}) \approx 0.85$

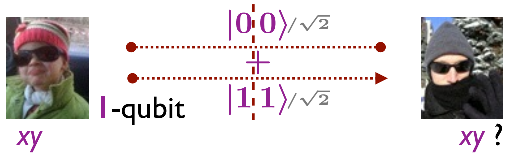


Realization: [Aspect-Grangier-Roger-Dalibard: Orsay'82]

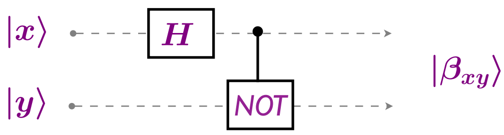
Superdense coding [1992]

Problem

- Alice & Bob share an EPR state: $|\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$
- Alice wants to send two bits *xy* to Bob
- But Alice can only send one qubit to Bob



Bell basis change



$$|\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$
$$|\beta_{01}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$
$$|\beta_{10}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$
$$|\beta_{11}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

Protocol

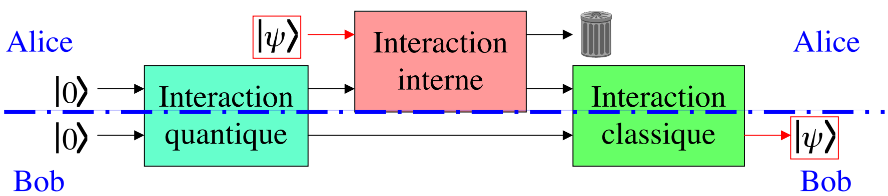
- Alice applies to its qubit *NOT*, if *y*=1; and *FLIP*, if *x*=1 $\quad FLIP = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
  - Alice sends its qubit to Bob
  - Bob performs the inverse of the Bell basis change, and observes *xy*

Quantum teleportation

Problem

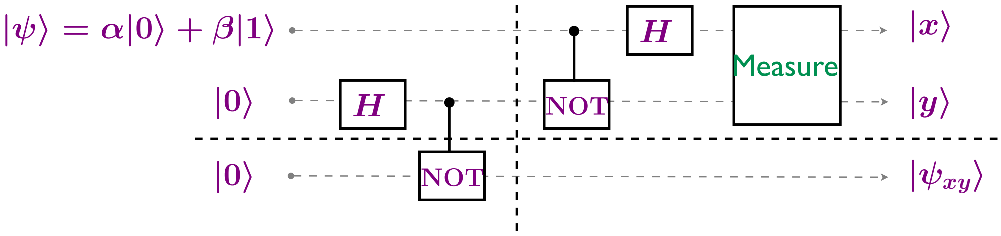
- Alice wants to transmit a qubit $|\psi\rangle$ to Bob
- Bob: far and unknown position to Alice

Realization



The quantum communication does not reveal anything on $|\psi\rangle$!

Circuit



Exercise

- Compute the state of the system before the measure
- Write the qubit state $|\psi_{xy}\rangle$ as a function of observed values *x,y*
- Explain the end of the protocol

Realizations

- 1 photon [Zeilinger et al : Innsbruck'97]
- 1 photon, 6 km [Gisin et al : Genève'02]
- 1 atom [Blatt et al : Innsbruck'04]
- Today: over 100km

Problem

- Alice and Bob are fare away
- They want to flip a coin in a fair way
  but they don't trust each other

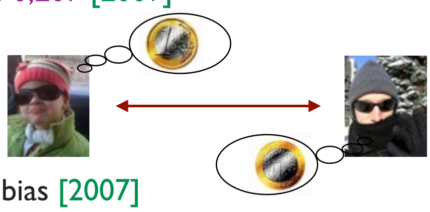
Classically

- Solutions based on harness assumptions of combinatorial problems
- No unconditionally secure solution

Quantumly

- There exists a protocol with maximal bias 0,25 [2001]
- There is no protocol with bias better than 0,207 [2002]
  - There exists a protocol with maximal bias 0,207 [2009]

Weak version: election

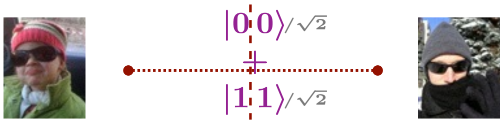


- Alice wants head
- Bob wants tail
- There exists a protocol with arbitrarily small bias [2007]

EPR based coin flipping

Main idea

- Assume Alice & Bob share an EPR state



- Alice & Bob observe their qubit and get bit a,b

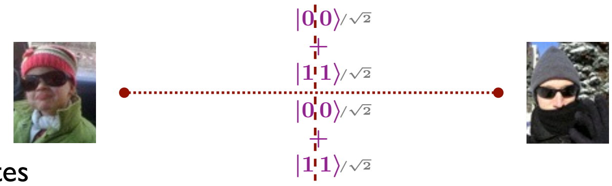
Fact

- $a=b$ with probability 1
- a (resp. b) is a uniform random bit

Problems

- Who create the EPR state?
- If Alice does, Bob needs to check that is an EPR state:
  And for instance not $|00\rangle \rightarrow a=b=0$ with probability 1
- In ordert o check the EPR state, Bob needs the 2 qubits
  Then Alice needs to check that Bob gives back the correct qubit

EPR based coin flipping



Protocol

- Initialization
  Alice prepares 2 EPR states
  Alice send the corresponding first qubits to Bob
- Selection
  Bob select the EPR state that will be use for flipping
  The other EPR state will be use for checking the honesty of Alice
  Alice and Bob observe their respective qubit of the *flipping* EPR state
- Checking
  Alice sends to Bob her qubit of the *checking* EPR state
  Bob measures the checking EPR state
    If the measure outcomes is correct, Bob accepts coin
    Otherwise, Bob declares that Alice has cheated

Theorem

- If both participant are honest, the outcome is a perfect random bit
- If one of the participants is dishonest, the maximal bias is 1/4

Attacks

Goal: increase the probability to get 0

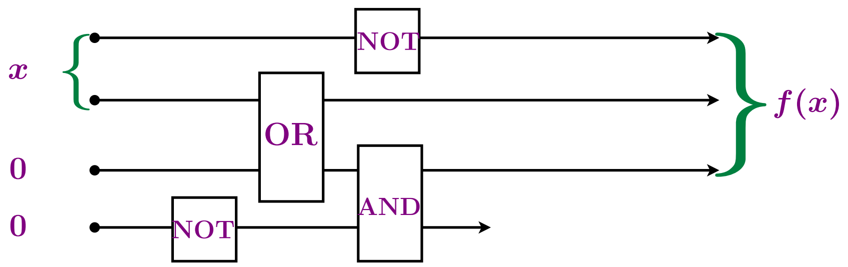
- Bob's attack: measure its 2 qubits, and select the EPR pair giving 0 (if any)
- Alice's attack: $\frac{|00\rangle|\mathrm{EPR\ state}\rangle}{\sqrt{3}} + \frac{|\mathrm{EPR\ state}\rangle|00\rangle}{\sqrt{3}}$

Gates

- A gate C is a function on at most **3** qubits
  Example: AND, OR, NOT, ...

Circuit

- A circuit is a sequence of gates $C = C_L \dots C_2 C_1$
  - The size of C is its number L of gates
- C computes a function f if for all input x: $C(x, 0^k) = (f(x), z)$



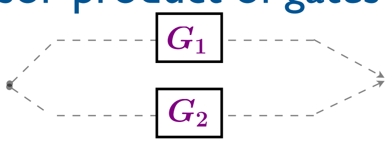
Theorem

- Any function can be computed by a circuit using only NOT, OR, AND gates

Gates $U \in \mathcal{U}(2^k), \quad k = 1, 2, 3$

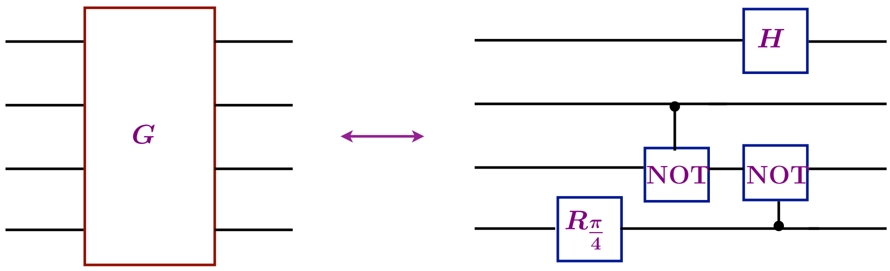
- A quantum gate is a unitary map that acts upon at most **3** qubits

Tensor product of gates

$|\psi_1\rangle|\psi_2\rangle$  $(G_1 \otimes G_2)|\psi_1\rangle|\psi_2\rangle = (G_1|\psi_1\rangle)(G_2|\psi_2\rangle)$

Circuit

- A quantum circuit is a sequence of gates (extended by $\otimes$ Id)



•Theorem

- Any unitary can be realized exactly by a circuit
  and approximated using only gates c-NOT and $\sqrt{}$H

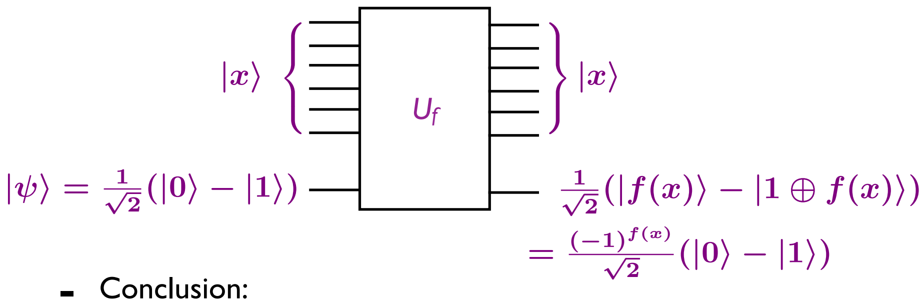
On the query operator S_f

Normal form

- Function: $f : \{0,1\}^n \longrightarrow \{0,1\}^m$
- Circuit: $U_f : |x\rangle|0\rangle \mapsto |x\rangle|f(x)\rangle$
  $|x\rangle|y\rangle \mapsto |x\rangle|y \oplus f(x)\rangle$

Circuit for S_f

- Boolean function: $f : \{0,1\}^n \longrightarrow \{0,1\}$
- Ancilla: $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$
- Circuit:



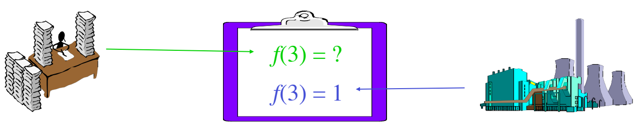
- Conclusion:

$$U_f(|x\rangle \otimes |\psi\rangle) = S_f(|x\rangle) \otimes |\psi\rangle$$

A first quantum algorithm [1992]

Deutsch-Jozsa problem

- Oracle input: $f : \{0,1\}^n \longrightarrow \{0,1\}$ a black-box function



such that f is either constant or balanced

- Output: 0 iff f is constant

Query complexity

- Deterministic: $2^{n-1}+1$
- Quantum: 1

Special case $n=1$

- No restriction on f
- Deterministic vs quantum: 2 queries vs 1 query

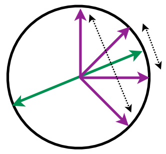
$x \mapsto f(x)$ can be nonreversible!

Reversible implementation of f

$$\alpha|0\rangle + \beta|1\rangle \dashrightarrow \boxed{S_f} \dashrightarrow (-1)^{f(0)}\alpha|0\rangle + (-1)^{f(1)}\beta|1\rangle$$

Hadamard gate: half-wave blade at 22,5°

$$|b\rangle \dashrightarrow \boxed{H} \dashrightarrow \tfrac{1}{\sqrt{2}}(|0\rangle + (-1)^b|1\rangle)$$



Quantum circuit

$$|0\rangle \dashrightarrow \boxed{H} \dashrightarrow \boxed{S_f} \dashrightarrow \boxed{H} \dashrightarrow \boxed{\text{Measure}} \dashrightarrow ?$$

$$|0\rangle \dashrightarrow \boxed{H} \dashrightarrow \boxed{S_f} \dashrightarrow \boxed{H} \dashrightarrow \boxed{\text{Measure}}$$

f constant $\dashrightarrow |0\rangle$

f balanced $\dashrightarrow |1\rangle$

Initialization: $|0\rangle$

Parallelization: $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

Query to f: $\frac{1}{\sqrt{2}}((-1)^{f(0)}|0\rangle + (-1)^{f(1)}|1\rangle)$

Interferences: $\frac{1}{2}((-1)^{f(0)}(|0\rangle + |1\rangle) + (-1)^{f(1)}(|0\rangle - |1\rangle))$

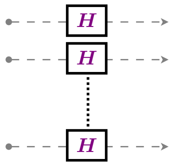
Final state: $\frac{1}{2}(((-1)^{f(0)} + (-1)^{f(1)})|0\rangle + ((-1)^{f(0)} - (-1)^{f(1)})|1\rangle)$

Reversible implementation of *f*

$$\sum_{x\in\{0,1\}^n} \alpha_x|x\rangle \dashrightarrow \boxed{S_f} \dashrightarrow \sum_{x\in\{0,1\}^n} (-1)^{f(x)}\alpha_x|x\rangle$$

Quantum Fourier transform

$QFT_n \equiv$



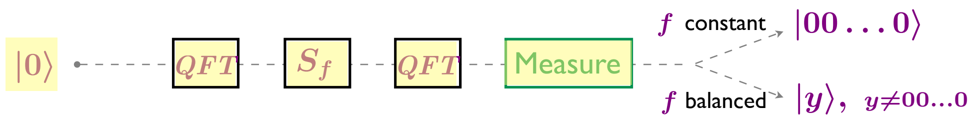
$$|b\rangle \dashrightarrow \boxed{H} \dashrightarrow \tfrac{1}{\sqrt{2}}(|0\rangle + (-1)^b|1\rangle)$$

$$QFT_n|x\rangle = \frac{1}{2^{n/2}}\sum_y (-1)^{x\cdot y}|y\rangle$$

where $x\cdot y = \sum_i x_i y_i \mod 2$

Quantum circuit

$$|0\rangle \dashrightarrow \boxed{QFT} \dashrightarrow \boxed{S_f} \dashrightarrow \boxed{QFT} \dashrightarrow \boxed{\text{Measure}} \dashrightarrow ?$$



Initialization: $|00\ldots0\rangle$

Parallelization: $\frac{1}{2^{n/2}}\sum_{x\in\{0,1\}^n}|x\rangle$

Query to *f*: $\frac{1}{2^{n/2}}\sum_{x\in\{0,1\}^n}(-1)^{f(x)}|x\rangle$

Interferences: $\frac{1}{2^n}\sum_{x,y\in\{0,1\}^n}(-1)^{f(x)+x\cdot y}|y\rangle$

Final state: $\left(\frac{1}{2^n}\sum_{x\in\{0,1\}^n}(-1)^{f(x)}\right)|00\ldots0\rangle + \sum_{y\neq 00\ldots0}\alpha_y|y\rangle$

Bernstein-Vazirani

Problem

- Oracle input: $f: \{0,1\}^n \rightarrow \{0,1\}$ a black-box function
  such that $f(x) = a \cdot x$
  for some fixed $a \in \{0,1\}^n$
- Output: a

Query complexity

- Randomized: n
  Query $f(0^{i-1}10^{n-i})=a_i$, for $i=1,2,\ldots,n$
- Quantum: 1

•Quantum circuit

$|0\rangle$ - - - QFT - - S_f - - QFT - - Measure - - - → $|a\rangle$

Analysis

$|0\rangle$ - - - QFT - - S_f - - QFT - - Measure - - - → $|a\rangle$

Initialization: $|00\ldots0\rangle$

Parallelization: $\frac{1}{2^{n/2}} \sum_{x \in \{0,1\}^n} |x\rangle$

Query to *f*: $\frac{1}{2^{n/2}} \sum_{x \in \{0,1\}^n} (-1)^{a \cdot x} |x\rangle = QFT|a\rangle$

Interferences: $QFT^2|a\rangle$

Final state: $|a\rangle$

On the difficulty of fatorizing

RSA Challenges

- http://www.rsasecurity.com/rsalabs

Challenge Number	Prize ($US)	Status	Submission Date	Submitter(s)
RSA-576	$10,000	Factored	December 3, 2003	J. Franke et al.
RSA-640	$20,000	Factored	November 2, 2005	F. Bahr et al.
RSA-704	$30,000	Not Factored		
RSA-768	$50,000	Not Factored		
RSA-896	$75,000	Not Factored		
RSA-1024	$100,000	Not Factored		
RSA-1536	$150,000	Not Factored		
RSA-2048	$200,000	Not Factored		

- RSA-640 (193 digits) :

  3107418240490043721350750035888567930037346022842727545720161948823206440518081504556346829671723286782437916272838
  033415471073108501919548529007337724822783525742386454014691736602477652346609

  =

  1634733645809253848443133883865090859841783670033092312181110852389333100104508151212118167511579

  x

  1900871281664822113126851573935413975471896789968515493666638539088027103802104498957191261465571

- RSA Algorithm (allows private communication)

  security based on the difficulty of factorizing

From period finding to factorization

•Theorem [Simon-Shor'94]

- Finding the period of *any* function on an abelian group can be done in quantum time poly $(\log |G|)$

Order finding

- Input: integers n and a such that $\gcd(a,n)=1$
- Output: the smallest integer $q \neq 0$ such that $a^q = 1 \mod n$
  - Reduction to period finding: the period of $x \to a^x \mod n$ is q

Factorization

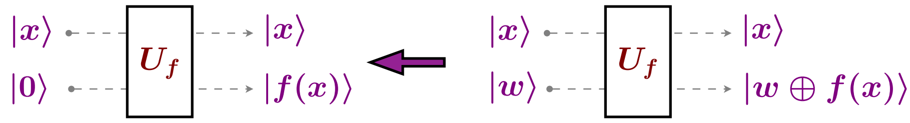
- Input: integer n
- Output: a nontrivial divisor of n

Reduction : Factorization $\leq_R$ Order finding

- Check that $\gcd(a,n)=1$
- Compute the order q of $a \mod n$
- Restart if q is odd or $a^{q/2} \neq -1 \mod n$
- Otherwise $(a^{q/2} - 1)\ (a^{q/2} + 1) = 0 \mod n$
- Return $\gcd(a^{q/2} \pm 1, n)$

Problem

- Oracle input: $f : \{0,1\}^n \to \{0,1\}^n$ a black-box function



such that $\exists s \in \{0,1\}^n : \forall x \neq y,\ f(x) = f(y) \iff y = x \oplus s$

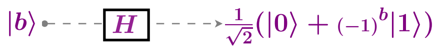
- Output: the period s

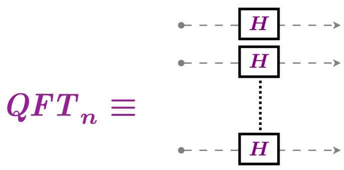
Complexity

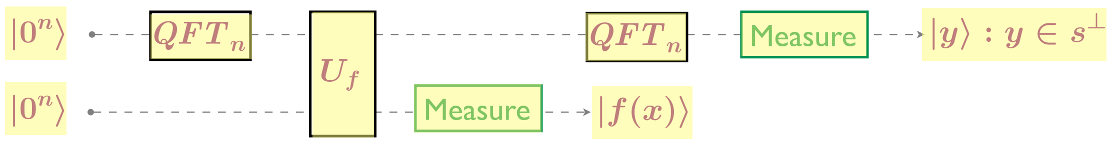
- Randomized: $2^{\Omega(n)}$ queries
- Quantum: $O(n)$ queries and time $O(n^3)$

Idea

- Use a Fourier transformation: $QFT_n|x\rangle = \frac{1}{2^{n/2}} \sum_y (-1)^{x \cdot y}|y\rangle$
  where $x \cdot y = \sum_i x_i y_i \mod 2$
- Realization of QFT_n using Hadamard gates:







Initialization: $|0^n\rangle|0^n\rangle$

Parallelization: $\frac{1}{2^{n/2}} \sum_x |x\rangle|0^n\rangle$

Query to f: $\frac{1}{2^{n/2}} \sum_x |x\rangle|f(x)\rangle$

Filter: $\frac{1}{\sqrt{2}}(|x\rangle + |x \oplus s\rangle)|f(x)\rangle$

Interferences:

$$\frac{1}{2^{(n+1)/2}} \sum_y ((-1)^{x \cdot y} + (-1)^{(x \oplus s)\cdot y})|y\rangle|f(x)\rangle$$

$$\frac{1}{2^{(n+1)/2}} \sum_y (-1)^{x \cdot y}(1 + (-1)^{s \cdot y})|y\rangle|f(x)\rangle$$

$$\frac{1}{2^{(n-1)/2}} \sum_{y : s \cdot y = 0} |y\rangle|f(x)\rangle$$

Construction of a linear system

- After $n+k$ iterations: $y^1, y^2, \ldots, y^{n+k} \in s^\perp$
- s is solution of the linear system in t:

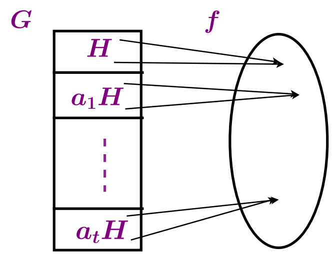
$$\begin{cases} y^1 \cdot t = 0 \\ y^2 \cdot t = 0 \\ \vdots \\ y^{n+k} \cdot t = 0 \end{cases} \leftrightarrow \begin{cases} y_1^1 t_1 + y_2^1 t_2 + \ldots + y_n^1 t_n = 0 \\ y_1^2 t_1 + y_2^2 t_2 + \ldots + y_n^2 t_n = 0 \\ \vdots \\ y_1^{n+k} t_1 + y_2^{n+k} t_2 + \ldots + y_n^{n+k} t_n = 0 \end{cases}$$

- If $s=0^n$ the y^i are of rank n with proba $\geq 1-1/2^k$
- If $s\neq 0^n$ the y^i are of rank $n-1$ with proba $\geq 1-1/2^{k+1}$
  - System solutions: 0^n and s

Complexity

- Constructing the system: $O(n)$ queries, time $O(n)$
- Solving the system: no query, time $O(n^3)$

Period Finding(G)



- Oracle input: function f on G such that

  f is *strictly* periodic for some unknown $H \leq G$:
  $$f(x) = f(y) \iff y \in xH$$
- Output: generator set for H

•Examples

- Simon Problem: $G = (\mathbb{Z}_2)^n,\ H = \{0, s\}$
- Factorization : $G = \mathbb{Z},\ H = r\mathbb{Z}$
- Discrete logarithm: $G = \mathbb{Z}^2,\ H = \{(rx, x) : x \in \mathbb{Z}\}$
- Pell's equations: $G = \mathbb{R}$
- Graph Isomorphism: $G = \mathcal{S}_n$

Quantum polynomial time algorithms (in log|G|)

  - Abelian groups G: *QFT*-based algorithm [1995]
  - Normal period groups H: *QFT*-based algorithm [2000]
  - Solvable groups G of constant exponent and constant length [2003]
- ...

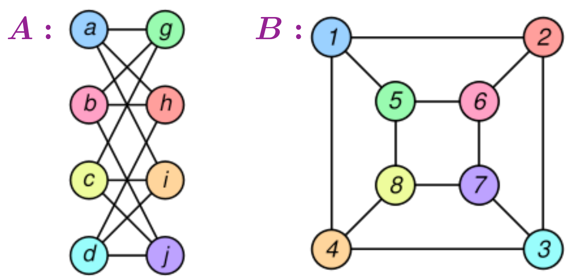
Shift problem

- Dihedral group $\mathbb{Z}_N \rtimes \mathbb{Z}_2$: sub-exponential time $2^{O(\sqrt{\log N})}$ [2003]

$f(\cdot,0)$	2	5	12	3	9	7	6	10	15	4
$f(\cdot,1)$	3	9	7	6	10	15	4	2	5	12

shift = -3

Graph Isomorphism



A	B
a	1
b	6
c	8
d	3
e	5
f	2
g	4
h	7

- Instance of Period Finding on the symmetric group
  $f : \pi \in \mathcal{S}_{2n} \mapsto \pi(A \cup B)$
- Symmetric group: we just know how to implement *QFT*... [1997]

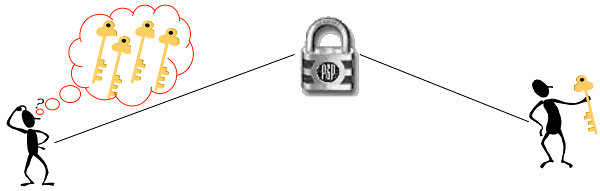
•General case

- Polynomial number of queries to *f*, but exponential post-processing time [1999]

Grover search algorithm

Grover problem

- Oracle input : $f : \{0,1\}^n \longrightarrow \{0,1\}$ such that $\exists! x_0 : f(x_0) = 1$
- Output : x_0
- Constraint : f is a black-box



Query complexity

- Randomized: $\Theta(2^n)$
- Quantum: $\Theta(\sqrt{2^n})$

$n = 2 \implies 1$ query

Preliminary remarks

Implementation of f

$$\sum_x \alpha_x |x\rangle \dashrightarrow \boxed{S_f} \dashrightarrow \sum_x (-1)^{f(x)} \alpha_x |x\rangle = \sum_x \alpha_x |x\rangle - 2\alpha_{x_0} |x_0\rangle$$

Double Hadamard gate

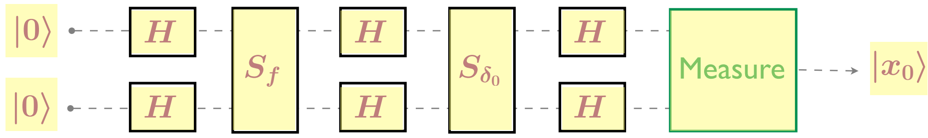
$$|x_1\rangle \dashrightarrow \boxed{H} \dashrightarrow \tfrac{1}{\sqrt{2}}(|0\rangle + (-1)^{x_1}|1\rangle)$$

$$|x_2\rangle \dashrightarrow \boxed{H} \dashrightarrow \tfrac{1}{\sqrt{2}}(|0\rangle + (-1)^{x_2}|1\rangle)$$

$$|x\rangle = |x_1 x_2\rangle \dashrightarrow \begin{array}{c}\boxed{H}\\ \boxed{H}\end{array} \dashrightarrow \tfrac{1}{2}\sum_y (-1)^{x\cdot y}|y\rangle$$

with $x \cdot y = x_1 y_1 + x_2 y_2 \mod 2$

Quantum solution ($n = 2$)



Initialization: $|00\rangle$

Parallelization: $\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$

Query to f: $\frac{1}{2}\sum_x |x\rangle - |x_0\rangle$

Interferences: $|00\rangle - \frac{1}{2}\sum_y (-1)^{x_0 \cdot y}|y\rangle$

Query to δ_0: $|00\rangle - \frac{1}{2}\left(\sum_y (-1)^{x_0 \cdot y}|y\rangle - 2|00\rangle\right) = -H \otimes H|x_0\rangle$

Final state: $-|x_0\rangle$

Geometrical analysis

Grover operator

$\mathrm{Vect}_{\mathbb{R}}(|x_0\rangle, |\mathrm{unif}\rangle)$

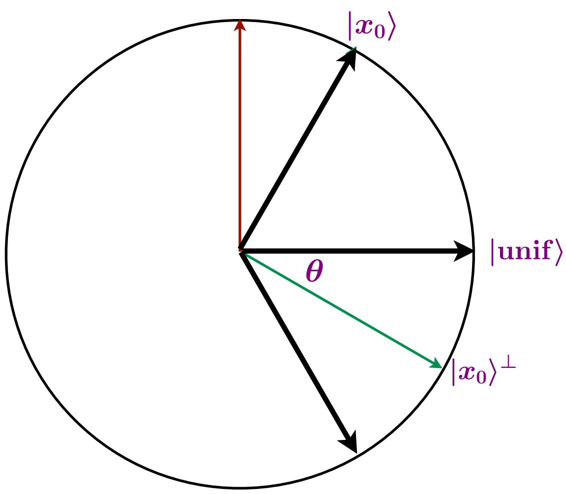
$S_f = -S_{|x_0\rangle} = S_{|x_0\rangle^\perp}$

$-S_{\delta_0} = S_{|00\rangle}$

$H^{\otimes 2} S_{|00\rangle} H^{\otimes 2} = S_{|\mathrm{unif}\rangle}$

$G = S_{|\mathrm{unif}\rangle} S_{|x_0\rangle^\perp} = R_{2\theta}$

with $\sin\theta = \langle \mathrm{unif}|x_0\rangle = \frac{1}{2}$



After 1 iteration

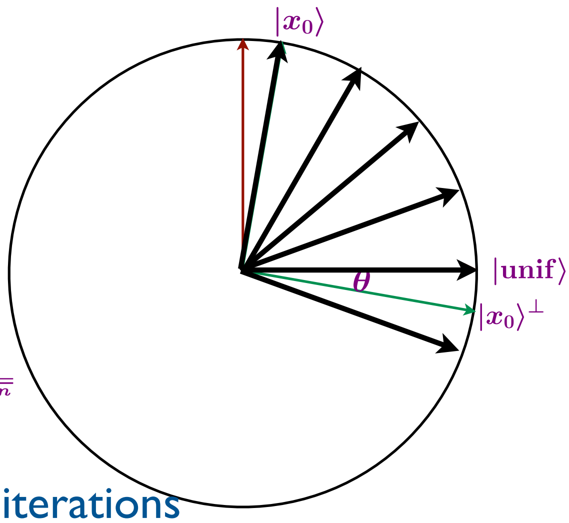
$|\mathrm{unif}\rangle \mapsto -G|\mathrm{unif}\rangle = -|x_0\rangle$

Geometrical analysis, general case

Grover operator

$$\boxed{G} \overset{\text{def}}{=} \boxed{S_f} - \boxed{H} - \boxed{-S_{\delta_0}} - \boxed{H}$$

$\mathrm{Vect}_{\mathbb{R}}(|x_0\rangle, |\mathrm{unif}\rangle)$



$S_f = -S_{|x_0\rangle} = S_{|x_0\rangle^\perp}$

$-S_{\delta_0} = S_{|00\rangle}$

$H^{\otimes 2} S_{|00\rangle} H^{\otimes 2} = S_{|\mathrm{unif}\rangle}$

$G = S_{|\mathrm{unif}\rangle} S_{|x_0\rangle^\perp} = R_{2\theta}$

with $\sin\theta = \langle \mathrm{unif} | x_0 \rangle = \frac{1}{\sqrt{2^n}}$

After $T = 2/\pi \cdot \sqrt{(2^n)}$ iterations

$|\mathrm{unif}\rangle \mapsto -G^T|\mathrm{unif}\rangle \approx -|x_0\rangle$

How many quantum algorithms exist?

Unstructured problems

- Grover algorithm [1996]

Algebraic problems

- Simon-Shor algorithm [1994]

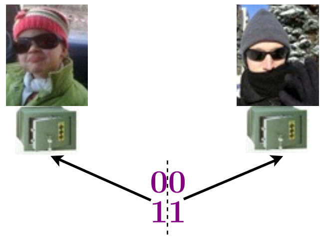
Well structured problems

- Classical algorithms are optimal!

Problems with few structures

- Quantum walk based algorithms [2003]
  quantum analogy of random walks
- Examples
  Element Distinctness, Commutativity: $n^{2/3}$ [2004]
  Triangle Finding: $n^{9/7}$ (lower bound n) [2013]
  Square Finding: $n^{1.25}$ (lower bound n) [2010]
  Matrix Multiplication: $n^{5/3}$ (lower bound $n^{3/2}$) [2006]
  AND-OR Tree evaluation: $\sqrt{n}$ [2007]

Where does the quantum superiority come from?



Entanglement?

- "Classical entanglement" exists: shared randomness
  - Flip a coin 00 or 11
  - Share each bit between Alice and Bob
  - Alice/Bob uses its bit when he/she wants, their result are correlated
- But quantum entanglement is "stronger"
  - Bell-CHSH inequality and applications

Complex amplitudes?

- No: they can be simulated using only real amplitude

$$\alpha|0\rangle + \beta|1\rangle \simeq \alpha_r|00\rangle + \alpha_i|01\rangle + \beta_r|10\rangle + \beta_i|11\rangle \qquad \mathcal{U}(2^n) \simeq \mathcal{O}(2^{n+1})$$

Negative amplitudes?

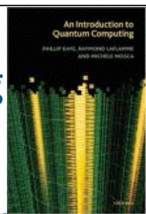
- Yes: they can induce destructive interferences

Hardness of amplitudes?

- No: amplitudes must be easily computable for being physically realizable

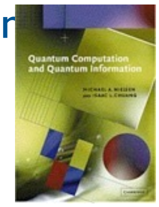
To continue...

An Introduction to Quantum Computing



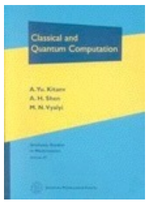
- Authors: Phillip Kaye, Raymond Laflamme, Michele Mosca
- Editor: Oxford University Press

Quantum Computation and Quantum Information



- Authors: Michael A. Nielsen, Isaac L. Chuang
- Editor: Cambridge University Press

Classical and Quantum Computation



- Authors: A. Yu. Kitaev, A. H. Shen, M. N. Vyalyi
- Editor: American Mathematical Society
- Collection: Graduate Studies in Mathematics

Lecture Notes for Quantum Computation

- Author: John Preskill
- Website: http://www.theory.caltech.edu/~preskill/ph229/

Quantum proofs for classical theorems

- Author: Andrew Drucker, Ronald de Wolf
- Website: http://arxiv.org/abs/0910.3376