Distributed colouring with non-local resources

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A distributed protocol may use:

- **no randomness** : \( P(y^*_i|x^*_i) = 1, P(y^*_i|x_i) = 0 \) for all \( x_i \neq x_i^* \).

- **local randomness** : \( P(y_1,\ldots,y_n|x_1,\ldots,x_n) = \prod_{i=1}^n P(y_i|x_i,\lambda_i) \).

- **shared randomness** : \( P(y_1,\ldots,y_n|x_1,\ldots,x_n) = \sum_{\lambda} P(\lambda) \prod_{i=1}^n P(y_i|x_i,\lambda) \).

- **quantum entanglement**
[Bell '64]: Existence of correlations arising from quantum mechanics that cannot be modelled by a "local hidden variable theory", i.e.,

"shared randomness ⪯ quantum entanglement"
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"shared randomness $\leq$ quantum entanglement"

**CHSH game**

Winning condition:

$$y_A \oplus y_B = x_A \land x_B$$

**Probability of winning:**
- Using shared randomness: at most 0.75.
- Using a quantum "Bell state": $\cos^2(\pi/8) \approx 0.86$. 
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CHSH game

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$$y_a \oplus y_b = x_a \land x_b$$

More precisely

$$y_A \oplus y_B = 0 \text{ if } (x_A, x_B) \in \{(0,0), (0,1), (1,0)\}$$

$$y_A \oplus y_B = 1 \text{ if } (x_A, x_B) = (1,1)$$

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- Using shared randomness: at most 0.75.
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Correlations arising from the quantum solution are **non-signalling**, i.e. the output of A doesn’t give any information on the input of B and vice-versa.

Mathematically

\[
\sum_{y_b} \mathbb{P}(y_a, y_b | x_a, x_b) = \sum_{y_b} \mathbb{P}(y_a, y_b | x_a, x'_b) = \mathbb{P}(y_a | x_a)
\]

and

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**Classical \(\subset\) Quantum \(\subset\) Non-Signalling**

- Not Non-Signalling implies not Quantum
- [Arfaoui '14] showed that for 2 players with binary input and output and output condition \(\neq y_a \oplus y_b\) the best non-signalling probability distribution is classical.
Suppose we have a graph $G = (V, E)$ modelling a communication network.

**LOCAL model**

- Every node has a (unique) **identifier**.
- One **round** of communication: send & receive information to neighbours & do computation.
- **Reliable synchronous** rounds (no crash nor fault).
- $k$ rounds of communication $\Leftrightarrow$ exchange with neighbours at distance $\leq k$ and do computation.
- Unbounded local computing power and bandwidth.
Introduction

The Colouring Problem: a fundamental symmetry breaking problem

Distributed Colouring Problem in the LOCAL model

How many rounds of communication are necessary and sufficient for $q$-colouring a graph?

$q = \Delta + 1$ and graph = cycle or path

[Cole & Vishkin '86]: $O(\log^*(n))$ rounds of communication are sufficient.

[Linial '92]: $\Omega(\log^*(n))$ rounds of communication are necessary.

$\log^* n = \min \{ i \geq 0 : \log^i(n) \leq 1 \}$
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$log^* n = \min\{i \geq 0: \log^{(i)} n \leq 1\}$
[Gavoille, Kosowki & Markiewicz ’09]: Non-Signalling + LOCAL = \( \phi \)-LOCAL

Non-Signalling

\( X_k \)  Choice of measurement \( \rightarrow \)  Non-Signalling Resources 1, \ldots, \( k \)  Measurement outcome \( \rightarrow \)  \( Y_k \)

\( X_{n-k} \)  Choice of measurement \( \rightarrow \)  Non-Signalling Resources \( k + 1, \ldots, n \)  Measurement outcome \( \rightarrow \)  \( Y_{n-k} \)

\( \phi \)-LOCAL

Input

Output
Consider a stochastic process \((X_n)_{n \in \mathbb{Z}}\) on \(\mathbb{Z}\).

\[
\cdots \quad \bigcirc \quad \bigcirc \quad \bigcirc \quad \bigcirc \quad \bigcirc \quad \cdots
\]

\(q\)-colouring process: \(X_i \in \{1, \ldots, q\}\) and \(X_n \neq X_{n+1}\).
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**\(k\)-localisability**

For all (possibly empty) connected sets \(I, J\) at distance at least \(k\) of each other, \(\mathbb{P}(X_I, X_J)\) depends only on \(|I|, |J|\).
A Probabilistic Formulation
Colouring the infinite path

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**k-localisability**

For all (possibly empty) connected sets \(I, J\) at distance at least \(k\) of each other, \(P(X_I, X_J)\) depends only on \(|I|, |J|\).

**k-dependence**

For every \(n \in \mathbb{Z}\),

\[
P(X_{\leq n}, X_{> n+k}) = P(X_{\leq n}) \cdot P(X_{> n+k})
\]

**k-dependence: an example**

If \((Z_n)_{n \in \mathbb{Z}}\) is iid, then \((X_n)_{n \in \mathbb{Z}}\) where each \(X_n := Z_n + \ldots + Z_{n+k}\) is \(k\)-dependent.
A Probabilistic Formulation

$k$-localisability and $k$-dependence

$k$-localisability

For all (possibly empty) connected sets $I$, $J$ at distance at least $k$ of each other, $\mathbb{P}(X_I, X_J)$ depends only on $\{|I|, |J|\}$.

$k$-dependence

For every $n \in \mathbb{Z}$, $\mathbb{P}(X_{\leq n}, X_{> n+k}) = \mathbb{P}(X_{\leq n}) \cdot \mathbb{P}(X_{> n+k})$.
A Probabilistic Formulation

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Remarks

- 0-dependent = independent
- $k$-dependent and stationary $\Rightarrow$ $k$-localisable
A Probabilistic Formulation

$k$-localisability and $k$-dependence

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Example

A random permutation of $\{1, \ldots, n\}$ is 0-localisable but not $k$-dependent for all $k \leq n$. 
A Probabilistic Formulation

$k$-dependent colouring

**Easy to check**: there is no $k$-dependent 2-colouring process for any $k \in \mathbb{N}$. 
**A Probabilistic Formulation**

**k-dependent colouring**

*Easy to check*: there is no \( k \)-dependent 2-colouring process for any \( k \in \mathbb{N} \).

### \( k \)-dependent colouring of \( \mathbb{Z} \) [Holroyd & Liggett ’15], [Holroyd & Liggett ’16]

- There is a 1-dependent and stationary \( q \)-colouring process for every \( q \geq 4 \).
- There is a 2-dependent and stationary 3-colouring process.
- There is no 1-dependent 3-colouring process.
A Probabilistic Formulation

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Iterative construction on the $n$-node path

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<thead>
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<th>$n = 1$</th>
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<td>121 1/48</td>
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<td>131 1/48</td>
<td>1213 1/120</td>
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Example: check that $\mathbb{P}(1 \ast 1) = 1/16 = \mathbb{P}(1)\mathbb{P}(1)$

- $n = 3$: $\mathbb{P}(1 \ast 1) = \mathbb{P}(121) + \mathbb{P}(131) + \mathbb{P}(141) = \frac{3}{48} = 1/16$
- $n = 4$:

  1. $\mathbb{P}(1 \ast 1) = \mathbb{P}(1 \ast 1 \ast)$  
     
     $= 3 \cdot \mathbb{P}(1212) + 6 \cdot \mathbb{P}(1213)$  
     
     $= \frac{3}{240} + \frac{6}{120} = \frac{15}{240}$  
     
     $= 1/16$

  2. $\mathbb{P}(1 \ast 1) = \mathbb{P}(1 \ast \ast 1) = 6 \cdot \mathbb{P}(1231) = \frac{6}{96} = 1/16
A Probabilistic Formulation
1-localisable colouring

Our results

Is there a 1-localisable 3-colouring process on $\mathbb{Z}$? **No.**
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Is there a 1-localisable 3-colouring process on $\mathbb{Z}$? **No.**

Proof technique

Relies on studying an induced hard-core process.

The supremum of the marginal probability $\rho(P_n)$ of the colour black appearing in $P_n$ gives a lower bound on the number of colours $q : q \geq 1/\rho(P_n)$. 

**Proof technique (continued)**

- **[Holroyd & Liggett ’16]**: $\rho(P_n) \to 1/4$ as $n \to \infty$ for a 1-dependent process.
- Our results: $\rho(P_n) = \text{Catalan} \left\lfloor \frac{n}{2} \right\rfloor \text{Catalan} + 1$ for a 1-localisable process.
  Therefore, $\rho(P_n) \to 1/4$ as $n \to \infty$ for a 1-localisable process.

- Our proof relies on combinatorics and linear programming.
A Probabilistic Formulation
1-localisable colouring

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Is there a 1-localisable 3-colouring process on $\mathbb{Z}$? No.

Proof technique

Relies on studying an induced hard-core process.

The supremum of the marginal probability $\rho(P_n)$ of the colour black appearing in $P_n$ gives a lower bound on the number of colours $q$:

\[ q \geq \frac{1}{\rho(P_n)}. \]

Proof technique (continued)

- [Holroyd & Liggett '16]: $\rho(P_n) \to 1/4$ as $n \to \infty$ for a 1-dependent process.

- **Our results**: $\rho(P_n) = \frac{\text{Catalan}_{\lfloor n/2 \rfloor}}{\text{Catalan}_{\lfloor n/2 \rfloor + 1}}$ for a 1-localisable process. Therefore, $\rho(P_n) \to 1/4$ as $n \to \infty$ for a 1-localisable process.

- Our proof relies on combinatorics and linear programming.


Alexander E. Holroyd and Thomas M. Liggett. Finitely dependent coloring. 
*Forum of Mathematics, Pi, 4 :e9, 43, 2016.*

Nathan Linial. Locality in Distributed Graph Algorithms. 