Distributed colouring with non-local resources

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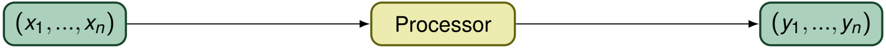
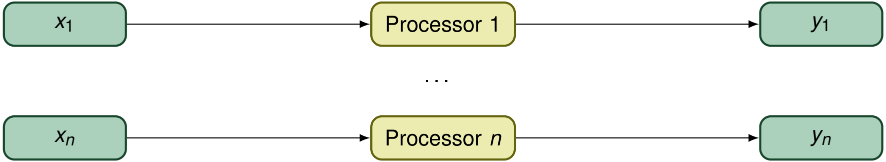
Centralised protocol



Distributed protocol



A distributed protocol may use :

- **no randomness** : $\mathbb{P}(y_i^*|x_i^*) = 1$, $\mathbb{P}(y_i^*|x_i) = 0$ for all $x_i \neq x_i^*$.
- **local randomness** : $\mathbb{P}(y_1, \ldots, y_n|x_1, \ldots, x_n) = \prod_{i=1}^n \mathbb{P}(y_i, |x_i, \lambda_i)$.
- **shared randomness** : $\mathbb{P}(y_1, \ldots, y_n|x_1, \ldots, x_n) = \sum_\lambda \mathbb{P}(\lambda) \prod_{i=1}^n \mathbb{P}(y_i|x_i, \lambda)$.
- **quantum entanglement**

[Bell ’64] : Existence of correlations arising from quantum mechanics that cannot be modelled by a ”local hidden variable theory”, i.e.,

”**shared randomness** $\lneq$ **quantum entanglement**”

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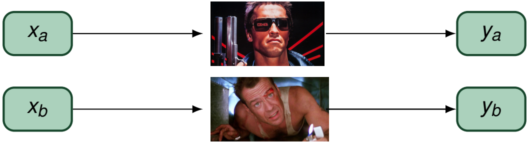
CHSH game



Winning condition :

$$y_a \oplus y_b = x_a \wedge x_b$$

Probability of winning :

- Using shared randomness : at most 0.75.
- Using a quantum "Bell state" : $\cos^2(\pi/8) \approx 0.86$.

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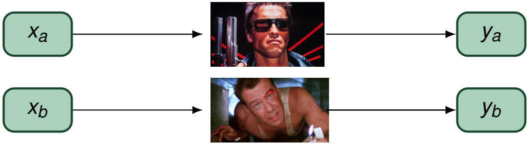

Winning condition :
$y_a \oplus y_b = x_a \wedge x_b$

More precisely

$y_A \oplus y_B = 0$ if
$(x_A, x_B) \in \{(0,0),(0,1),(1,0)\}$
$y_A \oplus y_B = 1$ if
$(x_A, x_B) = (1,1)$

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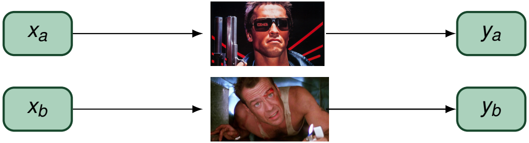

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Correlations arising from the quantum solution are **non-signalling**, i.e. the output of A doesn't give any information on the input of B and vice-versa.

Mathematically

$$\sum_{y_b} \mathbb{P}(y_a, y_b | x_a, x_b) = \sum_{y_b} \mathbb{P}(y_a, y_b | x_a, x_b') = \mathbb{P}(y_a | x_a)$$

and

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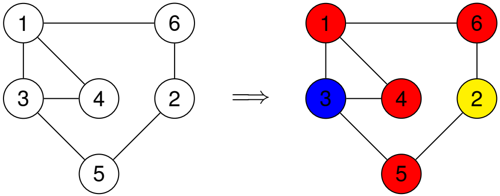
Classical $\subsetneq$ Quantum $\subsetneq$ Non-Signalling

- Not Non-Signalling implies not Quantum
- **[Arfaoui '14]** showed that for 2 players with binary input and ouput and output condition $\neq y_a \oplus y_b$ the best non-signalling probability distribution is classical.

Suppose we have a graph $G = (V, E)$ modelling a communication network.

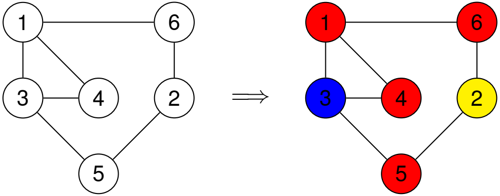
LOCAL model

- Every node has a (unique) **identifier**.
- One **round** of communication : send & receive information to neighbours & do computation.
- **Reliable synchronous** rounds (no crash nor fault).
- k rounds of communication $\Leftrightarrow$ exchange with neighbours at distance $\leq k$ and do computation.
- Unbounded local computing power and bandwith.



Distributed Colouring Problem in the LOCAL model

How many rounds of communication are necessary and sufficient for q-colouring a graph ?



Distributed Colouring Problem in the LOCAL model

How many rounds of communication are necessary and sufficient for q-colouring a graph ?

$q = \Delta + 1$ and graph=cycle or path

[Cole & Vishkin '86] : $O(\log^*(n))$ rounds of communcation are sufficient.

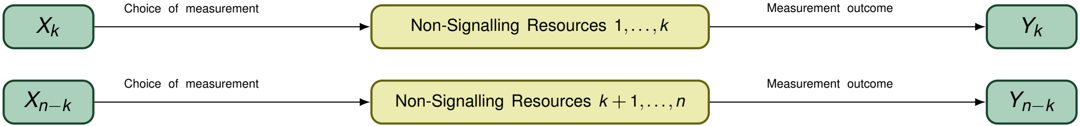
[Linial '92] : $\Omega(\log^*(n))$ rounds of communication are necessary.

$\log^* n = \min\{i \geq 0 : \log^{(i)} n \leq 1\}$

Physical Locality

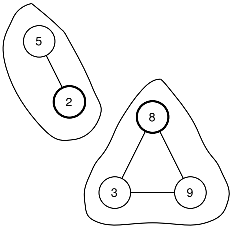
[Gavoille, Kosowki & Markiewicz ’09] : Non-Signalling + LOCAL = ϕ**-LOCAL**

Non-Signalling

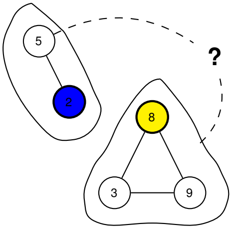


ϕ-LOCAL

Input



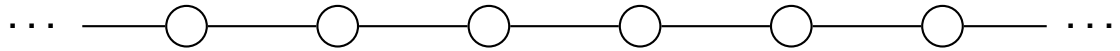
Output



A Probabilistic Formulation

Colouring the infinite path

Consider a stochastic process $(X_n)_{n\in\mathbb{Z}}$ on $\mathbb{Z}$.



q-colouring process : $X_i \in \{1, \dots, q\}$ and $X_n \neq X_{n+1}$.

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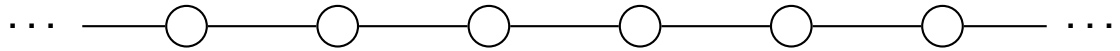

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k-localisability

For all (possibly empty) connected sets I, J at distance at least k of each other, $\mathbb{P}(X_I, X_J)$ depends only on $\{|I|,|J|\}$.

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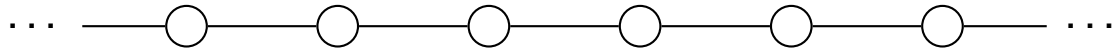

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k-dependence

For every $n \in \mathbb{Z}$,
$\mathbb{P}(X_{\leq n}, X_{>n+k}) = \mathbb{P}(X_{\leq n}) \cdot \mathbb{P}(X_{>n+k})$

k-dependence : an example

If $(Z_n)_{n\in\mathbb{Z}}$ is iid, then $(X_n)_{n\in\mathbb{Z}}$ where each $X_n := Z_n + \ldots + Z_{n+k}$ is k-dependent.

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- 0-dependent = independent
- k-dependent and stationary $\Rightarrow$ k-localisable

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Example

A random permutation of $\{1, \ldots, n\}$ is 0-localisable but not k-dependent for all $k \leq n$.

Easy to check : there is no k-dependent 2-colouring process for any $k \in \mathbb{N}$.

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k-dependent colouring of $\mathbb{Z}$ [Holroyd & Liggett '15], [Holroyd & Liggett '16]

- There is a 1-dependent and stationary q-colouring process for every $q \geq 4$.
- There is a 2-dependent and stationary 3-colouring process.
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Iterative construction on the n-node path

$n = 1$	$n = 2$	$n = 3$	$n = 4$
1 1/4 2 1/4 3 1/4 4 1/4	12 1/12 13 1/12 14 1/12 etc.	121 1/48 131 1/48 141 1/48 123 1/32 124 1/32 132 1/32 134 1/32 etc.	1212 1/240 1213 1/120 1231 1/96 etc.

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Example : check that $\mathbb{P}(1*1) = 1/16 = \mathbb{P}(1)\mathbb{P}(1)$

- $n=3$: $\mathbb{P}(1*1) = \mathbb{P}(121) + \mathbb{P}(131) + \mathbb{P}(141) = \frac{3}{48} = 1/16$
- $n=4$:

$$(1) \qquad \mathbb{P}(1*1) = \mathbb{P}(1*1*) = 3\cdot\mathbb{P}(1212) + 6\cdot\mathbb{P}(1213) = \frac{3}{240} + \frac{6}{120} = \frac{15}{240} = 1/16$$

$$(2) \qquad \mathbb{P}(1*1) = \mathbb{P}(1**1) = 6\cdot\mathbb{P}(1231) = \frac{6}{96} = 1/16$$

Our results

Is there a 1-localisable 3-colouring process on $\mathbb{Z}$? **No**.

A Probabilistic Formulation

1-localisable colouring

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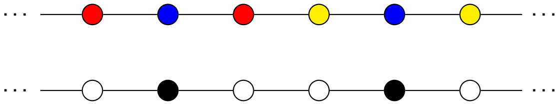
Proof technique

Relies on studying an induced hard-core process.



The supremum of the marginal probability $\rho(P_n)$ of the colour black appearing in P_n gives a lower bound on the number of colours q : $q \geq 1/\rho(P_n)$.

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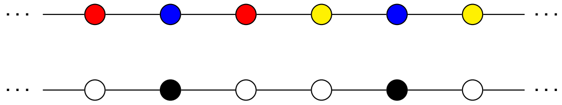

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Proof technique (continued)

- **[Holroyd & Liggett '16]** : $\rho(P_n) \to 1/4$ as $n \to \infty$ for a 1-dependent process.
- **Our results** : $\rho(P_n) = \frac{\text{Catalan}_{\lfloor n/2 \rfloor}}{\text{Catalan}_{\lfloor n/2 \rfloor + 1}}$ for a 1-localisable process.
  Therefore, $\rho(P_n) \to 1/4$ as $n \to \infty$ for a 1-localisable process.
- Our proof relies on combinatorics and linear programming.

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