Exercise Session 2 : $\boxed{7.10.2024}$

1) Saddle bound :

ⓐ Let f be analytic on $D(0,R)$, $R \in (0,\infty]$; with $f_n \geq 0$ $\forall n$.

Prove that $f_n \leq \frac{f(r)}{r^n}$ $\forall r \in (0,R)$

ⓑ Deduce that $\boxed{n! \geq \frac{n^n}{e^n}}$

(= weak version of Stirling inequality)

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2) Proof of $[z^n]\frac{1}{(1-z)^k} = \binom{n+k-1}{k-1}$

ⓐ 1^{st} proof: For F analytic at 0,

$[z^n]F(z) = \frac{1}{n!}\frac{\partial^n}{\partial z^n}F(z)$. Conclude

ⓑ 2^{nd} proof: Cauchy formula + change of variable $y = \frac{z}{1-z}$.

ⓒ 3^{rd} proof: Use the symbolic dictionary to define a combinatorial class with OGF $= \frac{1}{(1-z)^k}$.

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3) Longest runs .

ⓐ What is the OGF $f(z)$ of words on $\{0,1\}$ s.t # of consecutive 0s $< k$.

ⓑ What is the probability p_n that a random binary word has at most k consecutive 0s? (in terms of f).

ⓒ Give some bounds for the dominant singularity ρ_k of $f(z)$

ⓓ Prove than $p_n \sim \frac{1}{(2\rho_k)^n} \frac{1-\rho_k^k}{\rho_k(2-(k+1)\rho_k^k)}$.