Exercise session MPRI 2.15 – "Analytic combinatorics"

October 14th 2024

1 Boxed product and alternating permutations

The *boxed product* is defined as the modified labelled product:

$$\mathcal{A} = (\mathcal{B}^{\square} \star \mathcal{C}),$$

which denotes the subset of $\mathcal{B} \star \mathcal{C}$, formed with elements such that the smallest label is in the $\mathcal{B}$ component.

Question 1 Prove that:

$$\mathcal{A} = (\mathcal{B}^{\square} \star \mathcal{C}) \quad \Rightarrow \quad A(z) = \int_0^z \left(\frac{\partial}{\partial t} B(t)\right) \cdot C(t) dt.$$

Solution: The definition of the boxed product implies the coefficient relation:

$$A_n = \sum_{k=1}^{n} \binom{n-1}{k-1} B_k C_{n-k} = \frac{1}{n} \sum_{k=0}^{n} \binom{n}{k} (kB_k) C_{n-k}.$$

Hence,

$$\frac{nA_n}{n!} = \sum_{k=0}^{n} \frac{(kB_k)}{k!} \frac{C_{n-k}}{(n-k)!},$$

and the result follows. □

Question 2 What represents the class $\mathcal{A} = (\mathcal{Z}^{\square} \star \mathcal{C})$? What is its generating series in terms of $C(z)$?

Solution: It corresponds to the elements that can be obtained in the following way. Take an element γ of $\mathcal{C}$, preprend to it an atom of label say 0, and shift by 1 all the labels.

Its generating series is:

$$A(z) = \int_0^z C(t) dt$$

□

A permutation $\sigma = \sigma_1 \sigma_2 \dots \sigma_n$ is called an *alternating permutation* (also known as a *zig-zag permutation*) if:

$$\sigma_1 > \sigma_2 < \sigma_3 > \sigma_4 < \dots$$

For instance $\sigma = 6231745$ is alternating.

Question 3 Let $\mathcal{J}$ be the class of alternating permutation of odd length, prove that:

$$\mathcal{J} = \mathcal{Z} + (\mathcal{Z}^{\square} \star \mathcal{J} \star \mathcal{J}).$$

Solution: Either the permutation is reduced to an element, which accounts for the term $\mathcal{Z}$, or removing its smallest element decomposes it into 2 alternating permutations of odd length. Indeed, the left permutation ends with a maximum, and the right one starts with a maximum. □

Question 4 Deduce from the last question that $J(z) = \tan(z)$.

Solution: It follows from the two previous questions that:

$$J(z) = z + \int_0^z J(t)^2 dt \quad \text{and} \quad \frac{d}{dz} J(z) = 1 + J(z)^2.$$

Since $J(0) = 0$, we get that $\arctan(J(z)) = z$, and hence $J(z) = \tan(z)$. □

Remark : We could prove in a similar fashion that the generating series $K(z)$ of alternating permutations of even size satisfies $K'(z) = \tan(z)K(z)$, and hence that $K(z) = \frac{1}{\cos(z)}$.

2 Cycles in permutations

Let $P(z, u)$ be the (exponential) generating series of permutations with an additional variable u marking the number of cycles, i.e.

$$P(z, u) = \sum_{\sigma \text{ permutation}} u^{\text{cycles}(\sigma)} \frac{z^{|\sigma|}}{|\sigma|!}.$$

Question 5 Prove that

$$P(z, u) = \exp\left(u \log\left(\frac{1}{1-z}\right)\right).$$

What do you observe when $u = 1$? Is is surprising ?

Solution: A permutation is a set of cycles, hence we have:

$$\mathcal{P} = \text{SET}(\mathcal{U} \cdot \text{CYC}(\mathcal{Z}))$$

This gives the desired result. □

Question 6 Prove that the expected number of cycles in a random permutation of size n is equal to $H_n := \sum_{k=1}^{n} \frac{1}{k}$.

Solution: We have:

$$E_n(\text{cycles}) = \frac{n![z^n](\partial P(z,u)/\partial u)_{u=1}}{n!} = [z^n] \log\left(\frac{1}{1-z}\right) \frac{1}{1-z}.$$

By developping the logarithm as a series, we get:

$$[z^n] \log\left(\frac{1}{1-z}\right) \frac{1}{1-z} = [z^n] \sum_{k \geq 1} \frac{z^k}{k} \frac{1}{1-z} = \sum \frac{1}{k} [z^{n-k}] \frac{1}{1-z} = \sum_{k=1}^{n} \frac{1}{k}.$$

□

Question 7 Prove that the probability that all cycles of a random permutation of size n have length at most $n/2$ is equal to $1 - \sum_{k>n/2}^{n} 1/k$.

Remark: Since $H_n \sim \log(n) + \gamma + o(1)$, (where γ is Euler-Mascheroni's constant), this probability converges to $1 - \log(2) \approx 0.30685$.

Question 8 (From [Analytic Combinatorics by Flajolet and Sedgewick]) A hundred prisoners, each uniquely identified by a number between 1 and 100, have been sentenced to death. The director of the prison gives them a last chance. He has a cabinet with 100 drawers (numbered 1 to 100). In each, he'll place at random a card with a prisoner's number (all numbers different). Prisoners will be allowed to enter the room one after the other, and open, then close again, 50 drawers of their own choosing, but will not in any way be allowed to communicate with one another afterwards. The goal of each prisoner is to locate the drawer that contains his own number. If *all* prisoners succeed, then they will all be spared; if at least one fails, they will all be executed.

There are two mathematicians among the prisoners. The first one, declares that their overall chances of success are of order $1/2^{100} \approx 8 \cdot 10^{-31}$. The second one, a combinatorialist, claims she has strategy for the prisoners, which has a greater than 30% chance of success. On whose side do you stand ?