# THE SCALING LIMIT OF RANDOM SIMPLE TRIANGULATIONS AND RANDOM SIMPLE QUADRANGULATIONS

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ABSTRACT. Let $M_n$ be a simple triangulation of the sphere $\mathbb{S}^2$, drawn uniformly at random from all such triangulations with $n$ vertices. Endow $M_n$ with the uniform probability measure on its vertices. After rescaling graph distance by $(3/(4n))^{1/4}$, the resulting random measured metric space converges in distribution, in the Gromov–Hausdorff–Prokhorov sense, to the Brownian map. In proving the preceding fact, we introduce a labelling function for the vertices of $M_n$. Under this labelling, distances to a distinguished point are essentially given by vertex labels, with an error given by the winding number of an associated closed loop in the map. We establish similar results for simple quadrangulations. The appearance of a winding number suggests that a discrete complex-analytic approach to the study of random triangulations may lead to further discoveries.

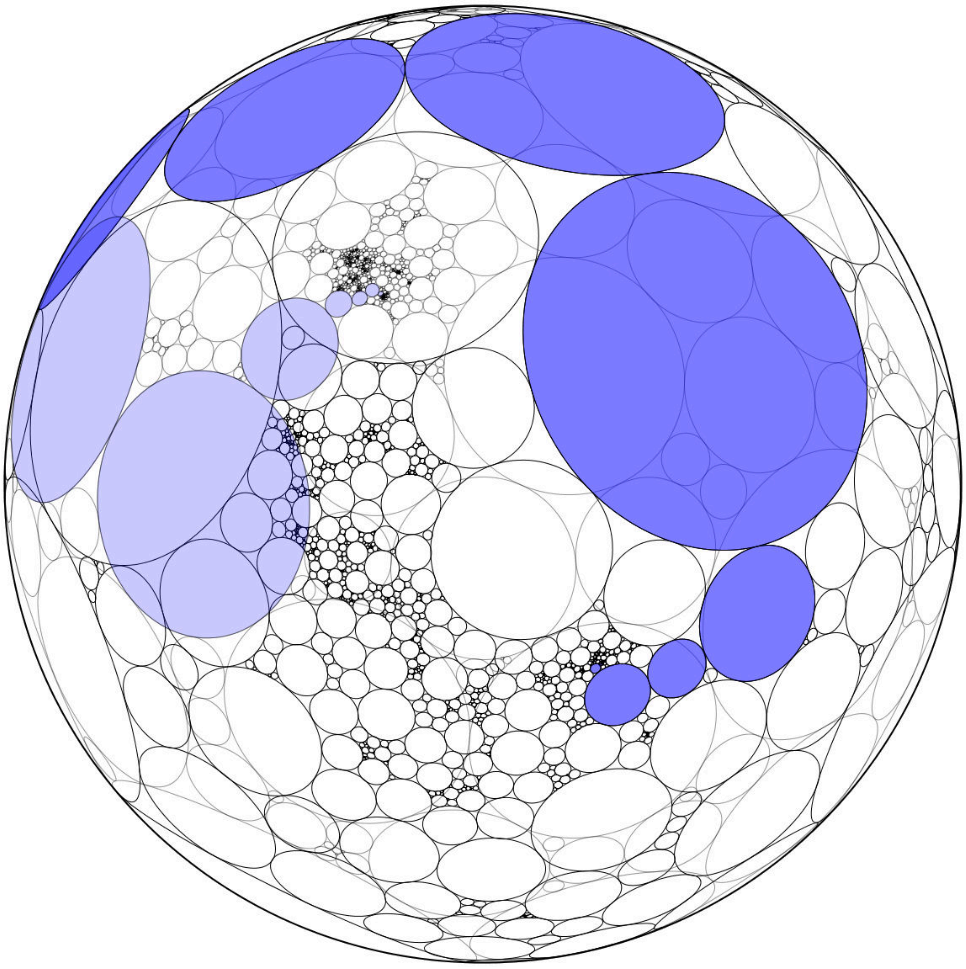

FIGURE 1. The circle packing associated to a uniformly random simple triangulation of $\mathbb{S}^2$ with $10^5$ vertices. Blue shaded circles form a shortest path between two uniformly random vertices (circles). Created using Ken Stephenson's CirclePack program; the file for the above packing is included with the arXiv posting of this manuscript.

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Contents



## 1. Introduction

We begin by heading straight for a statement of our main result.[1] A graph is *simple* if it has no loops or multiple edges. For integer $n \geq 3$, let $\triangle_n^\circ$ be the set of pairs $(M, \xi)$, where $M$ is an $n$-vertex simple triangulation of the sphere $\mathbb{S}^2$, and $\xi$ is a corner of $M$. Also, for integer $n \geq 4$, let $\square_n^\circ$ be the set of pairs $(M, \xi)$ with $M$ an $n$-vertex simple quadrangulation of $\mathbb{S}^2$ and $\xi$ a corner of $M$. Then let $\mathcal{M} = (\mathcal{M}_n, n \geq 4)$ be one of the sequences $(\triangle_n^\circ, n \geq 4)$ or $(\square_n^\circ, n \geq 4)$.

**Theorem 1.1.** *For $n \geq 4$, let $(M_n, \xi_n)$ be a uniformly random element of $\mathcal{M}_n$. Write $V(M_n)$ for the set of vertices of $M_n$, let $d_n : V(M_n) \to \mathbb{N}$ be graph distance in $M_n$ and let $\mu_n$ be the uniform probability measure on $V(M_n)$. Finally, let $c = (3/4)^{1/4}$ if $\mathcal{M} = (\triangle_n^\circ, n \geq 4)$ and let $c = (3/8)^{1/4}$ if $\mathcal{M} = (\square_n^\circ, n \geq 4)$. Then, as $n \to \infty$,*

$$(V(M_n), cn^{-1/4} d_n, \mu_n) \xrightarrow{\mathrm{d}} (S, d, \mu),$$

*for the Gromov–Hausdorff–Prokhorov distance, where $(S, d, \mu)$ is the Brownian map.*

We recall the definition of the Brownian map in Section 1.1, below. Our proof relies upon the remarkable work of Miermont [29] and, independently, Le Gall [22], which both established convergence for general (non-simple) random quadrangulations. In particular, our results do not constitute an independent proof of uniqueness of the limit object. A discussion of the constants in the above theorem, and their relation with those from [22, 29], appears in Appendix A.

The part of Theorem 1.1 pertaining to simple triangulations (sometimes called *type-III* triangulations; see [3]) answers a question of Le Gall [22] and Le Gall and Beltran [5]. One general motivation for establishing convergence to the Brownian map is its conjectured role as a universal limit object for a wide range of random map ensembles. However, the case of simple triangulations holds additional interest due to the conjectured link between the Brownian map and the Liouville quantum gravity constructed by Duplantier and Sheffield [12]; see [16] for further discussion of this connection. Le Gall [20] proved that the Brownian map is almost surely homeomorphic to the 2-sphere (see also [23, 27]). However,

[1]Precise definitions of almost all the terminology used in the introduction appear in Sections 2 and 3. After stating our main result, the remainder of introduction provides motivation and an overview of its proof, particularly the novel aspects of said proof.

homeomorphism equivalence is too weak, for example, to deduce conformal information or to prove dimensional scaling relations. For these, a canonical embedding of the Brownian map in $\mathbb{S}^2$ is needed (or at least would be very useful).

For any simple triangulation $M$ of $\mathbb{S}^2$, the Koebe-Andreev-Thurston theorem (see, e.g., [34], Chapter 7) provides a *canonical circle packing* in $\mathbb{S}^2$, unique up to conformal automorphism, whose tangency graph is $M$; see Figure 1 for an illustration of a random circle packing. (This uniqueness holds only for simple triangulations; for a uniformly random (non-simple) triangulation $N$ with $n$ vertices, for example, the number of degrees of freedom in a circle packing with tangency graph $N$ is typically linear in $n$.) The uniqueness provides hope that the conformal properties of the Brownian map can be accessed by studying the circle packings associated to large random simple triangulations

We deduce Theorem 1.1 from a result which provides more general sufficient conditions for a sequence $(M_n, n \in \mathbb{N})$ of random planar maps to converge in distribution to the Brownian map. More precisely, Theorem 4.1 states conditions under which, after suitably rescaling distances, and endowed with the uniform probability measure on its vertex set, $M_n$ converges in distribution to the Brownian map for the Gromov–Hausdorff–Prokhorov distance. The approach of Theorem 4.1 has its genesis in work of Chassaing and Schaeffer [11], and is based on bijective codings of maps by labelled plane trees. We refer to ensembles satisfying the conditions of Theorem 4.1 as *Chassaing–Schaeffer families.*

We hope Theorem 4.1 will be useful in proving convergence for other random map models, in particular for models falling within the framework of the "master bijection" of Bernardi and Fusy [6] and of the general bijection for blossoming trees, very recently established by Albenque and Poulalhon [1]. With this in mind, we have tried to state rather general conditions, which we summarize in Section 1.2. The proof of Theorem 4.1 is a fairly straightforward generalization of existing arguments (mostly due to Jean-François Le Gall), and we defer it to an appendix.

While the conditions under which we establish convergence to the Brownian map are rather general, *verifying* that a discrete random map ensemble satisfies these conditions can be rather involved. In many map ensembles of interest, the primary missing link is a labelling rule for the vertices of a canonical spanning tree of the map, such that vertex labels encode distances to a specified root vertex. For the case of random simple triangulations and quadrangulations, we provide a labelling that does not *precisely* encode distances, but we show that the error is insignificant in the limit. Intriguingly, for distances to a specified root vertex, the error in the label is bounded by the *winding number* of an associated closed loop in the map. In Section 1.3, we briefly describe the bijection between simple triangulations and certain labelled trees, on which our proof of Theorem 1.1 is based, and further discuss the role of winding numbers. The appearance of a winding number hints that a discrete complex-analytic perspective may shed further light on the shape of geodesics in random simple triangulations and eventually in the Brownian map.

One requirement of Theorem 4.1 is the convergence of a suitable spatial branching process, after renormalization, to the Brownian snake. Such convergence is known in many settings, but in others lack of symmetry (symmetry between the labels of children of a single node, in the coding of maps by labelled trees) has posed an obstacle. We introduce a technique we call *partial symmetrization*, in which we hold a "representative subtree" fixed while randomly permuting the children of individuals not within the subtree. This introduces enough symmetry that we may appeal to known results to establish convergence to the Brownian snake. On the other hand, fixing a large subtree allows the partially symmetrized process to be related to the original labelled tree and so to the associated map. A detailed explanation of the partial symmetrization technique is easier to provide for a specific bijection, and we defer it to Section 6.

We believe partial symmetrization may be used to show that the multi-type spatial branching processes coding random $p$-angulations (for odd $p \geq 5$) converge to the Brownian snake. Given the work of Miermont [29] and of Le Gall [22], this is the only missing element in a proof that $p$-angulations (and perhaps more general random maps with degrees given by suitable Boltzmann weights) converge to the Brownian map. We expect to return to this in a subsequent work.

1.1. **The Brownian map.** Given an interval $I \subset \mathbb{R}$ or $I \subset \mathbb{N}$ and a function $f : I \to \mathbb{R}$, for $s, t \in I$ with $s \leq t$ we write $\check{f}(s,t) = \inf_{x \in I \cap [s,t]} f(x)$ and write $\check{f}(t,s) = \inf_{x \in I \setminus (s,t)} f(x)$.

Let $\mathbf{e} = (\mathbf{e}(t), 0 \leq t \leq 1)$ be a standard Brownian excursion and, conditional on $\mathbf{e}$, let $Z = (Z(t), 0 \leq t \leq 1)$ be a centred Gaussian process such that $Z(0) = 0$ and for $0 \leq s \leq t \leq 1$,

$$\mathrm{Cov}(Z(s), Z(t)) = \check{\mathbf{e}}(s,t)\,.$$

We may and shall assume $Z$ is a.s. continuous; see [18, Section IV] for a more detailed description of the construction of the pair $(\mathbf{e}, Z)$.

Next, define an equivalence relation $\sim_{\mathbf{e}}$ as follows. For $0 \leq x \leq y \leq 1$ let $x \sim_{\mathbf{e}} y$ if $\mathbf{e}(x) = \mathbf{e}(y) = \check{\mathbf{e}}(x,y)$. It can be verified that almost surely, for all $x, y \in [0,1]$, if $x \sim_{\mathbf{e}} y$ then $Z(x) = Z(y)$, so we may view $Z$ as having domain $[0,1]/\sim_{\mathbf{e}}$. Next, for $x, y \in [0,1]$ let

$$d_Z(x,y) = Z(x) + Z(y) - 2\max(\check{Z}(x,y), \check{Z}(y,x))\,. \tag{1}$$

Then let $d^*$ be the largest pseudo-metric on $[0,1]$ satisfying that (a) for all $s, t \in [0,1]$, if $s \sim_{\mathbf{e}} t$ then $d_Z(s,t) = 0$, and (b) $d^* \leq d_Z$. Let $S = [0,1]/\{d^* = 0\}$, and let $d$ be the push-forward of $d^*$ to $S$. Finally, let $\mu$ be the push-forward of Lebesgue measure on $[0,1]$ to $S$. The (measured) *Brownian map* is (a random variable with the law of) the triple $(S, d, \mu)$. This name was first used by Marckert and Mokkadem [26], who considered a notion of convergence for random maps different from that of the present work.

For later use, let $\rho \in S$ be the equivalence class of the point 0, and, writing $s^* \in [0,1]$ for the point where $Z$ attains its minimum value (this point is almost surely unique), let $u^* \in S$ be the equivalence class of $s^*$. Then Corollary 7.3 of [22] states that for $U$ and $V$ uniformly distributed on $[0,1]$, independent of $Z$ and of each other,

$$d^*(U,V) \stackrel{\mathrm{d}}{=} d^*(U,s^*) \stackrel{\mathrm{d}}{=} -\check{Z}(0,1) \stackrel{\mathrm{d}}{=} Z(V) - \check{Z}(0,1). \tag{2}$$

1.2. **Sufficient conditions for convergence to the Brownian Map.** Our argument leans heavily on the *rerooting invariance* of the Brownian map ((2), above). Given the convergence of some discrete ensemble to the Brownian map, if the discrete ensemble possesses rerooting invariance then this can be transferred to the Brownian map. However, to date this is the only known technique for establishing rerooting invariance of the Brownian map (and the key reason why our results depend on those of [22, 29]).

Informally, to prove convergence we need that the random rooted map $M_n$ can in some sense be described by a suitable pair of random functions $C_n : [0,1] \to [0,\infty)$ and $Z_n : [0,1] \to \mathbb{R}$. Often $C_n$ will be the (spatially and temporally rescaled, clockwise) contour process of some canonical rooted spanning tree $(T_n, \xi_n)$ of $M_n$, and for the sake of this informal description we assume this to be so. To establish convergence we require (versions of) the following. In what follows let $r_n \in [0,1]$ be such that $Z_n(r_n) = \min(Z_n(x), 0 \leq x \leq 1)$, and write $d_{M_n}$ for (suitably rescaled) graph distance on $V(M_n)$.

1. **Distances to the minimum given by $Z_n$.** There is a vertex $u_n \in V(M_n)$ such that for all vertices $v$, if a clockwise contour exploration of $T_n$ visits $v$ at time $t$ then $Z_n(t) - Z_n(r_n)$ is $d_{M_n}(v, u_n) + o_n(1)$, where $o_n(1)$ represents an error that tends to zero in probability as $n \to \infty$.

2. **Distance bound via clockwise geodesics to the minimum.** For any pair of vertices $v, v'$ of $M_n$, if a clockwise contour exploration of $T_n$ visits $v$ and $v'$ at times $t$ and $t'$, respectively, then $d_{M_n}(v, v')$ is bounded from above by
$$Z_n(t) + Z_n(t') - 2\max(\check{Z}_n(t,t'), \check{Z}_n(t',t)) + o_n(1).$$
3. **Coding by the Brownian snake.** The pair $(C_n, Z_n)$ converges in distribution to $(\mathbf{e}, Z)$, for the topology of uniform convergence on $C([0,1], \mathbb{R})^2$.
4. **Invariance under rerooting.** If $U_n, V_n$ are independent, uniformly random vertices of $M_n$, then $d_{M_n}(U_n, V_n)$ is asymptotically equal in distribution to $d_{M_n}(u_n, V_n)$.

Briefly, given these properties the proof then proceeds as follows. Our argument roughly follows one used by Le Gall to prove convergence of rescaled random (non-simple) triangulations to the Brownian map, once convergence for quadrangulations is known ([22, Section 8]). It is useful to reparameterize so that all the metrics and pseudo-metrics under consideration are functions from $[0,1]^2$ to $[0,\infty)$; this can be accomplished by identifying the vertices of each metric space $M_n$ with a subset of $[0,1]$ and using bilinear interpolation.

First, 1. and 2. together can be used to prove tightness of the sequence of laws of the functions $(d_{M_n}, n \in \mathbb{N})$, which implies convergence along subsequences. Thus, let $d : [0,1]^2 \to [0,\infty)$ be a subsequential limit of $d_{M_n}$. Our aim is to show that almost surely $d$ and $d^*$ (defined in Section 1.1) are equal in law.

Next, 1. says that distances to the point of minimum label are given by $Z_m$, a limiting analogue of which is also true in the Brownian map. Invariance under rerooting 4. and (2) then yields that for $U, V$ independent and uniform on $[0,1]$, $d(U,V)$ is the limit in distribution of $-Z_n(r_n)$, so by 3. we obtain $d(U,V) \stackrel{\mathrm{d}}{=} -\min(Z(x), 0 \le x \le 1) = d^*(U,V)$.

Finally, 2. gives a bound on $d_{M_n}$ that is a finite-$n$ analogue of $d_Z$ (recall (1)). Since $d^*$ is maximal subject to $d^* \le d_Z$, 3. then yields that $d$ is stochastically dominated by $d^*$. In other words, by working in a suitable probability space (i.e. choosing an appropriate coupling), we may assume $d(x,y) \le d^*(x,y)$ for almost every $(x,y) \in [0,1]^2$. The fact that $d(U,V) \stackrel{\mathrm{d}}{=} d^*(U,V)$ then implies $d$ and $d^*$ are almost everywhere equal, so have the same law.

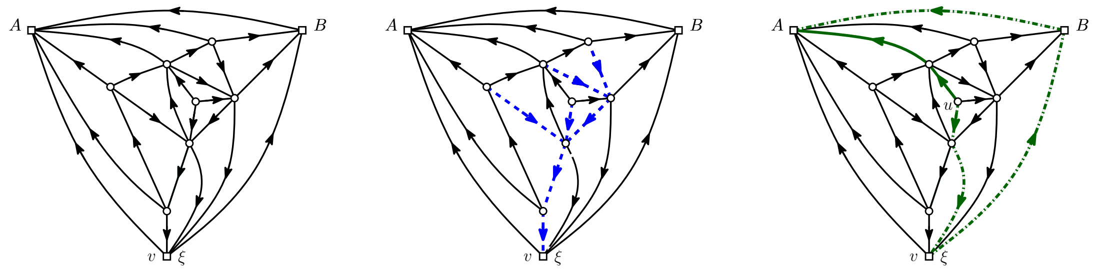
1.3. **Labels and geodesics, and an overview of the proof.** In this section (and throughout much of the rest of the paper), we restrict our attention to simple triangulations, as the details for simple quadrangulations are nearly identical.

Fix a pair $(G, \xi)$ with $G$ a simple triangulation of $\mathbb{S}^2$ and $\xi$ a corner of $G$. View $G$ as embedded in $\mathbb{R}^2$ so the face containing $\xi$ is the unique unbounded (outer) face. With this embedding, list the vertices of the face containing $\xi$ in clockwise order as $v, A, B$, with $v$ incident to $\xi$. A 3-*orientation* of $(G, \xi)$ is an orientation $\overrightarrow{E}$ of $E(G)$ such that in $\overrightarrow{E}$, $A, B$, and $v$ have outdegrees 0, 1, and 2, respectively, and all other vertices have outdegree three.[2] Schnyder [33] showed $(G, \xi)$ admits a 3-orientation if and only if $G$ is simple, and in this case admits a *unique* 3-orientation containing no counterclockwise cycles (we say an oriented cycle is *clockwise* if $\xi$ is on its left, and otherwise say it is counterclockwise); this 3-orientation is called *minimal*. Let $\overrightarrow{E}$ be the minimal 3-orientation of $(G, \xi)$.

The definitions of the following paragraph are illustrated in Figure 2. A subtree of $G$ containing the vertex $v$ incident to $\xi$ is *oriented* if all edges of the subtree are oriented towards $v$ in $\overrightarrow{E}$. It turns out there is a unique oriented subtree $T$ of $G$ on vertices $V(G) \setminus \{A, B\}$ which is *minimal* in the sense that for all edges $uw \in \overrightarrow{E}$ with $\{u, w\} \notin E(T)$, if $uw$ attaches to $u$ and $w$ in corners $c$ and $c'$, respectively, then $c$ precedes $c'$ in a clockwise contour exploration of $T$ starting from $\xi$. We endow this tree $T$ with a labelling $Y : V(T) \to \mathbb{N}$ as follows. For $e = uw \in \overrightarrow{E}$ with $\{u, w\} \in E(G)$, the *leftmost oriented path*

[2] This is equivalent to, but differs very slightly from, the standard definition.

from $e$ to $A$ is the unique oriented path $(u_0, u_1, \ldots, u_k)$ with the following two properties: (i) $u_0 = u$, $u_1 = w$; (ii) for $1 \le i < k$, if $\{u_i, y\} \in E(G)$ and this edge attaches to the path $(u_0, \ldots, u_k)$ on the left, then $yu_i \in \overrightarrow{E}$. For each vertex $u \in V(T)$ distinct from $v$, there are three such paths starting at $u$ (since $u$ has outdegree three in $\overrightarrow{E}$); we let $P(u) = P_{G,\xi}(u)$ be one of the shortest such paths. Then let $Y(u) = |P(u)|$, the number of vertices in $P(u)$.




(a) A simple triangulation endowed with its unique 3-orientation with no counterclockwise cycles.

(b) The minimal oriented tree is drawn in dashed blue lines.

(c) The thick green paths are both leftmost oriented paths from $u$ to $A$; the solid path is $P(u)$, so $Y(u) = 3$.

FIGURE 2. Orientations, spanning trees, and leftmost paths in simple triangulations

Surprisingly, $(G, \xi)$ may be recovered from the pair $(T, Y)$. More strongly, the above transformation is a bijection mapping planted simple planar triangulations to a certain set of "validly labelled" planted plane trees. This bijection is essentially due to Poulalhon and Schaeffer [31], but the connection of vertex labels with the lengths of certain oriented paths is new.

Since $Y(u)$ is the number of vertices on a certain path from $u$ to $A$, $Y(u) - 1$ is an upper bound on $d_G(u, A)$, the graph distance between $u$ and $A$ in $G$. It turns out that $Y(u) - d_G(u, A) - 1$ is bounded by twice the number of times a *shortest* path in $G$ from $u$ to $A$ winds clockwise around the *leftmost* path $P_{G,\xi}(u)$. More strongly, if $P(u) = (u_0, u_1, \ldots, u_k)$ and $Q$ is a path from $u_i$ to $u_j$ disjoint from $P(u)$ except at its endpoints, then $|Q| \ge j - i - 1$, and $|Q| \le j - i + 1$ (i.e. $Q$ is a shortcut from $u_i$ to $u_j$) only if $Q$ leaves $u_i$ on the right and rejoins $u_j$ on the left. This fact allows $Y(u) - d_G(u, A) - 1$ to be controlled as follows.

Let $n = |V(G)|$. If $Q$ is a shortcut from $u_i$ to $u_j$ then the union of $Q$ and $u_{i+1}, \ldots, u_{j-1}$ forms a cycle $C$ with $2(j - i) - 1$ or $2(j - i) - 2$ vertices. If there are $2k$ shortcuts between $u$ and $A$ and $Q$ is the $k$'th one, then all vertices of $C$ have distance at least $k$ both from $A$ and from $u$. It will follow that typically (i.e., for random $G$), when $k$ and $d_G(u_j, A)$ are both large (of order $n^{1/4}$) then $j - i$ should also be large (of order $n^{1/4}$), or else $G$ would contain a cycle of length $o(n^{1/4})$ separating two macroscopic regions. On the other hand, a "shortcut" of length of order $n^{1/4}$ is rather long; we will straightforwardly show that typically the diameter of $G$ will be $O(n^{1/4})$, in which case there can be at most a bounded number of such long shortcuts on any path. A rigorous version of this argument allows us to show that typically, for all $u \in V(T) \setminus \{v\} = V(G) \setminus \{v, A, B\}$, $Y(u) - d_G(u, A) - 1$ is much smaller than $n^{1/4}$. In other words, after rescaling, the labels $Y$ with high probability provide good approximations for distances to the root $A$. This essentially proves 1. from Section 1.2.

A modification of the above argument establishes without too much difficulty that for $u, w \in V(T)$ with $u$ preceding $w$ in lexicographic order, $d_G(u, w)$ is bounded by $Y(u) + Y(w) - 2\check{Y}(u, w) + 2$, where $\check{Y}(u, w)$ is the smallest value $Y(y)$ for any vertex $y$ following $u$ and preceding $w$ in lexicographic order. This will establish (2) from Section 1.2.

To establish (3) we use "partial symmetrization" as previously discussed. Finally, rerooting invariance, (4), will be a straightforward consequence of choosing a random root corner. Having verified all the conditions of our general convergence result (whose proof was already sketched), Theorem 1.1 for simple triangulations then follows immediately. An essentially identical development establishes Theorem 1.1 for simple quadrangulations.

1.4. **Outline.** We conclude the introduction by fixing some basic notation, in Section 1.5. In Section 2 we provide definitions related to planar maps and plane trees, many of which are standard. In Section 3 we introduce the Gromov–Hausdorff distance and mention some of its basic properties. In Section 4 we formally state our "universality" result, providing general sufficient conditions for a random map ensemble to converge to the Brownian map; proofs are deferred to Appendix B. In Section 5 we describe the bijections for simple triangulations and quadrangulations on which our proof of Theorem 1.1 is based. In Section 6 we prove convergence of the spatial branching process associated to a random simple triangulation to the Brownian snake; this is where partial symmetrization appears. In Section 7 we study the relation of distances with labels; this is where winding numbers appear. In Section 8, we use the bounds of Section 7 to show that our labelling provides a sufficiently close approximation of distances in random simple triangulations that the associated conditions of Theorem 4.1 are satisfied. In Section 9 we establish rerooting invariance and so complete the proof of Theorem 1.1. Finally, Section 10 proves Theorem 1.1 for quadrangulations, and Appendix A contains a derivation of the numerical constants from Theorem 1.1.

1.5. **Notation.** For the remainder of the paper, all graphs are connected, finite, simple (i.e. without loops nor multiple edges) and planar. Let $G = (V(G), E(G))$ be such a graph. Given a vertex $v \in V(G)$ we write $\deg_G(v) = |\{e \in E(G) : v \in e\}|$ for the degree of $v$ in $G$, and sometimes write $\deg(v)$ when $G$ is clear from context. If $v \in e$ we say $e$ is *incident* to $v$. We write $d_G : V(G) \times V(G) \to \mathbb{N}$ for graph distance on $G$. Given $W \subset V(G)$, we write $G[W]$ for the graph with vertices $W$ and edges $\{\{u, v\} \in E(G) : u, v \in W\}$.

An *oriented edge* of $G$ is an ordered pair $uw$, where $\{u, w\} \in E(G)$; we call $uw$ an *orientation* of $\{u, w\}$. An orientation of $G$ is a set $\overrightarrow{E} = \{\overrightarrow{e} : e \in E(G)\}$, where for each $e \in E(G)$, $\overrightarrow{e}$ is an orientation of $e$. The *outdegree* of $v \in V(G)$ (with respect to $\overrightarrow{E}$) is $\deg^+(v) = \deg^+_{\overrightarrow{E}}(v) = |\{w \in V(G) : vw \in \overrightarrow{E}\}|$.

If $S = (s_1, \ldots, s_r)$ is any sequence of objects, we say that $S$ has length $r$ and write $|S| = r$. A *path* in $G$ is a sequence $P = (u_0, u_1, \ldots, u_k)$ of vertices of $G$ with $\{u_i, u_{i+1}\} \in E(G)$ for $0 \le i < k$; we say $P$ is a path from $u_0$ to $u_k$, and note that $|P| = k + 1$. A path is *simple* if all its vertices are distinct. A *cycle* in $G$ is a path $(u_0, u_1, \ldots, u_k, u_{k+1})$ such that $u_{k+1} = u_0$; it is *simple* if $(u_0, \ldots, u_k)$ is a simple path. If $G$ is a *tree* (connected and acyclic) then for $u, w \in G$ we write $[\![u, v]\!]$ for the unique (shortest) path in $G$ from $u$ to $v$. Finally, for a non-negative integer $k$, write $[k] = \{0, 1, \ldots, k\}$,

## 2. Planar maps and plane trees

2.1. **Planar maps.** A *planar embedding* of $G$ is a function $\phi : V(G) \cup E(G) \to \mathbb{S}^2$ satisfying the following properties.

(1) The restriction $\phi|_{V(G)}$ is injective.
(2) For each $e = uv \in E(G)$, $\phi(e)$ is a simple curve with endpoints $\phi(u)$ and $\phi(v)$.
(3) For any two edges $e, f \in E(G)$, the curves $\phi(e)$ and $\phi(f)$ are disjoint except possibly at their endpoints.

The pair $(G, \phi)$ is called a *planar map*. The faces of $(G, \phi)$ are the connected components of $\mathbb{S}^2 \setminus \bigcup_{x \in V(G) \cup E(G)} \phi(x)$. Given a face $f$ the vertices and edges incident to $f$ are given by the set $\phi^{-1}(\partial f)$, where $\partial f$ is the boundary of $f$.

Two planar maps are *isomorphic* if there exists an orientation-preserving homeomorphism of $\mathbb{S}^2$ that sends one to the other. It is easily verified that planar map isomorphism is an equivalence relation.

For any planar map $(G, \phi)$, for each vertex $v \in V(G)$ there is a unique cyclic (clockwise) ordering $\mathcal{O}_v$ of the edges incident to $v$. Furthermore, up to isomorphism, the set of orderings $\{\mathcal{O}_v : v \in V(G)\}$ uniquely determines $(G, \phi)$. We may therefore specify the isomorphism equivalence class of $(G, \phi)$ by providing $G$ and the set of cyclic orderings associated to $(G, \phi)$. We will henceforth denote (a representative from the isomorphism equivalence class of) a planar map simply by $G$, leaving implicit both $\phi$ and its associated cyclic orderings.

For the remainder of Section 2.1, consider a fixed planar map $G$. A *corner of $G$ incident to $v$* is an ordered pair $\xi = (e, e')$ where $e$ and $e'$ are incident to $v$ and $e'$ follows $e$ in the clockwise order around $v$ and we also say that $e$ and $e'$ are incident to $\xi$.[3] We write $\mathrm{v}(\xi) = \mathrm{v}_G(\xi)$ for the vertex incident to $\xi$ in $G$. We write $\mathcal{C}(G)$ for the set of corners of $G$. Finally, if $e = \{u, v\}$ and $e' = \{v, w\}$, and $f$ is the face on the left when following $e$ and $e'$ from $u$ through $v$ to $w$, then we say $\xi = (e, e')$ is incident to $f$ and vice-versa. The *degree* of $f$ is the number of corners incident to $f$. The planar map $G$ is a *triangulation* or a *quadrangulation* if all its faces have respectively degree 3 or degree 4.

Given $e = \{u, v\} \in E(G)$, write $\kappa^\ell(u, v) = \kappa^\ell_G(u, v)$ (respectively, $\kappa^r(u, v) = \kappa^r_G(u, v)$) for the corner incident to $u$ and to $\{u, v\}$ that is on the left (respectively, on the right) when following $e$ from $u$ to $v$.

A *planted planar map* is a pair $(G, \xi)$, where $G$ is a planar map and $\xi \in \mathcal{C}(G)$. We call $\xi$ the *root corner* of $(G, \xi)$, call $\mathrm{v}(\xi)$ its *root vertex*, and call the face of $G$ incident to $\xi$ its *root face*. If $G'$ is a connected subgraph of $G$ containing $\xi$, then $(G', \xi)$ is again a planar map, and we call it a *planted submap* of $(G, \xi)$.

2.2. **Plane trees.** A plane tree (resp. planted plane tree) is a planar map $G$ (resp. planted planar map $(G, \xi)$) such that $G$ is a tree[4]. If $\mathrm{T} = (T, \xi)$ is a planted plane tree then recalling that $\mathrm{v}(\xi)$ is the root vertex of T, we may speak of parents, children, ancestors, descendants in the usual way. In particular, for each $w \in V(T) \setminus \{\mathrm{v}(\xi)\}$ we write $p(w) = p_{\mathrm{T}}(w)$ for the parent of $w$.

The *Ulam–Harris encoding* is the injective function $U = U_{\mathrm{T}} : V(T) \to \bigcup_{i \geq 0} \mathbb{N}^i$ defined as follows (let $\mathbb{N}^0 = \{\emptyset\}$ by convention). First, set $U(\mathrm{v}(\xi)) = \emptyset$. For every other vertex $w \in V(T)$, consider the unique path $\mathrm{v}(\xi) = v_0, v_1, \ldots, v_k = w$ from $\mathrm{v}(\xi)$ to $w$. For $1 \leq i \leq k$ let $n_i$ be such that $v_i$ is the $n_i$'th child of $v_{i-1}$, in cyclic order around $v_{i-1}$ starting from $\kappa^r(v_{i-1}, v_{i-2})$ if $i \geq 2$ or from $\xi$ if $i = 1$. Then set $U(w) = n_1 n_2 \ldots n_k \in \mathbb{N}^k$. In other words, the root receives label $\emptyset$ and for each $i \geq 1$ the label of any $i$'th child is obtained recursively by concatenating the integer $i$ to the label of its parent. It is easily verified that (the isomorphism class of) T can be recovered from the set of labels $\{U(v) : v \in V(T)\}$.

The *lexicographic ordering* $\preceq_{\mathrm{lex}} = \preceq_{\mathrm{lex},\mathrm{T}}$ of $V(T)$ is the total order of $V(T)$ induced by the lexicographic order on $\{U(v) : v \in V(T)\}$. This ordering induces a lexicographic ordering of $E(T)$ (also denoted $\preceq_{\mathrm{lex}} = \preceq_{\mathrm{lex},\mathrm{T}}$ by a slight abuse of notation) by defining $\{u, v\} \preceq_{\mathrm{lex},\mathrm{T}} \{u', v'\}$ if and only if $u, v \preceq_{\mathrm{lex},\mathrm{T}} u'$ or $u, v \preceq_{\mathrm{lex},\mathrm{T}} v'$. These are the orders in which a clockwise contour exploration of the plane tree $T$ starting from $\xi$ first visits the vertices and edges of $T$, respectively. For $u \in V(T)$, list the children of $u$ in lexicographic order as $c_{\mathrm{T}}(u, 1), \ldots, c_{\mathrm{T}}(u, k)$, where $k = k_{\mathrm{T}}(u) = \deg_T(u) - \mathbf{1}_{[u \neq \mathrm{v}(\xi)]}$ is the number of children of $u$ in $T$.

[3] We allow that $e = e'$, which can happen if $d_G(v) = 1$.

[4] It is relatively common to define a planted plane tree as a pair $(T, v)$ where $T$ is a plane tree and $v$ is a degree-one vertex of $T$. Our definition, which is equivalent, can be recovered by deleting the plant vertex and its incident edge, and rooting at the corner thereby created.

The *contour* exploration $r_{\mathrm{T}} : [2|V(T)|-2] \to V(T)$ is inductively defined as follows. Let $r_{\mathrm{T}}(0) = \mathrm{v}(\xi)$. Then, for $1 \le i \le 2|V(T)| - 2$, let $r_{\mathrm{T}}(i)$ be the lexicographically first child of $r_{\mathrm{T}}(i-1)$ that is not an element of $\{r_{\mathrm{T}}(0), \ldots, r_{\mathrm{T}}(i-1)\}$, or let $r_{\mathrm{T}}(i)$ be the parent of $r_{\mathrm{T}}(i-1)$ if no such node exists. Note that each vertex $v \in V(T) \setminus \{\mathrm{v}(\xi)\}$ appears $\deg_T(v)$ times in the contour exploration, and $\mathrm{v}(\xi)$ appears $\deg_T(\mathrm{v}(\xi)) + 1$ times.

The contour exploration induces an ordering of $\mathcal{C}(T)$, as follows. For $0 \le i < 2|V(T)|-2$, let $e_{\mathrm{T}}(i) = \{r_{\mathrm{T}}(i), r_{\mathrm{T}}(i+1)\}$. Then let $\xi_{\mathrm{T}}(0) = \xi$, and for $1 \le i < 2|V(T)| - 2$ let $\xi_{\mathrm{T}}(i) = (e_{\mathrm{T}}(i-1), e_{\mathrm{T}}(i))$. The *contour ordering*, denoted $\preceq_{\mathrm{ctr}}=\preceq_{\mathrm{ctr,T}}$, is the total order of $\mathcal{C}(T)$ induced by $(\xi_{\mathrm{T}}(i), 0 \le i < 2|V(T)|-2)$. For convenience, also let $\xi_{\mathrm{T}}(2|V(T)|-2) = \xi$. Finally, write $\preceq_{\mathrm{cyc}}=\preceq_{\mathrm{cyc,T}}$ for the cyclic order on $\mathcal{C}(T)$ induced by $\preceq_{\mathrm{ctr,T}}$. It can be verified that $\preceq_{\mathrm{cyc}}$ does not depend on the choice of root corner $\xi$.

Given $u, v \in V(T)$, we say that $v$ is the *successor* of $u$ if $u \preceq_{\mathrm{lex}} v$ and for all $w \in V(T)$, if $u \preceq_{\mathrm{lex}} w \preceq_{\mathrm{lex}} v$ then $w = u$ or $w = v$. We define successorship for corners in a similar fashion.

Given a plane tree $\mathrm{T} = (T, \xi)$ and a set $R \subset V(T)$ with $\mathrm{v}(\xi) \in R$, the *reduced tree* $\mathrm{T}(R)$ is the unique planted plane tree $(T', \xi')$ such that the following hold: (i) $V(T') = R$; (ii) for $u, v \in R$, $\{u, v\} \in E(T')$ if and only if one of $u, v$ is an ancestor of the other in T and $[\![u, v]\!] \cap R = \{u, v\}$; and (iii) the order $\preceq_{\mathrm{lex,T'}}$ of $R$ is the restriction of $\preceq_{\mathrm{lex,T}}$ to $R$. Also, the subtree of T *spanned* by $R$, denoted $\mathrm{T}\langle R\rangle$, is the subtree of T induced by the union of the shortest paths between all pairs of vertices in $R$. Note that $\mathrm{T}\langle R\rangle$ naturally inherits a planted plane tree structure from T.

2.3. **The contour process and spatial planted plane trees.** A *labelled planted plane tree* is a triple $\mathrm{T} = (T, \xi, D)$, where $(T, \xi)$ is a planted plane tree and $D : E(T) \to \mathbb{R}$ is an arbitrary function. Given a labelled plane tree, define a function $X := X_{\mathrm{T}} : V(T) \to \mathbb{R}$ as follows. First, let $X(\mathrm{v}(\xi)) = 0$. Next, given $u \in V(T)$ with $X(u)$ already defined, for $1 \le i \le k_{(T,\xi)}(u)$ let $X(c_{(T,\xi)}(u, i)) = X(u) + D(\{u, c_{(T,\xi)}(u, i)\})$.

Now define $C([0, 1], \mathbb{R})$ functions $C_{\mathrm{T}}$ and $Z_{\mathrm{T}}$ by setting

$$C_{\mathrm{T}}(i/(2|V(T)| - 2)) = d_T(\mathrm{v}(\xi), r_{(T,\xi)}(i)) \quad \text{and} \quad Z_{\mathrm{T}}(i/(2|V(T)| - 2)) = X_{\mathrm{T}}(r_{(T,\xi)}(i)) \,,$$

for $i \in \{0, 1, \ldots, 2|V(T)|-2\}$, and extending each function to $[0, 1]$ by linear interpolation.

We refer to $C_{\mathrm{T}}$ as the *contour process* of T. Note that the definition of $C_{\mathrm{T}}$ does not depend on the function $D$, so we may in fact view $C$ as a function of the planted plane tree $(T, \xi)$ and write $C_{(T,\xi)}$ instead of $C_{\mathrm{T}}$.

2.4. **Spanning trees in planar maps.** Given a planar map $G$, a *spanning tree* of G is a subgraph $T$ of $G$ such that $T$ is a tree with $V(T) = V(G)$. If $(G, \xi)$ is a planted planar map and $T$ is a spanning tree of $G$ then we call $(T, \xi)$ a planted spanning tree of $(G, \xi)$.

Finally, given a planted planar map $\mathrm{G} = (G, \xi)$ and an orientation $\overrightarrow{E}$ of $E(G)$, we say that a planted spanning tree $(T, \xi)$ of G is *oriented* with respect to $\overrightarrow{E}$ if in the orientation of $E(T)$ obtained from $\overrightarrow{E}$ by restriction, all edges are oriented towards $\mathrm{v}(\xi)$.

## 3. Distances between metric spaces: Gromov, Hausdorff, and Prokhorov

**The Gromov–Hausdorff distance.** For proofs of the assertions in this section, and for further details, we refer the reader to [10, 28]. Let $\mathrm{X} = (X, d)$ and $\mathrm{X}' = (X', d')$ be compact metric spaces. Given $C \subset X \times X'$, the *distortion* of $C$, denoted $\mathrm{dis}(C)$, is the quantity

$$\mathrm{dis}(C) = \sup\{|d(x, y) - d'(x', y')| : (x, x') \in C, (y, y') \in C\}.$$

A *correspondence* between X and X′ is a set $C \subset X \times X'$ such that for every $x \in X$ there is $x' \in X'$ such that $(x, x') \in C$ and vice versa. We write $C(X, X')$ for the set of correspondences between $X$ and $X'$. The Gromov–Hausdorff distance $d_{\mathrm{GH}}(\mathrm{X},\mathrm{X}')$ between metric spaces $\mathrm{X}=(X,d)$ and $\mathrm{X}'=(X',d')$ is

$$d_{\mathrm{GH}}(\mathrm{X},\mathrm{X}')=\frac{1}{2}\inf\{\mathrm{dis}(C):C\in C(X,X')\}.$$

We list without proof some basic properties of $d_{\mathrm{GH}}$. Let $\mathcal{M}$ be the set of isometry classes of compact metric spaces.

(1) Given metric spaces $\mathrm{X}=(X,d)$ and $\mathrm{X}'=(X',d')$, there exists $C\in C(X,X')$ such that $d_{\mathrm{GH}}(\mathrm{X},\mathrm{X}')=\mathrm{dis}(C)/2$.
(2) If $\mathrm{X}_1$ and $\mathrm{X}_2$ are isometric, and $\mathrm{X}_1'$ and $\mathrm{X}_2'$ are isometric, then $d_{\mathrm{GH}}(\mathrm{X}_1,\mathrm{X}_1')=d_{\mathrm{GH}}(\mathrm{X}_2,\mathrm{X}_2')$. In other words, $d_{\mathrm{GH}}$ is a *class function* for $\mathcal{M}$.
(3) The push-forward of $d_{\mathrm{GH}}$ to $\mathcal{M}$ (which we continue to denote $d_{\mathrm{GH}}$) is a distance on $\mathcal{M}$, and $(\mathcal{M},d_{\mathrm{GH}})$ is a complete separable metric space.

A *$k$-pointed metric space* is a triple $(X,d,(x_1,\ldots,x_k))$ where $(X,d)$ is a metric space and $x_i\in X$ for $1\le i\le k$. We say $k$-pointed metric spaces $\mathrm{X}=(X,d,(x_1,\ldots,x_k))$ and $\mathrm{X}'=(X',d',(x_1',\ldots,x_k'))$ are *isometry-equivalent* if there exists a bijective isometry $f:X\to X'$ such that $f(x_i)=x_i'$ for $1\le i\le k$. The *$k$-pointed Gromov–Hausdorff distance* $d_{\mathrm{GH}}^k$ between $\mathrm{X},\mathrm{X}'$ is given by

$$d_{\mathrm{GH}}^k(\mathrm{X},\mathrm{X}')=\frac{1}{2}\inf\left\{\mathrm{dis}(C):C\in C(X,X')\text{ and }(x_i,x_i')\in C,1\le i\le k\right\}.$$

Much as before, if $\mathcal{M}^{(k)}$ is the set of isometry-equivalence classes of $k$-pointed compact metric spaces, then $d_{\mathrm{GH}}^k$ is a class function for $\mathcal{M}^{(k)}$ so may be viewed as having domain $\mathcal{M}^{(k)}$, and $(\mathcal{M}^{(k)},d_{\mathrm{GH}}^k)$ then forms a complete separable metric space.

**The Gromov–Hausdorff–Prokhorov distance.** Following [28], a *weighted metric space* is a triple $(X,d,\mu)$ such that $(X,d)$ is a metric space and $\mu$ is a Borel probability measure on $(X,d)$. Weighted metric spaces $(X,d,\mu)$ and $(X',d',\mu')$ are *isometry-equivalent* if there exists a measurable bijective isometry $\phi:X\to X'$ such that $\phi_*\mu=\mu'$, where $\phi_*\mu$ denotes the push-forward of $\mu$ under $\phi$. Write $\mathcal{M}_w$ for the set of isometry-equivalence classes of weighted compact metric spaces.

Given weighted metric spaces $\mathrm{X}=(X,d,\mu)$ and $\mathrm{X}'=(X',d',\mu')$, a *coupling* between $\mu$ and $\mu'$ is a Borel measure $\nu$ on $X\times X'$ (for the product metric) with $\pi_*\nu=\mu$ and $\pi_*'\nu=\mu'$, where $\pi:X\times X'\to X$ and $\pi':X\times X'\to X'$ are the projection maps. Let $M(\mu,\mu')$ be the set of couplings between $\mu$ and $\mu'$. The *Gromov–Hausdorff–Prokhorov distance* is defined by

$$d_{\mathrm{GHP}}(\mathrm{X},\mathrm{X}')=\inf\left\{\epsilon>0:\exists C\in C(X,X'),\exists\nu\in M(\mu,\mu'),\nu(C)\ge 1-\epsilon,\mathrm{dis}(C)\le 2\epsilon\right\}.$$

The push-forward of $d_{\mathrm{GHP}}$ to $\mathcal{M}_w$, which we again denote $d_{\mathrm{GHP}}$, is a distance on $\mathcal{M}_w$, and $(\mathcal{M}_w,d_{\mathrm{GHP}})$ is a complete separable metric space (see [28, Section 6] and [13, Section 2]).

## 4. Chassaing–Schaeffer families

In this section we describe conditions under which a random map ensemble converges to the Brownian map. A *spatial map-tree pair* is a 4-tuple of the form $\mathrm{P}=(\mathrm{M},\mathrm{T},R,X)$, such that

(i) $\mathrm{T}=(T,\xi)$ is a planted plane tree,
(ii) $\mathrm{M}=(M,\zeta)$ is a planted planar map,
(iii) $R\subset V(M)\cap V(T)$, $\mathrm{v}_M(\zeta)=\mathrm{v}_T(\xi)$ and $\mathrm{v}_T(\xi)\in R$, and
(iv) $X:R\to\mathbb{R}$ is a labelling function with $X(\mathrm{v}_T(\xi))=0$.

Note that $T$ need not be a subgraph of $M$. A *marked* spatial map-tree pair is a 5-tuple $\mathrm{P}=(\mathrm{M},\mathrm{T},R,X,u)$ such that $(\mathrm{M},\mathrm{T},R,X)$ is a spatial map-tree pair and $u\in R$.

Given a 4- or 5-tuple P as above, let $C_{\mathrm{P}} : [0,1] \to Z_{\geq 0}$ and $Z_{\mathrm{P}} : [0,1] \to \mathbb{R}$ be defined by setting

$$C_{\mathrm{P}}(x) = d_T(v_T(\xi), r_{\mathrm{T}(R)}(x \cdot (2|R|-2))) , \quad Z_{\mathrm{T}}(x) = X(r_{\mathrm{T}(R)}(x \cdot (2|R|-2)))$$

for $x \in \{0, 1/(2|R|-2), 2/(2|R|-2), \ldots, 1\}$ and extending both functions to $[0,1]$ by linear interpolation (recall that $\mathrm{T}(R)$ is the *reduced* tree defined in Section 2.2).

For the remainder of the section, let $\mathcal{P} = (\mathcal{P}_n, n \in \mathbb{N})$ be a sequence of finite sets of marked spatial map-tree pairs, such that $\min(|R| : (\mathrm{M}, \mathrm{T}, R, X, u) \in \mathcal{P}_n) \to \infty$ as $n \to \infty$. Let $\mathrm{P}_n = (\mathrm{T}_n, \mathrm{M}_n, R_n, X_n, u_n)$ be a uniformly random element of $\mathcal{P}_n$, and write $\mathrm{T}_n = (T_n, \xi_n)$ and $\mathrm{M}_n = (M_n, \zeta_n)$. We say that $\mathcal{P}$ is a *Chassaing–Schaeffer* or *CS* family if there exist sequences $(a_n, n \in \mathbb{N})$ and $(b_n, n \in \mathbb{N})$ such that the following three properties hold.

**1.** As $n \to \infty$, $(a_n C_{\mathrm{P}_n}, b_n Z_{\mathrm{P}_n}) \xrightarrow{\mathrm{d}} (\mathbf{e}, Z)$ in the topology of uniform convergence on $C([0,1], \mathbb{R})^2$, where $(\mathbf{e}, Z)$ is as described in Section 1.1.
**2.** (i) For all $\epsilon > 0$,

$$\lim_{n\to\infty} \mathbf{P}\left\{ b_n \max_{v \in V(M_n)} d_{M_n}(v, R_n) > \epsilon \right\} = 0 .$$

(ii) Write $d_{\mathrm{Prok}}$ for the Prokhorov distance between Borel measures on $\mathbb{R}$. For each $n$, conditional on $\mathrm{P}_n$, let $U_n, V_n$ be independent uniformly random elements of $R_n$. Then

$$\lim_{n\to\infty} d_{\mathrm{Prok}}(b_n d_{M_n}(\mathrm{v}_{M_n}(\zeta_n), u_n), b_n d_{M_n}(U_n, V_n)) = 0 .$$

**3.** (i) Let $m = m(n) = 2|R_n| - 2$. Then for all $\epsilon > 0$,

$$\lim_{n\to\infty} \mathbf{P}\left\{ \exists i, j \in [m] : d_{M_n}(r_{\mathrm{T}_n(R_n)}(i), r_{\mathrm{T}_n(R_n)}(j)) \geq Z_{\mathrm{P}_n}(i/m) + Z_{\mathrm{P}_n}(j/m) - 2\max\left(\check{Z}_{\mathrm{P}_n}(i/m, j/m), \check{Z}_{\mathrm{P}_n}(j/m, i/m)\right) + \epsilon b_n^{-1} \right\} = 0 .$$

(ii) For all $\epsilon > 0$,

$$\lim_{n\to\infty} \mathbf{P}\left\{ \exists j \in [m] : d_{M_n}(r_{\mathrm{T}_n(R_n)}(j), u_n) \leq Z_{\mathrm{P}_n}(j/m) - \check{Z}_{\mathrm{P}_n}(0,1) - \epsilon b_n^{-1} \right\} = 0 .$$

For later use, we note one consequence of **3.** Let $I_n \in [2|R_n| - 2]$ be minimal such that $X_n(r_{\mathrm{T}_n(R_n)}(I_n)) = \check{Z}_n(0,1)$. Letting $i$ be such that $u_n = r_{\mathrm{T}_n(R_n)}(i)$, **3.**(ii) implies that

$$\lim_{n\to\infty} \mathbf{P}\left\{ |X_n(r_{\mathrm{T}_n(R_n)}(i)) - X_n(r_{\mathrm{T}_n(R_n)}(I_n))| > \epsilon b_n^{-1} \right\} = 0 .$$

In other words, $X_n(r_{\mathrm{T}_n(R_n)}(i))$ is, up to a $o(b_n^{-1})$ correction, the smallest displacement in $R_n$. Together with **3.**(i) and **3.**(ii) this yields that, for all $\epsilon > 0$,

$$\mathbf{P}\left\{ \exists j \in [2|R_n| - 2] : |d_{M_n}(r_{\mathrm{T}_n(R_n)}(j), u_n) - (X_n(r_{\mathrm{T}_n(R_n)}(j)) - X_n(r_{\mathrm{T}_n(R_n)}(I_n)))| > \frac{\epsilon}{b_n} \right\} \to 0 , \tag{3}$$

as $n \to \infty$. In other words, for $u \in R_n$, the distance $d_{M_n}(u, u_n)$ is essentially given by the difference in labels between the associated tree vertices.

**Theorem 4.1.** *If $\mathcal{P}$ is a CS family then, writing $\mu_n$ for the uniform probability measure on $R_n$,*

$$(V(M_n), b_n d_{M_n}, \mu_n) \xrightarrow{\mathrm{d}} (S, d, \mu)$$

*for $d_{\mathrm{GHP}}$, where $(S, d, \mu)$ is as defined in Section 1.1.*

The proof of Theorem 4.1 appears in Appendix B. We conclude the section by mentioning one corollary of the theorem; we are slightly informal to avoid notational excess and as the argument is straightforward. For $n, k \geq 1$, conditional on $\mathrm{P}_n$, let $U_{n,1}, \ldots, U_{n,k}$ be independent with law $\mu_n$. Proposition 10 of [28] implies that if the convergence in Theorem 4.1 holds then also

$$(V(M_n), b_n d_{M_n}, (U_{n,1}, \ldots, U_{n,k})) \xrightarrow{\mathrm{d}} (S, d, (U_1, \ldots, U_k)),$$

for $d_{\mathrm{GH}}^k$, where conditional on $(S, d, \mu)$, $U_1, \ldots, U_k$ are independent with law $\mu$. By Proposition 8.2 of [21], conditional on $(S, d, \mu)$, the points $\rho, u^* \in S$ are independent with law $\mu$; by **2.**(ii) it follows that $(V(M_n), b_n d_{M_n}, (\mathrm{v}_{M_n}(\zeta_n), u_n)) \xrightarrow{\mathrm{d}} (S, d, (\rho, u^*))$ for $d_{\mathrm{GH}}^2$.

## 5. Bijections for simple triangulations

We start with a summary of the results of the section; to do so some definitions are needed. For integer $k \geq 1$, a plane tree $T$ is a *$k$-blossoming tree* if each vertex of degree greater than one is incident to exactly $k$ vertices of degree one. If $T$ is a $k$-blossoming tree (for some $k$), we write $\mathcal{B} = \mathcal{B}(T)$ for the set of degree-one vertices of $T$. Note that both $k$ and $\mathcal{B}$ are uniquely determined by $T$. We call $\mathcal{B}$ the *blossoms* of $T$, and $V(T) \setminus \mathcal{B}$ the inner vertices of $T$. Also, an edge between two inner vertices is called an inner edge, and an edge between an inner vertex and a blossom is a *stem*. A corner $c$ is an inner corner if $\mathrm{v}(c) \notin \mathcal{B}$. A *planted* $k$-blossoming tree is a planted plane tree $(T, \xi)$ such that $T$ is a $k$-blossoming tree. The bijections of Section 5 concern 2-blossoming trees, which we simply call blossoming trees for the remainder of the section.

Write $\mathcal{T}_n$ for the set of planted blossoming trees $(T, \xi)$ with $n$ inner vertices and with $\mathrm{v}(\xi)$ an inner vertex. Fix $(T, \xi) \in \mathcal{T}_n$, and note that $|E(T)| = |V(T)| - 1 = 3n - 1$ so $|\mathcal{C}(T)| = 6n - 2 = 3|\mathcal{B}(T)| - 2$. We say $(T, \xi)$ is *balanced* if $\xi = (e, e')$ for distinct stems $e, e'$, and for all $c' \in \mathcal{C}(T)$,

$$3\left(|\{k \in \mathcal{C}(T) : \xi \preceq_{\mathrm{cyc}} k \preceq_{\mathrm{cyc}} c', \mathrm{v}(k) \in \mathcal{B}\}|\right) + 1 \geq |\{k \in \mathcal{C}(T) : \xi \preceq_{\mathrm{cyc}} k \preceq_{\mathrm{cyc}} c'\}|$$

(recall the definition of $\preceq_{\mathrm{cyc}}$) from Section 2.2). For $n \geq 1$ let $\mathcal{T}_n^{\circ} \subset \mathcal{T}_n$ be the set of balanced blossoming trees with $n$ inner vertices. Also, write $\mathcal{T}_n^{\bullet}$ for the set of triples $(T, \xi, \hat{\xi})$ with $(T, \xi) \in \mathcal{T}_n^{\circ}$ and $(T, \hat{\xi}) \in \mathcal{T}_n$.

A *valid labelling* of a planted plane tree $\mathrm{T} = (T, \xi')$ is a labelling $d = (d_e, e \in E(T))$ of the edges of $T$ by elements of $\{-1, 0, 1\}$ such that for all $v \in V(T)$, writing $k = k_{\mathrm{T}}(v)$, the sequence $d_{\{v, c_{\mathrm{T}}(v,1)\}}, \ldots, d_{\{v, c_{\mathrm{T}}(v,k)\}}$ is non-decreasing. Let $\mathcal{T}_n^{\mathrm{vl}}$ be the set of validly labelled plane trees with $n$ vertices. We emphasize that a validly labelled plane tree is a "normal" tree, not a blossoming tree.

Finally, recall that for $n \geq 3$, $\triangle_n^{\circ}$ is the set of planted triangulations with $n$ inner vertices. In Section 5.1, below, we associate to each $(G, \xi) \in \triangle_n^{\circ}$ a canonical set $\hat{\mathcal{C}}(G, \xi) \subset \mathcal{C}(G)$ with $|\hat{\mathcal{C}}(G, \xi)| = 4n - 2$, and let $\triangle_n^{\bullet} = \{(G, c, \hat{c}) : (G, c) \in \triangle_n^{\circ}, \hat{c} \in \hat{\mathcal{C}}(G, c)\}$.

The following diagram summarizes the bijective relations between $\mathcal{T}_n, \mathcal{T}_n^{\bullet}, \mathcal{T}_n^{\circ}, \triangle_{n+2}^{\bullet}$, and $\triangle_{n+2}^{\circ}$ established in [32] and in the current section, primarily in Propositions 5.1, 5.2, and 5.5. (We may already verify that the projection from $\mathcal{T}_n^{\bullet}$ to $\mathcal{T}_n^{\circ}$ is $(4n-2)$-to-1, since a blossoming tree with $n$ inner vertices has $4n - 2$ inner corners.)

$$\begin{array}{ccccccc}
\mathcal{T}_n^{\mathrm{vl}} & \xleftarrow[\text{bij}]{\phi_n;\ \text{Prop.5.5}} & \mathcal{T}_n & \xleftarrow[2-\text{to}-1]{\psi_n;\ \text{Prop.5.2}} & \mathcal{T}_n^{\bullet} & \xrightarrow[(4n-2)-\text{to}-1]{\text{projection}} & \mathcal{T}_n^{\circ} \\
 & & & & \chi_n^{\bullet} \downarrow \text{bij} & & \text{bij} \downarrow \chi_n;\ \text{Prop.5.1} \\
 & & & & \triangle_{n+2}^{\bullet} & \xrightarrow[(4n-2)-\text{to}-1]{\text{projection}} & \triangle_{n+2}^{\circ}
\end{array} \quad (4)$$

After concluding with bijective arguments, in Section 5.4 we explain how to sample uniformly random triangulations using conditioned Galton-Watson trees. We end the section by describing the inverse of the bijection $\chi_n : \mathcal{T}_n^\circ \to \triangle_{n+2}^\circ$, which will be needed later.

5.1. **A bijection between triangulations and blossoming trees.** We first describe a bijection of Poulalhon and Schaeffer [31] between balanced blossoming trees and simple, planted triangulations of the sphere (see Figure 3; the orientations of the arrows in the figure are explained in Section 5.5). Fix a blossoming tree $T$. Given a stem $\{b, u\}$ with $b \in \mathcal{B}(T)$, if $bu$ is followed by two inner edges in a clockwise contour exploration of $T$ – $uv$ and $vw$, say – then the *local closure* of $\{b, u\}$ consists in removing the blossom $b$ and its stem, and adding a new edge $\{u, w\}$ (such that $\kappa^r(u, w) = (\{u, w\}, \{u, v\})$ and $\kappa^\ell(w, u) = (\{w, v\}, \{w, u\})$). After performing the local closure, $uw$ always has a triangle on its right. The edge $\{u, w\}$ is considered to be an inner edge in subsequent local closures.

The *partial closure* of a blossoming tree is the planar map obtained by performing all possible local closures. Equivalently, for each corner $c$ with $\mathrm{v}(c) \in \mathcal{B}$, let $s(c)$ be the corner $c'$ minimizing $|\{k \in \mathcal{C}(T), c \preceq_{\mathrm{cyc}} k \prec_{\mathrm{cyc}} c'\}|$ subject to the condition that

$$3|\{k \in \mathcal{C}(T) : c \preceq_{\mathrm{cyc}} k \preceq_{\mathrm{cyc}} c', \mathrm{v}(k) \in \mathcal{B}\}| < |\{k \in \mathcal{C}(T), c \preceq_{\mathrm{cyc}} k \preceq_{\mathrm{cyc}} c'\}|, \tag{5}$$

if such a corner exists (recall the definition of $\preceq_{\mathrm{cyc}}$) from Section 2.2). The partial closure operation identifies $\mathrm{v}(c)$ with $\mathrm{v}(s(c))$ whenever $\mathrm{v}(c) \in \mathcal{B}$ and $s(c)$ is defined; it follows from the latter description that the partial closure does not depend on the order in which local closures take place. Say $\mathrm{v}(c)$ is *closed* if $s(c)$ is defined, and otherwise say $\mathrm{v}(c)$ is *unclosed*.

It can be checked that the partial closure is a simple map and contains precisely one face $f$ of degree greater than three, and all unclosed blossoms are incident to $f$. Furthermore, simple counting arguments show that each inner corner incident to $f$ is adjacent to at least one unclosed blossom, and that there are precisely two corners, say $\xi^C$ and $\xi^D$, that are incident to two unclosed blossoms. Note that $\xi^C$ and $\xi^D$ are both corners of $T$ (i.e., they are not created while performing the partial closure). Let $C = \mathrm{v}(\xi^C)$ and $D = \mathrm{v}(\xi^D)$.

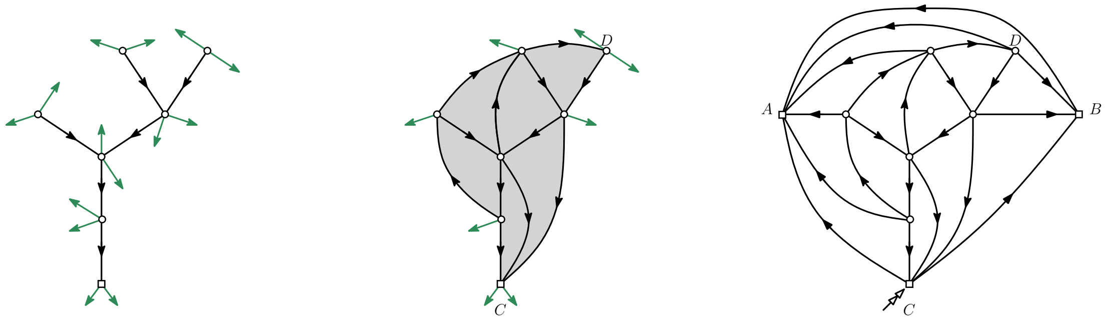
Given $\xi \in \mathcal{C}(T)$, we say the planted blossoming tree $(T, \xi)$ is *balanced* if $\xi = \xi^C$ or $\xi = \xi^D$, and in this case call $(T, \xi)$ a *balanced blossoming tree*. It follows straightforwardly from (5) that this definition of balanced agrees with the one given at the start of the section. We now suppose $\xi \in \{\xi^C, \xi^D\}$. Let $S_{CD}$ (resp. $S_{DC}$) be the set of non-blossom vertices $v$ of the distinguished face $f$ of the partial closure such that in the planted tree $(T, C)$ (resp. $(T, D)$) we have $v \preceq_{\mathrm{ctr}} D$ (resp. $v \preceq_{\mathrm{ctr}} C$). In other words, vertices of $S_{CD}$ lie after $C$ and before $D$ in a clockwise tour of $f$, and likewise for $S_{DC}$.

To finish the construction, remove the remaining blossoms and their stems. Add two additional vertices $A$ and $B$ within $f$, then add an edge between $A$ (resp. $B$) and each of the vertices of $S_{CD}$ (resp. of $S_{DC}$). In the resulting map, define a corner $c$ by $c = (\{C, B\}, \{C, A\})$ if $\mathrm{v}(\xi) = C$ or $c = (\{D, A\}, \{D, B\})$ if $\mathrm{v}(\xi) = D$. Finally, add an edge between $A$ and $B$ in such a way that, after its addition, $A, B$, and $\mathrm{v}(\xi)$ lie on the same face $f$. The result is a planar map, rooted at $\xi$, called the *closure* of $T$. For later use, define a function $s' : V(T) \to V(T)$ as follows. First, set $s'(v) = v$ for $v \in V(T) \setminus \mathcal{B}$. For $v \in \mathcal{B}$, let $u$ be the unique neighbour of $v$ and let $k$ be the unique corner incident to $v$. If $s(k)$ is defined then let $s'(v) = \mathrm{v}(s(k))$; otherwise, if $u \in S_{CD}$ let $s'(v) = A$ and if $u \in S_{DC}$ let $s'(v) = B$.

Write $\chi : \bigcup_{n \geq 1} \mathcal{T}_n^\circ \to \bigcup_{n \geq 1} \triangle_{n+2}^\circ$ for the function sending a balanced blossoming tree to its closure, and for $n \geq 1$ let $\chi_n : \mathcal{T}_n^\circ \to \triangle_{n+2}^\circ$ be the restriction of $\chi$ to $\mathcal{T}_n^\circ$.

**Proposition 5.1** ([32]). *For all $n \geq 1$, $\chi_n$ is a bijection between $\mathcal{T}_n^\circ$ and $\triangle_{n+2}^\circ$.*

It bears emphasis that we only consider balanced blossoming trees $(T, \kappa)$ up to isomorphism of planted planar maps. In particular, if $(T, \xi_C)$ and $(T, \xi_D)$ are isomorphic then $T$ only corresponds to one planted triangulation.




(a) A balanced blossoming tree

(b) all the local closures have been performed

(c) The resulting rooted simple triangulation, endowed with its unique minimal 3-orientation

FIGURE 3. The closure of a balanced tree into a simple triangulation.

Note that if $(T,\xi)$ is a blossoming tree and $\chi(T,\xi)=(G,c)$ then it is natural to identify the inner vertices and inner edges of $T$ with subsets of $V(G)$ and $E(G)$, respectively. More formally, we may choose representatives from the isomorphism equivalence classes of the tree and its closure so that $V(T)\setminus\mathcal{B}(T)=V(G)\setminus\{A,B\}$ and $\{\{u,v\}\in E(T): u,v\notin\mathcal{B}\}\subset E(G)$. We will adopt this perspective in the remainder of the paper.

Now let let $\psi_n$ be the map from $\mathcal{T}_n^\bullet$ to $\mathcal{T}_n$ which sends $(T,\xi,\hat{\xi})$ to $(T,\hat{\xi})$.

**Proposition 5.2.** *$\psi_n:\mathcal{T}_n^\bullet\to\mathcal{T}_n$ is a two-to-one map.*

*Proof.* Fix $(T,\hat{\xi})\in\mathcal{T}_n$, and let $\xi^1,\xi^2$ be the two corners of $T$ for which $(T,\xi^1)$ and $(T,\xi^2)$ are balanced blossoming trees. We consider two cases depending on the symmetries of $T$.

First, if $(T,\xi^1)$ and $(T,\xi^2)$ are not isomorphic (as planted plane trees) then $(T,\xi^1,\hat{\xi})$ and $(T,\xi^2,\hat{\xi})$ are distinct elements of $\mathcal{T}_n^\bullet$ and so $|\psi_n^{-1}(T,\hat{\xi})|=2$.

Next suppose that $(T,\xi^1)$ and $(T,\xi^2)$ are isomorphic, and fix an automorphism $a:T\to T$ with $a(\xi^1)=\xi^2$. Then $(T,\hat{\xi})$ and $(T,a(\hat{\xi}))$ are necessarily isomorphic. In this case $(T,\xi^1,\hat{\xi})$ and $(T,\xi^2,a(\hat{\xi}))$ are distinct elements of $\mathcal{T}_n^\bullet$ and so again $|\psi_n^{-1}(T,\hat{\xi})|=2$. $\square$

Fix $(T,\xi)\in\mathcal{T}_n^\circ$. Note that necessarily $\mathrm{v}(\xi)$ is an inner vertex and that $\xi$ is adjacent to two stems. Let $(G,c)=\chi_n(T,\xi)\in\triangle_{n+2}^\circ$ and list the vertices of the root face of $(G,c)$ in clockwise order as $(\mathrm{v}(c),A,B)$ (i.e. such that $\kappa^\ell(\mathrm{v}(c),A)=\kappa^r(\mathrm{v}(c),B)=c$). Define a function $\hat{\chi}_n$ from the inner corners of $T$ to $\mathcal{C}(G)$ as follows. Recall the definition of $s':V(T)\to V(T)$ from Page 13. Every corner in $c=\mathcal{C}(T)$ may be written as $c=\kappa^\ell(u,v)$ for a unique edge $\{u,v\}$ of $T$ with $u\notin\mathcal{B}(T)$; let $\hat{\chi}(c)=\kappa^\ell(s'(u),s'(v))$, and write $\hat{\mathcal{C}}(G,c)=\{\hat{\chi}(c): c \text{ an inner corner of } T\}$. Since $T$ is 2-blossoming it has $4n-2$ inner corners, so also $|\hat{\mathcal{C}}(G,\xi)|=4n-2=4|V(G)|-10$. Also, having defined $\hat{\mathcal{C}}(G,c)$, the definition of $\triangle_{n+2}^\bullet$ from the start of the section is complete. Furthermore, it is clear that the projection from $\triangle_{n+2}^\bullet$ to $\triangle_{n+2}^\circ$ sending $(G,c,\hat{c})$ to $(G,c)$ is $(4n-2)$-to-1.

Now let $\chi_n^\bullet:\mathcal{T}_n^\bullet\to\triangle_{n+2}^\bullet$ be defined as follows. For $(T,\xi,\hat{\xi})\in\mathcal{T}_n^\bullet$, let $(G,c)=\chi_n(T,\xi)$, let $\hat{\chi}_n(\hat{\xi})=\hat{c}$, and set $\chi_n^\bullet(T,\xi,\hat{\xi})=(G,c,\hat{c})$. For all $(G,c,\hat{c})\in\triangle_{n+2}^\bullet$ there is then a unique triple $(T,\xi,\hat{\xi})\in\mathcal{T}_n^\bullet$ such that $\chi(T,\xi)=(G,c)$ and $\hat{\chi}(\hat{\xi})=\hat{c}$. In other words, $\chi_n^\bullet$ is a bijection.

5.2. **Bijection with labels.** We now present an alternative description of the bijection from Proposition 5.1, based on (5). Given a blossoming tree $(T,\xi)$, write $\mathrm{T}=(T,\xi)$ and define $\lambda:=\lambda_{\mathrm{T}}:\mathcal{C}(T)\to\mathbb{Z}$ as follows. Recall the definition of the contour ordering $(\xi_{\mathrm{T}}(i), 0\le i\le 2|V(T)|-2)$ from Section 2.2, and in particular that $\xi_{\mathrm{T}}(0)=\xi$. Let

$\lambda_{\mathrm{T}}(\xi_{\mathrm{T}}(0)) = 2$ and, for $0 \le i < 2|V(T)| - 3$, set

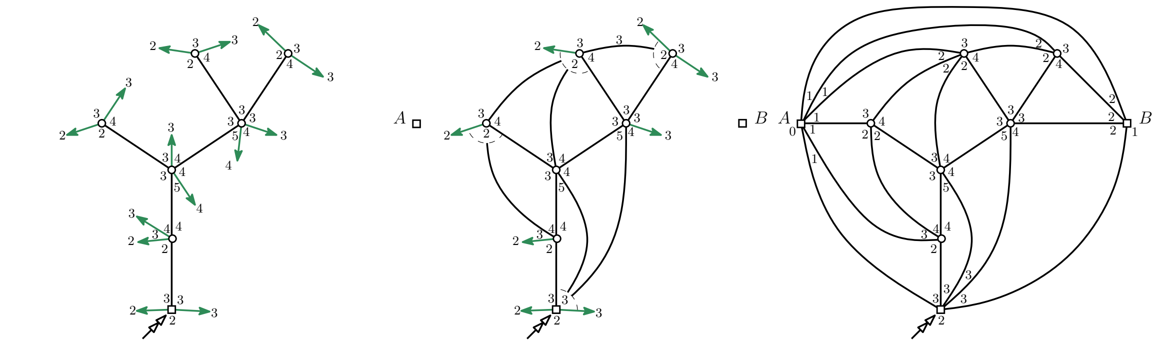
$$\lambda_{\mathrm{T}}(\xi_{\mathrm{T}}(i+1)) = \begin{cases} \lambda_{\mathrm{T}}(\xi_{\mathrm{T}}(i)) - 1 & \text{if } \mathrm{v}(\xi_{\mathrm{T}}(i)) \notin \mathcal{B}(T), \mathrm{v}(\xi_{\mathrm{T}}(i+1)) \notin \mathcal{B}(T), \\ \lambda_{\mathrm{T}}(\xi_{\mathrm{T}}(i)) & \text{if } \mathrm{v}(\xi_{\mathrm{T}}(i)) \notin \mathcal{B}(T), \mathrm{v}(\xi_{\mathrm{T}}(i+1)) \in \mathcal{B}(T), \\ \lambda_{\mathrm{T}}(\xi_{\mathrm{T}}(i)) + 1 & \text{if } \mathrm{v}(\xi_{\mathrm{T}}(i)) \in \mathcal{B}(T), \mathrm{v}(\xi_{\mathrm{T}}(i+1)) \notin \mathcal{B}(T), \end{cases}$$

This labelling is depicted in Figure 4(a). Informally, we perform a clockwise contour exploration of the tree and label the corners as we go. When leaving an inner vertex and arriving at an inner vertex, decrease the label by one; when leaving an inner vertex and arriving at a blossom, leave the label unchanged; when the leaving a blossom and arriving at an inner vertex, increase the label by one.

It is not hard to see that $\mathrm{T} = (T, \xi)$ is balanced if and only if $\xi$ is incident to two stems and $\lambda_{\mathrm{T}}(c) \ge 2$ for all $c \in \mathcal{C}(T)$ (see Figure 4(a)). Assume $(T, \xi)$ is balanced and write $\xi'$ for the unique corner in $\mathcal{C}(T) \setminus \{\xi\}$ for which $(T, \xi')$ is also balanced. Given a corner $c \in \mathcal{C}(T)$ with $\mathrm{v}(c) \in \mathcal{B}(T)$, recall the definition of $s(c)$ from (5). A counting argument shows that when $s(c)$ is defined, it is equal to the first corner $c'$ following $c$ in clockwise order for which $\lambda_{(T,\xi)}(c') < \lambda_{(T,\xi)}(c)$ (and in fact $\lambda_{(T,\xi)}(s(c)) = \lambda_{(T,\xi)}(c) - 1$). Furthermore, $s(c)$ is defined if and only if either $\lambda_{\mathrm{T}}(c) > 2$ and $c \preceq_{\mathrm{ctr,T}} \xi'$, or $\lambda_{\mathrm{T}}(c) > 3$ and $\xi' \preceq_{\mathrm{ctr,T}} c$.

Next, add two vertices, say $A$ and $B$, within the unique face of the partial closure with degree greater than three. For each $c \in \mathcal{C}(T)$ with $\mathrm{v}(c) \in \mathcal{B}(T)$ and $s(c)$ undefined, identify $\mathrm{v}(c)$ with $A$ if $\lambda_{\mathrm{T}}(c) = 2$, and with $B$ if $\lambda_{\mathrm{T}}(c) = 3$. At this point, the unique face of degree greater than three is incident to $\xi, A, \xi'$ and $B$ in cyclic order. Finally, add a single edge between $A$ and $B$. The following fact, whose straightforward proof is omitted, states that the resulting planar map is $\chi(T, \xi)$.

**Fact 5.3.** *The triangulation obtained from a balanced blossoming tree by iterating local closures and the one obtained by the label procedure coincide.* □




(a) The corner labelling of a balanced blossoming tree

(b) The labelled partial closure

(c) The resulting corner-labelled simple triangulation

FIGURE 4. Closing a balanced tree via the corner labelling.

The closure contains corners not present in the blossoming tree, and the new corners are labelled as follows. For any bud corner $c$ with $s(c)$ defined, closing $\mathrm{v}(c)$ may be viewed as splitting a single corner in two, and the two new corners inherit the label of the corner that was split. An example is shown in Figure 4(b); the dashed arcs denote corners that are "split" by the partial closure operation. Let $f$ be the face of $\chi(T, \xi)$ incident to $\xi$. Give the corner of $A$ (resp. $B$) incident to $f$ label 0 (resp. 1), and give all other corners incident to $A$ (resp. $B$) label 1 (resp. 2). We write $\lambda^* = \lambda^*_{(T,\xi)}$ for this corner labelling of $\chi(T, \xi)$, and note that $\lambda^* : \mathcal{C}(\chi(T, \xi)) \to \mathbb{Z}^{\ge 0}$ since we have assumed $(T, \xi)$ is balanced. An example of the resulting corner-labelled triangulation is depicted in Figure 4(c).

5.3. **From labels to displacement vectors.** We next explain the connection between blossoming trees and validly labelled plane trees. Fix $n \geq 1$, let $(T, \hat{\xi}) \in \mathcal{T}_n$ and let $\lambda = \lambda_{T,\hat{\xi}} : \mathcal{C}(T) \to \mathbb{Z}$ be as defined in Section 5.2. We define a function $Y = Y_{(T,\hat{\xi})} : V(T) \to \mathbb{Z}$ by setting $Y(v) = \min\{\lambda(c) : c \in \mathcal{C}(T), \mathrm{v}(c) = v\}$ for all $v \in V(T)$. Next, for each inner edge $e \in E(T)$, writing $e = \{v, p(v)\}$, with $v \in V(T) \setminus \{\mathrm{v}(\hat{\xi})\}$, set $D_e = D_e(T, \hat{\xi}) = Y(v) - Y(p(v))$. The following easy fact, whose proof is omitted, allows us to recover the locations of stems from the edge labels.

**Fact 5.4.** *For all $e = \{v, p(v)\} \in E(T)$, $D_e = |\{e' \preceq_{\mathrm{lex}} e : e' \text{ a stem incident to } p(v)\}| - 1$.* □

Now fix $v \in V(T)$, let $k = k_{(T,\hat{\xi})}$, and for $1 \leq i \leq k$ let $e_i = \{v, c_{(T,\hat{\xi})}(v, i)\}$. It follows from the above fact that for $1 \leq i \leq k$ the number of stems $e$ incident to $v$ with $e \preceq_{\mathrm{lex}} e_i$ is $D_{e_i} + 1$. In particular $(D_{e_i}, 1 \leq i \leq k)$ is a non-decreasing sequence of elements of $\{-1, 0, 1\}$; this is what allows us to connect blossoming trees with validly labelled trees.

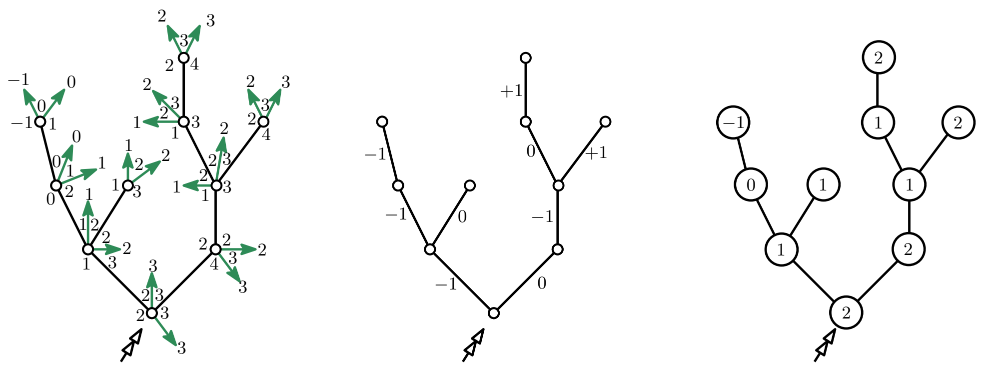
For $n \geq 1$ define a map $\phi_n : \mathcal{T}_n \to \mathcal{T}_n^{\mathrm{vl}}$ as follows. Given $(T, \hat{\xi}) \in \mathcal{T}_n$, write $\hat{\xi} = (e_-, e_+)$. Let $e$ be the last inner edge incident to $\mathrm{v}(\hat{\xi})$ preceding $e_-$ in clockwise order (with $e = e_-$ if $e_-$ is an inner edge), and let $e'$ be the first inner edge incident to $\mathrm{v}(\hat{\xi})$ following $e_+$ in clockwise order (with $e = e_+$ if $e_+$ is an inner edge). Write $\xi' = (e, e')$, let $T'$ be the subtree of $T$ induced by the inner vertices, let $D = (D_e(T, \hat{\xi}), e \in E(T))$, and let $\phi_n(T, \hat{\xi}) = (T', \xi', D)$. The following proposition is an immediate consequence of Fact 5.4.

**Proposition 5.5.** *The map $\phi_n : \mathcal{T}_n \to \mathcal{T}_n^{\mathrm{vl}}$ is a bijection. Furthermore, given $(T', \xi', D) \in \mathcal{T}_n^{\mathrm{vl}}$, the inverse $\phi_n^{-1}(T', \xi', D)$ is obtained as follows. For each corner $c = (\{u, v\}, \{v, w\})$ with $\deg_{T'}(v) \geq 2$:*

(i) *if $u = p(v)$ then attach $D_{\{v,w\}} + 1$ stems to $\mathrm{v}(c)$ in corner $c$;*
(ii) *if $w = p(v)$ then attach $1 - D_{\{u,v\}}$ stems to $\mathrm{v}(c)$ in corner $c$;*
(iii) *if $p(u) = v = p(w)$ then attach $D_{\{v,w\}} - D_{\{u,v\}}$ stems to $\mathrm{v}(c)$ in corner $c$.*

*Finally, attach two stems to each vertex $v$ with $k_{(T',\xi')}(v) = 0$.* □

The above bijection and definitions are illustrated in Figure 5. In the next section, we explain how the above functions can be used to sample random simple triangulations with the aid of conditioned Galton–Watson trees.



(a) The corner labelling of a corner-rooted blossoming tree.

(b) The tree $T'$ with values $D^e$ marked on edges

(c) The tree $T'$ with values $Y(v)$ marked on vertices.

FIGURE 5. The equivalence between blossoming trees and validly vector-labelled plane trees. The root corner is indicated via a double arrow.

5.4. **Corner-rooted triangulations via conditioned Galton–Watson trees.** Let $(T_n, \xi_n)$ be uniformly distributed on $\mathcal{T}_n$. We are now able to describe the law of $(T_n, \xi_n)$ as a modification of the law of a critical Galton–Watson tree conditioned to have a given size. (Galton–Watson trees are naturally viewed as planted plane trees; see e.g. Le Gall [19].) Let $G \overset{\mathrm{d}}{=} \mathrm{Geometric}(3/4)$, and let $B$ have law given by

$$\mathbf{P}\{B = c\} = \frac{\binom{c+2}{2}\mathbf{P}\{G = c\}}{\mathbf{E}\binom{G+2}{2}}, \text{ for } c \in \mathbb{N}. \tag{6}$$

**Fact 5.6.** *The distribution $B$ is critical, i.e. $\mathbf{E}B = 1$.* □

This fact follows from simple computations involving the 3 first moments of a geometric law; its proof is omitted.

**Proposition 5.7.** *Let $(T', \xi')$ be a Galton–Watson tree with branching factor $B$ conditioned to have $n$ vertices. For each vertex $v$ of $T'$, writing $B_v$ for the number of children of $v$ in $T'$, add two stems incident to $v$, uniformly at random from among the $\binom{B_v+2}{2}$ possibilities. The resulting planted plane tree $(T, \hat{\xi})$ is uniformly distributed over $\mathcal{T}_n$.*

*Proof.* Fix $\mathrm{t} \in \mathcal{T}_n$ and let $\mathrm{t}' = \mathrm{t}(V(T) \setminus \mathcal{B}(T))$ be the tree t with its blossoms removed. List the vertices of $\mathrm{t}'$ in lexicographic order as $v_1, \ldots, v_n$ and recall that $k_{\mathrm{t}'}(v_i)$ is the number of children of $v_i$ in $\mathrm{t}'$.

Then $(T, \hat{\xi})$ is equal to t if and only if $(T', \xi') = \mathrm{t}'$ and for each $v \in V(T')$, the blossoms are inserted at the right place. Hence:

$$\begin{aligned}
\mathbf{P}\left\{(T, \hat{\xi}) = \mathrm{t}\right\} &\propto \prod_{i=1}^{n} \frac{1}{\binom{k_{\mathrm{t}'}(v_i)+2}{2}} \mathbf{P}\{B = k_{\mathrm{t}'}(v_i)\} \\
&= \prod_{i=1}^{n} \frac{1}{\binom{k_{\mathrm{t}'}(v_i)+2}{2}} \binom{k_{\mathrm{t}'}(v_i) + 2}{2} \mathbf{P}\{G = k_{\mathrm{t}'}(v_i)\} \\
&= \frac{3^{n-1}}{4^{2n-1}}.
\end{aligned}$$

The last equality holds since $G$ is geometric and $\sum_{i=1}^{n} k_{\mathrm{t}'}(v_i) = n - 1$. Since the last term does not depend on the shape of t, all blossoming trees with $n$ vertices appear with the same probability. □

**Corollary 5.8.** *With $(T, \hat{\xi})$ as in Proposition 5.7, let $\xi^1, \xi^2 \in \mathcal{C}(T)$ be such that $(T, \xi^i)$ is balanced for $i \in \{1, 2\}$. Conditional on $(T, \hat{\xi})$ choose $\xi \in \{\xi^1, \xi^2\}$ uniformly at random. Then $(G, c, c') = \chi^\bullet(T, \xi, \hat{\xi})$ is uniformly distributed in $\triangle^\bullet_{n+2}$, and so $(G, c)$ is uniformly distributed in $\triangle^\circ_{n+2}$.*

*Proof.* Conditional on $(T, \hat{\xi})$ the triple $(T, \xi, \hat{\xi})$ is a uniformly random element of the preimage of $(T, \hat{\xi})$ under $\phi_n$. By Proposition 5.7 $(T, \hat{\xi})$ is uniformly distributed in $\mathcal{T}_n$, and the result is then immediate from (4). □

Proposition 5.5 now allows us to describe the distribution of a uniformly random element $(T', \xi', D)$ of $\mathcal{T}_n^{\mathrm{vl}}$. For each $k \geq 1$, let $\nu_k$ be the uniform law over non-decreasing vectors $(d_1, \ldots, d_k) \in \{-1, 0, 1\}^k$.

**Corollary 5.9.** *Let $(T', \xi')$ be a Galton–Watson tree with branching factor $B$ conditioned to have $n$ vertices. Conditional on $(T', \xi')$, independently for each $v \in V(T')$ let $(D_{\{v, c_{(T',\xi')}(v,j)\}}, 1 \leq j \leq k_{(T',\xi')}(v))$ be a random vector with law $\nu_{k_{(T',\xi')}(v)}$. Finally, let $D = (D_e, e \in E(T'))$. Then $(T', \xi', D)$ is uniformly distributed in $\mathcal{T}_n^{\mathrm{vl}}$.*

*Proof.* By Proposition 5.5, $(T', \xi')$ is uniformly distributed in $\mathcal{T}_n$. The result then follows from Proposition 5.7. □

For later use, we note the following fact. Recall the definition of $X_{\mathrm{T}}$ for T a labelled planted plane tree, from Section 2.3.

**Fact 5.10.** *Fix $\xi_1, \xi_2 \in \mathcal{C}(T)$ with $\mathrm{v}(\xi_1), \mathrm{v}(\xi_2)$ inner corners, and let $\mathrm{T}_1 = (T', \xi_1', D_1) = \phi_n(T, \xi_1)$ and $\mathrm{T}_2 = (T', \xi_2', D_2) = \phi_n(T, \xi_2)$. Then for all $v \in V(T')$, $X_{\mathrm{T}_1}(v) = Y_{(T,\xi_1)}(v) - 2$, and $\left|\left(Y_{(T,\xi_1)}(v) - Y_{(T,\xi_2)}(v)\right) - X_{(T,\xi_1)}(\mathrm{v}(\xi_2))\right| \leq 3$.*

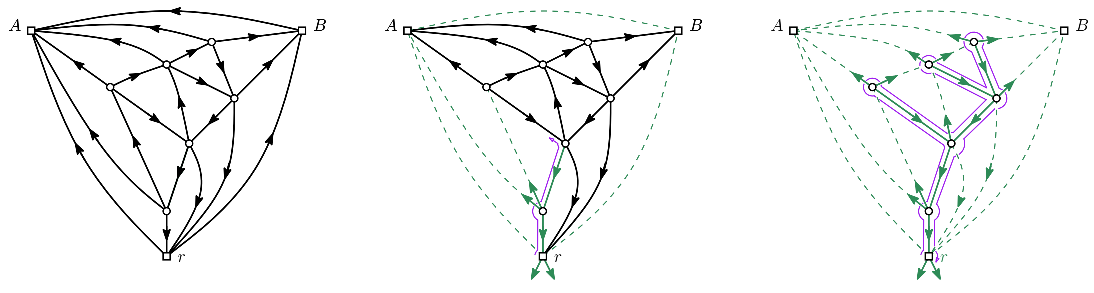
In other words the labellings $X_{\mathrm{T}}$ and $Y_{\mathrm{T}}$ are related by an additive constant of 2, and rerooting shifts all labels according to the label of the new root under the old labelling, up to an additive error of 3. This is a direct consequence of Fact 5.4 and the definitions of $X_{\mathrm{T}}$ and $Y_{\mathrm{T}}$; its proof is omitted.

We conclude Section 5 by explaining the inverse of the bijection $\chi_n$. The description of the inverse relies the properties of so called 3-orientations for simple triangulations. We make use of such orientations in Section 7 when studying the relation between vertex labels and geodesics.

5.5. **Orientations and the opening operation.** In a planted map endowed with an orientation, a directed cycle is said to be clockwise if the root corner is situated on its left and counterclockwise otherwise. An orientation is called *minimal* if it has no counterclockwise cycles. Let $(G, \xi)$ be a planted planar triangulation, and recall from Section 1.3 that that $(G, \xi)$ admits a unique minimal 3-orientation. We next describe how to obtain this 3-orientation via the bijection described in Proposition 5.1.

Given a balanced 2-blossoming tree $\mathrm{T} = (T, \xi)$, orient all stems towards their incident blossom, and orient all other edges towards $\mathrm{v}(\xi)$. In the triangulation $\chi(\mathrm{T})$, all edges except $\{A, B\}$ inherit an orientation from $T$; orient $\{A, B\}$ from $B$ to $A$. Then all inner vertices of T not incident to $\xi$ have outdegree 3 in $T$ and the closure operation does not change this outdegree. It follows easily that the resulting orientation of $\chi(\mathrm{T})$ is a 3-orientation. Furthermore, the "clockwise direction" of the local closures implies that closure never creates counterclockwise cycles, so the 3-orientation is minimal.

Given a planted planar triangulation $\mathrm{G} = (G, \xi)$, the balanced blossoming tree $\chi^{-1}(G, \xi)$ can be recovered as follows. Let $\overrightarrow{E}$ be the unique minimal 3-orientation of $E(G)$. Let $v = \mathrm{v}(\xi)$ and list the vertices of the face incident to $\xi$ in clockwise order as $(v, A, B)$. Remove the edge $\{A, B\}$, and perform a clockwise contour exploration of G starting from $\xi$. Each time we see an edge $uv$ for the first time, if it is oriented in the opposite direction from the contour process then keep it; otherwise replace it by a stem $\{u, b_{uv}\}$. This procedure is depicted in Figure 6.




(a) A simple triangulation endowed with its 3-orientation.

(b) Here the root edges have been removed and the contour process has started.

(c) The blossoming tree obtained after the completion of the algorithm.

FIGURE 6. The opening of a simple triangulation into a 2-blossoming tree.

## 6. Convergence to the Brownian snake

Fix a probability distribution $\mu$ on $\mathbb{N}$, and a sequence $\nu = (\nu_k, k \geq 1)$ with $\nu_k = (\nu_k^i, 1 \leq i \leq k)$ a probability distribution on $\mathbb{R}^k$ for $k \geq 1$. For $n \in \mathbb{N}$, we then write $\mathrm{LGW}(\mu, \nu, n)$ for the law on labelled planted plane trees $\mathrm{T} = (T, \xi, D)$ such that:

- The planted plane tree $(T, \xi)$ has the law of the genealogical tree of a Galton-Watson process with reproduction law $\mu$, conditioned to have total progeny $n$.[5]
- Conditionally on $(T, \xi)$, $D : E(T) \to \mathbb{R}$ has the following law. Independently for each $u \in V(T)$, $(D(\{u, c_{(T,\xi)}(u, 1)\}), \ldots, D(\{u, c_{(T,\xi)}(u, k))\})$ is distributed according to $\nu_k$.

Here is the connection with random simple triangulations. If $(T_n, \xi_n, D_n)$ is uniformly distributed in $\mathcal{T}_n^{\mathrm{vl}}$, then Corollary 5.9 states that the law of $(T_n, \xi_n, D_n)$ is $\mathrm{LGW}(\mu, \nu, n)$, where $\mu$ is the law defined in (6) and for $k \geq 1$, $\nu_k$ is the uniform law on non-decreasing vectors in $\{-1, 0, 1\}^k$.

Recall the definition of the pair $(\mathbf{e}, Z)$ from Section 1.1, and the definitions of the functions $C_{\mathrm{T}}$, $X_{\mathrm{T}}$ and $Z_{\mathrm{T}}$ from Section 2.3. Note that $(T_n, \xi_n, D_n)$ does not correspond to a balanced tree (when planted at $\xi_n$) so $X_{\mathrm{T}}$ (which equals $Y_{\mathrm{T}} - 2$) may take negative values. However, by Fact 5.10, re-planting so that $X_{\mathrm{T}}$ is balanced corresponds to changing all labels by an additive shift of $\min_{v \in V(T)} X_{\mathrm{T}}(v)$, up to an additive error of at most 3. We establish the following convergence.

**Proposition 6.1.** *For $n \geq 1$ let $\mathrm{T}_n = (T_n, \xi_n, D_n)$ be uniformly random in $\mathcal{T}_n^{\mathrm{vl}}$. Then as $n \to \infty$,*

$$\left((3n)^{-1/2} C_{\mathrm{T}_n}(t), (4n/3)^{-1/4} Z_{\mathrm{T}_n}(t)\right)_{0 \leq t \leq 1} \xrightarrow{\mathrm{d}} (\mathbf{e}(t), Z(t))_{0 \leq t \leq 1}, \tag{7}$$

*for the topology of uniform convergence on $C([0,1], \mathbb{R})^2$.*

Before proving this theorem, we place it in the context of the existing literature on invariance principles for spatial branching processes. Fix $\mu$ and $\nu$ and let $(\mathrm{T}_n, n \in \mathbb{N})$ be such that $\mathrm{T}_n = (T_n, \xi_n, D_n)$ has law $\mathrm{LGW}(\mu, \nu, n)$ for $n \in \mathbb{N}$. In what follows, given a measure $\eta$ on $\mathbb{R}$ and $p > 0$ write $|\eta|_p = (\int_{\mathbb{R}} |x|^p \mathrm{d}\eta)^{1/p}$. Aldous ([2], Theorem 2) showed that if $|\mu|_1 = 1$ and $\sigma_\mu^2 := |\mu|_2^2 - |\mu|_1^2 \in (0, \infty)$, then

$$\left(\frac{\sigma_\mu}{2} \frac{C_{\mathrm{T}_n}(t)}{n^{1/2}}\right)_{0 \leq t \leq 1} \xrightarrow{\mathrm{d}} \mathbf{e} \tag{8}$$

as $n \to \infty$, for the topology of uniform convergence on $C([0,1], \mathbb{R})$. Now additionally suppose that the random variables $\{\nu_k(i) : k \in \mathbb{N}, 1 \leq i \leq k\}$ are all identically distributed, that $|\nu_k(1)|_1 < \infty$, that $\nu_k$ is *centred* (i.e., $\int_{\mathbb{R}} x \mathrm{d}\nu_k^1(x) = 0$) for every $1 \leq i \leq k$, and that

$$\sup_k \mathbf{P}\left\{|\nu_k^1| \geq y\right\} = o(y^{-4}) \quad \text{for every } k \geq 1.$$

Under these conditions, writing $\sigma_\nu = \sigma_{\nu_1^1}$, Janson and Marckert ([17], Theorem 2) prove that

$$\left(\frac{\sigma_\mu}{2} \frac{C_{\mathrm{T}_n}(t)}{n^{1/2}}, \frac{(\sigma_\mu/2)^{1/2}}{\sigma_\nu} \frac{Z_{\mathrm{T}_n}(t)}{n^{1/4}}\right)_{0 \leq t \leq 1} \xrightarrow[n \to \infty]{(d)} (\mathbf{e}(t), Z(t))_{0 \leq t \leq 1}. \tag{9}$$

in the same topology as in Proposition 6.1 (In fact Theorem 2 of [17] is stated with the additional assumption that for each $k$, the entries of the vector $\nu_k = (\nu_k(i), 1 \leq i \leq k)$ are independent. However, it is not difficult to see, and was explicitly noted in [17], that straightforward modifications of the proof allow this additional assumption to be removed.) Under the same assumptions, the convergence in (9) can also be obtained as a

[5]To avoid trivial technicalities, we assume $\mu$ is such that the support of $\mu$ has greatest common divisor 1, so that such conditioning is well-defined for all $n$ sufficiently large.

special case of [25, Theorem 8]. In the latter article, the marginals of $\nu_k$ are not required to be identically distributed but they are assumed to be *locally centred* meaning that for all $1 \le i \le k$, $\int_{\mathbb{R}} x \mathrm{d}\nu_k^i(x) = 0$. In our setting, the law of the labelled planted plane tree is given by Corollary 5.9. In this case the entries are clearly not identically distributed, and neither are they locally centred: observe for instance that $\int_{\mathbb{R}} x \mathrm{d}\nu_2^1(x) = -1/6$.

In [24], the "locally centred" assumption is replaced by a *global* centering assumption, namely that

$$\sum_{k\ge 0} \mu(\{k\}) \sum_{i=1}^{k} \int_{\mathbb{R}} x \mathrm{d}\nu_k^i(x) = 0,$$

which is satisfied by our model. However, [24] requires that $\mu$ has bounded support, which is not the case in Corollary 5.9.

We expect that the technique we use to prove Proposition 6.1 can be used to extend the results of [24] to a broad range of laws $\mathrm{LGW}(\mu,\nu,n)$ for which $\mu$ does not have compact support. However, for the sake of concision we have chosen to focus on the random labelled planted plane trees that arise from random simple triangulations (the treatment for random simple quadrangulations differs only microscopically and is omitted).

For the remainder of the section, let $\mu$ and $\nu = (\nu_k, k \ge 1)$ be given by Proposition 6.1, and for $n \ge 1$ let $\mathrm{T}_n = (T_n, \xi_n, D_n)$ have law $\mathrm{LGW}(\mu,\nu,n)$. To prove Proposition 6.1, we establish the following facts.

**Lemma 6.2** (Random finite-dimensional distributions)**.** *Let* $(U_i, i \ge 1)$ *be independent Uniform*$[0,1]$ *random variables, independent of the trees* $(\mathrm{T}_n, n \ge 1)$*. Then for all* $j \ge 1$*,*

$$\Big((3n)^{-1/2} C_{\mathrm{T}_n}(U_i), (4n/3)^{-1/4} Z_{\mathrm{T}_n}(U_i)\Big)_{1\le i\le j} \xrightarrow{\mathrm{d}} (\mathbf{e}(U_i), Z(U_i))_{1\le i\le j}\,,$$

*for the topology of uniform convergence on* $C([0,1],\mathbb{R})^2$*.*

**Lemma 6.3** (Tightness)**.** *The family of laws of the processes* $((4n/3)^{-1/4} Z_{\mathrm{T}_n}, n \ge 1)$ *is tight for the space of probability measures on* $C([0,1)$*.*

Given the two preceding lemmas, Proposition 6.1 follows by standard arguments, which we only briefly sketch.

*Proof of Proposition 6.1.* In Appendix A we calculate $|\mu|_1 = 1$ and $\sigma_\mu/2 = 3^{-1/2}$. By (8) and Skorohod's representation theorem, we may thus work in a space in which almost surely

$$\limsup_{n\to\infty} \sup_{0\le t\le 1} |(3n)^{-1/2} C_{\mathrm{T}_n}(t) - \mathbf{e}(t)| = 0\,.$$

By Lemma 6.2 and another application of Skorohod's representation theorem, we may further assume that for each $i \ge 1$, $(4n/3)^{-1/4} Z_{\mathrm{T}_n}(U_i) \xrightarrow{\mathrm{a.s.}} Z(U_i)$ as $n \to \infty$, In this space, for all $j \ge 1$ and $\epsilon > 0$ we have

$$\limsup_{n\to\infty} \mathbf{P}\left\{ \max_{1\le i\le j} |(4n/3)^{-1/4} Z_{\mathrm{T}_n}(U_i) - Z(U_i)| \ge \epsilon \right\} = 0.$$

The tightness given by Lemma 6.3 implies that in this space, we almost surely have

$$\limsup_{n\to\infty} \sup_{0\le t\le 1} |(4n/3)^{-1/4} Z_{\mathrm{T}_n}(t) - Z(t)| = 0\,.$$

Combined with the a.s. convergence of $C_{\mathrm{T}_n}$ to $\mathbf{e}$, it follows that in this space we have $(C_{\mathrm{T}_n}, Z_{\mathrm{T}_n}) \xrightarrow{\mathrm{a.s.}} (\mathbf{e}, Z)$ as $n \to \infty$, which implies the claimed convergence in distribution. ☐

The remainder of the section is thus devoted to proving Lemmas 6.2 and 6.3. The proofs of both rely on a coupling, defined in Section 6.1, between $\mathrm{LGW}(\mu,\nu,n)$ and $\mathrm{LGW}(\mu,\hat{\nu},n)$, where $\hat{\nu}$ is a symmetrized version of $\nu$ which is locally centred.

6.1. **Partial symmetrization and locally centred displacements.** Fix a labelled planted plane tree $\mathrm{T} = (T, \xi, D)$, and a family of permutations $\sigma = (\sigma^v : v \in V(T), k_{\mathrm{T}}(v) > 0)$ with $\sigma^v$ a permutation of $\{1, \ldots, k_{\mathrm{T}}(v)\}$. Then define the *permuted* labelled planted plane tree $\mathrm{T}^\sigma = (T^\sigma, \xi^\sigma, D^\sigma)$ by permuting the order of the subtrees rooted at the children of $v$ according to $\sigma^v$, for each $v \in V(T)$, and permuting edge labels accordingly. More precisely, there is a bijection $\mathrm{b} = \mathrm{b}_{\mathrm{T},\sigma}$ between $V(T)$ and $V(T^\sigma)$ such that in the Ulam-Harris encoding $U_{\mathrm{T}}$, if $v \in V(T)$ has $U_{\mathrm{T}}(v) = n_1 n_2 \ldots n_k$ then the vertex $\mathrm{b}_{\mathrm{T},\sigma}(v) \in V(T^\sigma)$ satisfies

$$U_{\mathrm{T}^\sigma}(\mathrm{b}_{\mathrm{T},\sigma}(v)) = \sigma^{\emptyset}(n_1)\sigma^{n_1}(n_1 n_2) \ldots \sigma^{n_1 \ldots n_{k-1}}(n_1 \ldots n_k)\,,$$

where for $w \in V(T)$ we abuse notation and write $\sigma^{U_{\mathrm{T}}(w)}$ in place of $\sigma^w$ (and recall that $U_{\mathrm{T}}(\mathrm{v}(\xi)) = \emptyset$). Furthermore, for this bijection, for all $\{u, v\} \in E(T)$ with $u = p(v)$ we have $D^\sigma(\{\mathrm{b}_{\mathrm{T},\sigma}(u), \mathrm{b}_{\mathrm{T},\sigma}(v)\}) = D(\{u, v\})$.

Given $R \subset V(T)$ with $\mathrm{v}(\xi) \in R$, the *$R$-symmetrization* $\mathrm{T}^R = (T^R, \xi^R, D^R)$ of T is defined as follows. Let $T\langle R\rangle$ be the subtree of $T$ spanned by $R$, and observe that $T\langle R\rangle = T\langle V(T\langle R\rangle)\rangle$. Independently for each vertex $v \in V(T)$, let $\sigma^v$ be a uniformly random permutation of $\{1, \ldots, k_{\mathrm{T}}(v)\}$. Then let $\sigma = (\sigma^v : v \in V(T), k_{\mathrm{T}}(v) > 0)$, and define $\tau = (\tau^v : v \in V(T), k_{\mathrm{T}}(v) > 0)$ by

$$\tau^v = \begin{cases} \mathrm{Id}^{k_{\mathrm{T}}(v)} & \text{if } v \in V(T\langle R\rangle)\,, \\ \sigma^v & \text{otherwise.} \end{cases}$$

where $\mathrm{Id}^{k_{\mathrm{T}}(v)}$ denotes the identity permutation on $\{1, \ldots, k_{\mathrm{T}(v)}(v)\}$. Then the set of vertices $V(T^R)$ is the image of $V(T)$ by $\mathrm{b}_{\mathrm{T},\tau}$. Also, for all $\{u, v\} \in E(T)$ with $u = p(v)$, let

$$D^R(\{\mathrm{b}_{\mathrm{T},\sigma(u)}, \mathrm{b}_{\mathrm{T},\sigma(v)}\}) = \begin{cases} D(\{u, v\}) & \text{if } u \notin V(T\langle R\rangle) \\ D(\{u, \mathrm{b}_{\mathrm{T},\sigma}(v)\}) & \text{otherwise.} \end{cases}$$

Note that in forming $\mathrm{T}^R$, displacements from vertices of $T\langle R\rangle$ to their children are permuted, but the order of the children is not. The $R$-symmetrization is depicted in Figure 7.

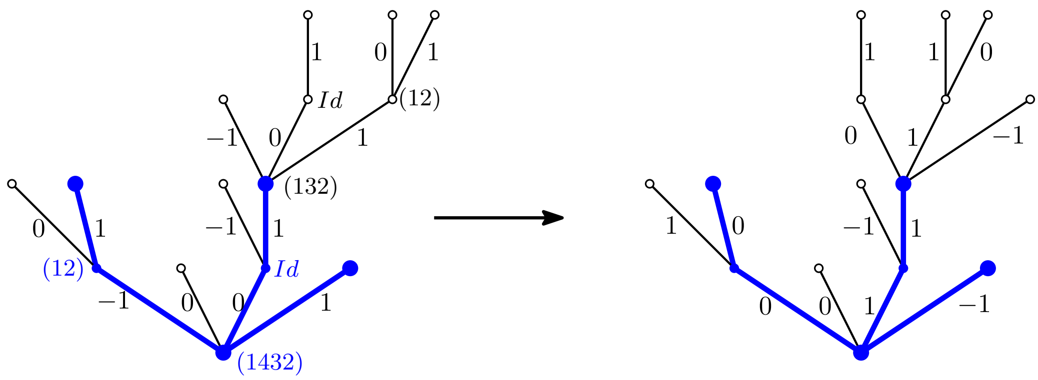


FIGURE 7. Illustration of the symmetrization procedure: the vertices in $R$ are represented by bigger blue circles and $T\langle R\rangle$ by blue fat lines.

In what follows, given $v \in V(T)$ write $v^R = \mathrm{b}_{\mathrm{T},\tau}(v)$ for the image of $v$ in $V(T^R)$ and for $e = \{u, v\} \in E(T)$, let $e^R = \{u^R, v^R\}$. Note that $(T, \xi)$ and $(T^R, \xi^R)$ are isomorphic as rooted trees but need not be isomorphic as plane trees. However, writing $S = \{v^R, v \in V(T\langle R\rangle)\}$, we do have that $T\langle R\rangle$ and $T^R\langle S\rangle$ are isomorphic as planted plane trees, and the map sending $v$ to $v^R$ is an isomorphism between them.

Now recall the definition of the contour exploration $(r_{\mathrm{T}}(i), 0 \le i \le 2|V(T)|-2)$ from Section 2.2. The following fact states that vertices of $T\langle R\rangle$ and $T^R\langle S\rangle$ are explored at the same times in the contour explorations of $T$ and of $T^R$, respectively

**Fact 6.4.** *For any set $R \subset V(T)$ with $\mathrm{v}(\xi) \in R$, for all $v \in V(T\langle R\rangle)$ and all $0 \le j \le 2|V(T)|-2$, we have $r_{\mathrm{T}}(j) = v$ if and only if $r_{\mathrm{T}^R}(j) = v^R$.* ☐

*Proof.* For $v \in V(T\langle R\rangle)$, since $\tau^v = \mathrm{Id}^{k_{\mathrm{T}}(v)}$, the list of sizes of the subtrees rooted at the children of $v$ (listed in lexicographically increasing order of child) are identical to the list of sizes of the subtrees rooted at the children of $v^R$ (again in lexicographic order). Since $\mathrm{v}(\xi) \in R$ the result follows. ☐

6.2. **Symmetrization and uniform sampling.** We now apply the partial symmetrization procedure to the study of conditioned spatial Galton–Watson trees. For the remainder of Section 6, let $\mu$ be the law of $B$ where $B$ is as in Corollary 5.9. Also, for $k \ge 1$ let $\nu_k$ be as in Corollary 5.9, and define a new sequence $\hat{\nu} = (\hat{\nu}_k, k \ge 1)$ as follows. Fix $k \ge 1$ and let $D = (D_1, \ldots, D_k)$ be a random vector in $\mathbb{R}^k$ with law $\nu_k$. Then let $\sigma$ be a uniformly random permutation of $\{1, \ldots, k\}$, independent of $D$, and let $\hat{\nu}_k = (\hat{\nu}_k(i), 1 \le i \le k)$ be the law of $(D_{\sigma(1)}, \ldots, D_{\sigma(k)})$. Note that since $\sum_i \int_{\mathbb{R}} x \mathrm{d}\nu_k^i(x) = 0$, we have $\int_{\mathbb{R}} x \mathrm{d}\hat{\nu}_k^i(x) = 0$ for each $1 \le i \le k$; in other words, $\hat{\nu}$ is locally centred.

**Claim 6.5.** *Let $\mathrm{T} = (T, \xi, D)$ have law $\mathrm{LGW}(\mu, \nu, n)$. Then for any set $I \subset \{0, 1, \ldots, 2n-2\}$ with $0 \in I$, writing $R = \{r_{\mathrm{T}}(i), i \in I\}$, $(T^R, \xi^R, D^R)$ has law $\mathrm{LGW}(\mu, \hat{\nu}, n)$.*

*Proof.* It follows from the branching property of Galton-Watson processes that $(T^R, \xi^R)$ and $(T, \xi)$ have the same law. The fact that the label process of $(T^R, \xi^R, D^R)$ is driven by $\hat{\nu}$ then follows directly from the construction of $(T^R, \xi^R, D^R)$ and by the definion of $\hat{\nu}$. ☐

**Corollary 6.6.** *For $n \ge 1$ let $\mathrm{T}_n = (T_n, \xi_n, D_n)$ be uniformly random in $\mathcal{T}_n^{\mathrm{vl}}$. Fix any sequence $(I_n, n \ge 1)$ with $I_n \subset \{0, 1, \ldots, 2n-2\}$ and $0 \in I_n$, and write $R_n = \{r_{\mathrm{T}_n}(i), i \in I_n\}$. Then as $n \to \infty$,*

$$\left((3n)^{-1/2} C_{\mathrm{T}_n^{R_n}}(t), (4n/3)^{-1/4} Z_{\mathrm{T}_n^{R_n}}(t)\right)_{0 \le t \le 1} \xrightarrow{\mathrm{d}} (\mathbf{e}(t), Z(t))_{0 \le t \le 1}, \tag{10}$$

*for the topology of uniform convergence on $C([0,1], \mathbb{R})^2$.*

*Proof.* Since $0 \in I_n$ we have $\mathrm{v}(\xi_n) \in R_n$. By Corollary 5.9, $\mathrm{T}_n$ has law $\mathrm{LGW}(\mu, \nu, n)$, so $\mathrm{T}_n^{R_n}$ has law $\mathrm{LGW}(\mu, \hat{\nu}, n)$. Since $\hat{\nu}$ is locally centred, the result follows by (9). ☐

The next proposition, together with Corollary 6.6, will allow us to establish convergence of random FDDs without first symmetrizing, and is the reason why we study random rather than deterministic FDDs. For $u \in V(T\langle R\rangle)$, write $A_{\mathrm{T},R}(u) = \{i : 1 \le i \le k_{\mathrm{T}}(u), c_{\mathrm{T}}(u,i) \in V(T\langle R\rangle)\}$ for the indices of the children of $u$ that are vertices of $T\langle R\rangle$.

**Proposition 6.7.** *Given integers $n, j \ge 1$, let $\mathrm{T}_n = (T_n, \xi_n, D_n)$ be uniformly random in $\mathcal{T}_n^{\mathrm{vl}}$. Let $v_1, \ldots, v_j$ be independent and uniformly random elements of $V(T_n)$, and let $R = \{\mathrm{v}(\xi_n), v_1, \ldots, v_j\}$. Conditional on $\mathrm{T}_n$, on $\{k_{\mathrm{T}}(u) : u \in V(T_n\langle R\rangle)\}$ and on $\{|A_{\mathrm{T},R}(u)| : u \in V(T_n\langle R\rangle)\}$, the sets $\{A_{\mathrm{T},R}(u) : u \in V(T_n\langle R\rangle)\}$ are independent and, for each $u \in V(T_n\langle R\rangle)$, $A_{\mathrm{T},R}(u)$ is a uniformly random subset of $\{1, \ldots, k_{\mathrm{T}}(u)\}$ of size $|A_{\mathrm{T},R}(u)|$.*

*Proof.* Independence follows easily from the branching property, so we focus on a single vertex $u \in V(T_n\langle R\rangle)$. Now condition further: on everything except the ordering of the subtrees rooted at the children of $u$. More precisely, for $w \in V(T_n)$ temporarily write $T_n^w$ for the subtree of $T_n$ rooted at $w$. Condition further on the sets $A_{\mathrm{T},R}(w)$ for $w \in V(T_n\langle R\rangle) \setminus \{u\}$, on the ordered sequences of plane trees $(T_n^{c_{\mathrm{T}}(w,i)} : 1 \le i \le k_{\mathrm{T}}(w))$ for $w \in V(T) \setminus \{u\}$, and on the *unordered* set $\{T_n^{c_{\mathrm{T}}(u,i)}, 1 \le i \le k_{\mathrm{T}}(w)\}$. Each possible ordering of $\{T_n^{c_{\mathrm{T}}(u,i)}, 1 \le i \le k_{\mathrm{T}}(w)\}$ then specifies $(T_n, \xi_n)$ and the vertices $v_1, \ldots, v_j$, and distinct orderings lead to different results for the triple $(T_n, \xi_n, (v_1, \ldots, v_j))$.

Since planted plane trees with the same child sequence are equally likely to arise as the genealogical tree of any fixed Galton–Watson process, and $v_1, \ldots, v_j$ are uniformly random vertices, it follows that under this conditioning, each possible ordering of the children of $u$ is equally likely. The result follows. □

From the preceding proposition, it will follow straightforwardly that, distributionally, the effect of partial symmetrization on the displacements along edges of $T_n\langle R\rangle$ is insignificant.

*Proof of Lemma 6.2.* Let $\mathrm{T}_n, n \ge 1$ have law $\mathrm{LGW}(\mu, \nu, n)$, fix $j \ge 1$ and let $U_1, \ldots, U_j$ be independent Uniform$[0,1]$ random variables independent of the trees $\mathrm{T}_n$. In what follows we suppress $n$-dependence whenever possible to lighten notation.

For $1 \le i \le j$, let $\{u_i, w_i\} \in E(T_n)$ be the edge of $T_n$ traversed at time $U_i$ by $C_{\mathrm{T}_n}$. More formally, let $\{u_i, w_i\}$ be such that $u_i = p(w_i)$ and

$$\{r_{\mathrm{T}_n}(\lfloor (2n-2)\cdot U_i \rfloor), r_{\mathrm{T}_n}(\lceil (2n-2)\cdot U_i \rceil)\} = \{u_i, w_i\}\,.$$

Then $|C_{\mathrm{T}_n}(U_i) - d_{\mathrm{T}_n}(\mathrm{v}(\xi_n), w_i)| \le 1$ and $|Z_{\mathrm{T}_n}(U_i) - X_{\mathrm{T}_n}(w_i)| \le 1$. Furthermore, $\{u_i, w_i\}$ is a uniformly random edge of $T_n$, so $w_i$ is a uniformly random element of $V(T_n) \setminus \{\mathrm{v}(\xi_n)\}$. For $n$ large, then, $(w_1, \ldots, w_j)$ are essentially independent, uniformly random elements of $V(T_n)$; we may couple the sequence $(w_1, \ldots, w_j)$ with a sequence $(v_1, \ldots, v_j)$ of independent uniformly random elements of $V(T_n)$ so that $\mathbf{P}\{(w_1, \ldots, w_j) \ne (v_1, \ldots, v_j)\} \to 0$ as $n \to \infty$. It thus suffices to show that

$$((3n)^{-1/2}(d_{\mathrm{T}_n}(\mathrm{v}(\xi_n), v_i), (4n/3)^{-1/4} X_{\mathrm{T}_n}(v_i)))_{1\le i\le j} \xrightarrow{\mathrm{d}} (\mathbf{e}(U_i), Z(U_i))_{1\le i\le j}\,. \tag{11}$$

Now write $R = \{v_1, \ldots, v_j\}$, and note that Fact 6.4 implies the indices of $v_1^R, \ldots, v_j^R$ in lexicographic order agree with those of $v_1, \ldots, v_j$; it follows that $v_1^R, \ldots, v_j^R$ are independent, uniformly random elements of $V(T_n^R)$ conditional on $\mathrm{T}_n^R$. By construction we have $(d_{\mathrm{T}_n}(\mathrm{v}(\xi_n), v_i), 1 \le i \le j) = (d_{\mathrm{T}_n^R}(\mathrm{v}(\xi_n^R), v_i^R), 1 \le i \le j)$, so we turn our attention to the second coordinate.

For $1 \le i \le j$ let $B_i$ be the set of edges $\{u, w\} \in E(T_n)$ with $u = p(w)$, with $w$ an ancestor of $v_i$, and with $|A_{\mathrm{T},R}(u)| = 1$. Then let

$$A_i = \sum_{e \in B_i} D_n(e), \quad \text{and} \quad A_i^R = \sum_{e \in B_i} D_n^R(e^R)\,.$$

By Corollary 6.6 we have

$$((3n)^{-1/2}(d_{\mathrm{T}_n^R}(\mathrm{v}(\xi_n^R), v_i^R), (4n/3)^{-1/4} X_{\mathrm{T}_n^R}(v_i^R)))_{1\le i\le j} \xrightarrow{\mathrm{d}} (\mathbf{e}(U_i), Z(U_i))_{1\le i\le j}\,.$$

For all $1 \le i \le j$ we have $|A_i^R - X_{\mathrm{T}_n^R}(v_i^R)| \le j-1$ since $T_n\langle R\rangle$ has at most $j-1$ vertices with more than one child, and all displacements are at most 1 in absolute value. Since $(j-1)$ is constant it follows that we may replace $X_{\mathrm{T}_n^R}(v_i^R)$ by $A_i^R$ without changing the distributional limit.

Next, by Claim 6.5, $(T_n^R, \xi_n^R, D_n^R)$ has law $\mathrm{LGW}(\mu, \hat{\nu}, n)$. By the definition of the symmetrized law $\hat{\nu}$, each summand $D_n^R(e^R)$ of $A_i^R$ is uniformly distributed on $\{-1, 0, 1\}$. Also, for $e = \{u, w\} \in B_i$ with $u = p(w)$, by Proposition 6.7, $w$ is a uniformly random child of $u$, so by the definition of $\nu$, $D_n(e)$ is uniformly distributed on $\{-1, 0, 1\}$. It follows that $(A_i, 1 \le i \le j)$ and $(A_i^R, 1 \le i \le j)$ are identically distributed (even conditional on $T_n\langle R\rangle$), so

$$((3n)^{-1/2} d_{\mathrm{T}_n}(\mathrm{v}(\xi_n), v_i), (4n/3)^{-1/4} A_i)_{1\le i\le j} \xrightarrow{\mathrm{d}} (\mathbf{e}(U_i), Z(U_i))_{1\le i\le j}\,.$$

Finally, for all $1 \le i \le j$ we have $|A_i - X_{\mathrm{T}_n}(v_i)| \le j-1$ so we may replace $A_i$ by $X_{\mathrm{T}_n}(v_i)$ and convergence in distribution still occurs. This establishes (11). □

6.3. **Tightness.** The argument for Lemma 6.3 hinges on a simple deterministic bound relating fluctuations of the displacements in a tree $\mathrm{T}$ and in its symmetrization $\mathrm{T}^R$. Fix $\mathrm{T} = (T, \xi, D)$, and a set $R \subset V(T)$ with $\mathrm{v}(\xi) \in R$. As earlier, we write $S = \{v^R, v \in V(T\langle R\rangle)\}$. For $0 \le j \le 2|V(T)| - 2$, let $f_{\mathrm{T},R}(j) = \min\{j' \ge j : r_{\mathrm{T}}(j') \in V(T\langle R\rangle)\}$. By Fact 6.4 we have $f_{\mathrm{T},R}(j) = \min\{j' \ge j : r_{\mathrm{T}^R}(j') \in V(T^R\langle S\rangle)\}$, and if $r_{\mathrm{T}}(f_{\mathrm{T},R}(j)) = v$ then $r_{\mathrm{T}^R}(f_{\mathrm{T},R}(j)) = v^R$. Now let $\delta_{\mathrm{T},R}(j) = |X_{\mathrm{T}}(j) - X_{\mathrm{T}}(f_{\mathrm{T},R}(j))|$, and let $\delta'_{\mathrm{T},R} = |X_{\mathrm{T}^R}(j) - X_{\mathrm{T}^R}(f_{\mathrm{T},R}(j))|$. Finally, let $\Delta_{\mathrm{T},R} = \max\{\delta_{\mathrm{T},R}(j), 0 \le j \le 2|V(T)| - 2\}$ and let $\Delta'_{\mathrm{T},R} = \max\{\delta'_{\mathrm{T},R}(j), 0 \le j \le 2|V(T)| - 2\}$.

**Claim 6.8.** *For $R \subset V(T)$ with $\mathrm{v}(\xi) \in R$, $|\Delta_{\mathrm{T},R} - \Delta'_{\mathrm{T},R}| \le 2\max\{|D(e)|, e \in E(T)\}$.*

*Proof.* First, if $f_{\mathrm{T},R}(j) = j$ then $r_{\mathrm{T}}(j) \in V(T\langle R\rangle)$, so $r_{\mathrm{T}^R}(j) \in V(T^R\langle S\rangle)$ and so $\delta_{\mathrm{T},R}(j) = \delta'_{\mathrm{T},R}(j) = 0$. Next, if $f_{\mathrm{T},R}(j) \ne j$ then $r_{\mathrm{T}}(f_{\mathrm{T},R}(j))$ is the most recent ancestor of $r_{\mathrm{T}}(j)$ in $V(T\langle R\rangle)$. Let $u$ be the unique child of $r_{\mathrm{T}}(f_{\mathrm{T},R}(j))$ that is an ancestor of $r_{\mathrm{T}}(j)$. By construction, the displacements on the path from $r_{\mathrm{T}}(j)$ to $u$ in $\mathrm{T}$ are identical to the displacements on the path from $r_{\mathrm{T}^R}(j)$ to $u^R$ in $\mathrm{T}^R$. It follows that

$$\begin{aligned}|\delta_{\mathrm{T},R}(j) - \delta'_{\mathrm{T},R}(j)| &= |D(\{r_{\mathrm{T}}(f_{\mathrm{T},R}(j)), u\}) - D^R(\{r_{\mathrm{T}^R}(f_{\mathrm{T},R}(j)), u^R\})| \\ &\le 2\max\{D(e), e \in E(T)\},\end{aligned}$$

from which the claim follows immediately. □

*Proof of Lemma 6.3.* Given any sequence $R_n, n \ge 1$ with $R_n \subset V(\mathrm{T}_n)$ and $\mathrm{v}(\xi_n) \in R_n$, by Claim 6.5, $\mathrm{T}_n^{R_n}$ has law $\mathrm{LGW}(\mu, \hat{\nu}, n)$, so by Corollary 6.6 we have

$$((4n/3)^{-1/4} Z_{\mathrm{T}_n^{R_n}}(t))_{0 \le t \le 1} \xrightarrow{\mathrm{d}} (Z(t))_{0 \le t \le 1}\,.$$

It follows that the family of laws of the processes $(Z_{\mathrm{T}_n^{R_n}}(t))_{0 \le t \le 1}$ are tight; in other words, for all $\epsilon > 0$ there exists $\alpha = \alpha(\epsilon) > 0$ such that

$$\limsup_{n\to\infty} \mathbf{P}\left\{\sup_{x,y \in [0,1], |x-y| \le \alpha} |Z_{\mathrm{T}_n^{R_n}}(x) - Z_{\mathrm{T}_n^{R_n}}(y)| > \epsilon n^{1/4}\right\} < \epsilon\,. \tag{12}$$

We emphasize that this bound is uniform over the choice of the sets $R_n$.

Next, let $U_i, i \ge 1$ be independent, uniformly random elements of $[0,1]$. Fix $\delta > 0$ and let $J = J(\delta)$ be minimal so that $\{U_1, \ldots, U_J\}$ forms a $\delta$-net in $[0,1]$[6]. As in the proof of Lemma 6.2, we couple $U_1, \ldots, U_J$ with a sequence $(v_1, \ldots, v_J)$ of independent uniformly random elements of $V(T_n)$ so that with probability tending to one, for each $1 \le i \le J$ either $v_i = r_{\mathrm{T}_n}(\lfloor U_i \cdot (2n-2)\rfloor)$ or $v_i = r_{\mathrm{T}_n}(\lceil U_i \cdot (2n-2)\rceil)$. Then let $R_n = R_n(\delta) = \{v_1, \ldots, v_{J(\delta)}\}$.

Given $x \in [0,1]$ let $u(x) = r_{\mathrm{T}_n}(\lfloor x \cdot (2n-2)\rfloor)$, and let $u_1(x) = r_{\mathrm{T}_n}(f_{\mathrm{T}_n,R_n}(\lfloor x \cdot (2n-2)\rfloor))$. By the definitions of $f_{\mathrm{T}_n,R_n}$ and of $R_n = R_n(\delta)$ we have

$$|\lfloor x \cdot (2n-2)\rfloor - f_{\mathrm{T}_n,R_n}(\lfloor x \cdot (2n-2)\rfloor)| < \delta \cdot (2n-2) + 2\,,$$

which is less than $2\delta \cdot (2n-2)$ for $n$ large. Applying Claim 6.8, since $\max\{|D(e)|, e \in E(T)\} \le 1 < \epsilon n^{1/4}$ for $n$ large, we obtain

$$\begin{aligned}&\limsup_{n\to\infty} \mathbf{P}\left\{\sup_{x\in[0,1]} |X_{\mathrm{T}_n}(u(x)) - X_{\mathrm{T}_n}(u_1(x))| \ge 2\epsilon n^{1/4}\right\} \\ &\le \limsup_{n\to\infty} \mathbf{P}\left\{\sup_{x,y\in[0,1],|x-y|<2\delta} |Z_{\mathrm{T}_n^{R_n}}(x) - Z_{\mathrm{T}_n^{R_n}}(y)| > \epsilon n^{1/4}\right\}\,. \end{aligned}\tag{13}$$

[6]In other words, so that $\min\{U_i, 1 \le i \le J\} \le \delta$, $\max\{U_i, 1 \le i \le J\} \ge 1-\delta$, and for all $1 \le i \le J$ there is $1 \le i' \le J$ such that $|U_i - U'_i| < \delta$.

Next, fix $x, y \in [0,1]$. Recall that $[\![u_1(x), u_1(y)]\!]$ is the shortest path from $u_1(x)$ to $u_1(y)$ in $T_n$, and note that all vertices of $[\![u_1(x), u_1(y)]\!]$ lie in $T_n\langle R_n\rangle$. Let

$$B(u_1(x), u_1(y)) = \{\{w, w'\} \in E(T_n) : w, w' \in [\![u_1(x), u_1(y)]\!] : w = p(w'), |A_{\mathrm{T}_n,R_n}(w)| = 1\}$$

be the set of edges of $[\![u_1(x), u_1(y)]\!]$ for which the parent vertex has only one child in $T_n\langle R_n\rangle$, let $A = |\sum_{e\in B(u_1(x),u_1(y))} D_n(e)|$, and let $A^{R_n} = \sum_{e\in B(u_1(x),u_1(y))} D_n^{R_n}(e^{R_n})$. Arguing from Proposition 6.7 as in the proof of Lemma 6.2 shows that $A$ and $A^{R_n}$ have the same distribution[7], that $|A - |X_{\mathrm{T}_n}(u_1(x)) - X_{\mathrm{T}_n}(u_1(y))|| \le J$ and that $|A^{R_n} - |X_{\mathrm{T}_n^{R_n}}(u_1(x)) - X_{\mathrm{T}_n^{R_n}}(u_1(y))|| \le J$. (The bound is $J$ rather than $J-1$ since the most recent common ancestor of $u_1(x)$ and $u_1(y)$ may be the parent vertex of two edges of $[\![u_1(x), u_1(y)]\!]$.) Furthermore, by the definitions of $f_{\mathrm{T}_n,R_n}$ and of $R_n = R_n(\delta)$ we have

$$|f_{\mathrm{T}_n,R_n}(\lfloor x\cdot(2n-2)\rfloor) - f_{\mathrm{T}_n,R_n}(\lfloor y\cdot(2n-2)\rfloor)| \le (|x-y|+\delta)\cdot(2n-2)+4,$$

and if $x - y < \delta$ then for any $\epsilon > 0$, the above difference is less than $(4+\epsilon)n\delta$ for $n$ sufficiently large. Also, Fact 6.4 states that vertices of $T_n\langle R_n\rangle$ are visited at the same time in the contour explorations of $\mathrm{T}_n$ and of $\mathrm{T}_n^{R_n}$. Since $J/n^{1/4} \to 0$ almost surely, this yields

$$\begin{aligned}
&\limsup_{n\to\infty} \mathbf{P}\left\{\sup_{x,y\in[0,1]:|x-y|<\delta} |X_{\mathrm{T}_n}(u_1(x)) - X_{\mathrm{T}_n}(u_1(y))| \ge 2\epsilon n^{1/4}\right\}\\
\le&\limsup_{n\to\infty} \mathbf{P}\left\{\sup_{x,y\in[0,1]:|x-y|<5\delta} |X_{\mathrm{T}_n}(u(x)) - X_{\mathrm{T}_n}(u(y))| \ge 2\epsilon n^{1/4} - 2J\right\}\\
\le&\limsup_{n\to\infty} \mathbf{P}\left\{\sup_{x,y\in[0,1],|x-y|<5\delta} |Z_{\mathrm{T}_n^{R_n}}(x) - Z_{\mathrm{T}_n^{R_n}}(y)| > \epsilon n^{1/4}\right\}. \qquad (14)
\end{aligned}$$

Finally, for $x, y \in [0,1]$, if $|Z_{\mathrm{T}_n}(x) - Z_{\mathrm{T}_n}(y)| > 6\epsilon n^{1/4} + 2$ then one of $|X_{\mathrm{T}_n}(u(x)) - X_{\mathrm{T}_n}(u_1(x))|$, $|X_{\mathrm{T}_n}(u_1(x)) - X_{\mathrm{T}_n}(u_1(y))|$, or $|X_{\mathrm{T}_n}(u_1(y)) - X_{\mathrm{T}_n}(u(y))|$ is at least $2\epsilon n^{1/4}$. It follows from (13) and (14) that

$$\begin{aligned}
&\limsup_{n\to\infty} \mathbf{P}\left\{\sup_{x,y\in[0,1]:|x-y|<\delta} |Z_{\mathrm{T}_n}(x) - Z_{\mathrm{T}_n}(y)| > 6\epsilon n^{1/4} + 2\right\}\\
\le& 3\limsup_{n\to\infty} \mathbf{P}\left\{\sup_{x,y\in[0,1],|x-y|<5\delta} |Z_{\mathrm{T}_n^{R_n}}(x) - Z_{\mathrm{T}_n^{R_n}}(y)| > \epsilon n^{1/4}\right\}.
\end{aligned}$$

This bound holds for any $\epsilon > 0$. Taking $\delta = \alpha(\epsilon)/3$, by (12) the final bound is at most $3\epsilon$, which establishes the requisite tightness. □

## 7. Blossoming trees, labelling, and distances

The goal of this section is to *deterministically* relate labels in a validly-labelled plane tree with the distances in the corresponding triangulation. For the remainder of Section 7, we fix $n \in \mathbb{N}$ and $(T, \xi, \hat{\xi}) \in \mathcal{T}_n^\bullet$, let $(G, c, \hat{c}) = \chi_n^\bullet(T, \xi, \hat{\xi})$ and let $(T', \xi', D) = \phi_n(\psi_n(T, \xi, \hat{\xi}))$, and write $\mathrm{G} = (G, c)$ and $\mathrm{T} = (T, \xi)$. Writing $\mathcal{B}$ for the buds of $T$, we suppose throughout that $V(T') = V(T)\setminus\mathcal{B} = V(G)\setminus\{A, B\}$. Finally, define $Y_{\mathrm{T}}$ as in Section 5.3, and note that since $\mathrm{T}$ is balanced, $Y_{\mathrm{T}}(v) \ge 2$ for all $v \in V(T)$. It will be useful to extend the domain of $Y_{\mathrm{T}}$ by setting $Y_{\mathrm{T}}(A) = 1$ and $Y_{\mathrm{T}}(B) = 2$, and we adopt this convention.

[7] This is the only place in this proof where we require that $v_1, \ldots, v_J$ are uniformly random.

7.1. **Bounding distances using leftmost paths.** To warm up, we prove a basic lemma bounding the difference between labels of adjacent vertices.

**Lemma 7.1.** *For all* $\{u,w\} \in E(G)$, $|Y_{\mathrm{T}}(u) - Y_{\mathrm{T}}(w)| \le 3$.

*Proof.* First, recall from Page 15 that if $u \in V(T)$ and $\{u,A\} \in E(G)$ or $\{u,B\} \in E(G)$ then there is a corner $c$ incident to $u$ with $\lambda_{\mathrm{T}}(c) \le 3$, so $Y_{\mathrm{T}}(u) \le 3$. From this, if $\{u,w\} \cap \{A,B\} \neq \emptyset$ then the result is immediate. Next, if $\{u,v\} \in E(T)$ then it is an inner edge of $T$, in which case $Y(u) - Y(v) = D_{\{u,v\}}(T,\xi) \in \{-1,0,1\}$. Finally, if $\{u,w\} \notin E(T)$ but $u,w \in V(T)$ then there are corners $c^1, c^2$ of $T$ such that $\mathrm{v}(c^1) = u$, $\mathrm{v}(c^2) = w$, and either $c^2 = s_{\mathrm{T}}(c^1)$ or $c^1 = s_{\mathrm{T}}(c^2)$. Assuming by symmetry that $c^2 = s(c^1)$, we have $\lambda_{\mathrm{T}}(c^2) = \lambda_{\mathrm{T}}(c^1) - 1$. Since the labels on corners incident to a single vertex differ by at most two, the result follows in this case. □

The above lemma, though simple, already allows us to prove the labels provide a lower bound for the graph distance to $A$ in $G$, up to a constant factor.

**Corollary 7.2.** *For all* $u \in V(G)$, $d_G(u,A) \ge Y_{\mathrm{T}}(u)/3$.

*Proof.* Let $(u_0, u_1, \ldots, u_l)$ be a shortest path from $u = u_0$ to $A = u_l$ in $G$. Then by Lemma 7.1, since $Y_{\mathrm{T}}(A) = 1$ we have $Y_{\mathrm{T}}(u) = |Y_{\mathrm{T}}(u_0) - Y_{\mathrm{T}}(u_l) - 1| < 3l$, so $d_G(u,A) = l \ge Y_{\mathrm{T}}(u)/3$. □

We next aim to prove a corresponding upper bound. For this we use the leftmost paths briefly introduced in Section 1.3. Let $(G,c) = \chi(T,\xi)$ as above, and let $\overrightarrow{E}$ be its unique minimal 3-orientation (defined in Section 5.5). Given an oriented edge $e = uw$ with $\{u,w\} \in E$ and $x \in V(G)$, a *path* from $e$ to $x$ is a path $Q = (v_0, v_1, \ldots, v_m)$ in $G$ with $v_0 v_1 = uw$ and $v_m = x$. (In the preceding, we do not require that $uw \in \overrightarrow{E}$.) Given $e = \{u_0, u_1\} \in E(G)$ with $u_0 u_1 \in \overrightarrow{E}$, the *leftmost path* from $e$ to $A$ is the unique directed path $P(e) = P_{(G,c)}(e) = (u_0, u_1, \ldots, u_\ell)$ with $u_\ell = A$ such that for each $1 \le i \le \ell - 1$, $u_i u_{i+1}$ is the first outgoing edge incident to $u_i$ when considering the edges incident to $u_i$ in clockwise order starting from $\{u_{i-1}, u_i\}$. The following fact establishes two basic properties of leftmost paths.

**Fact 7.3.** *For all* $e \in E(G)$, $P(e)$ *is a simple path. Furthermore, if* $P(e) = (u_0, u_1, \ldots, u_\ell)$ *and* $P(e') = (v_0, v_1, \ldots, v_m)$ *are distinct leftmost paths to* $A$ *with* $u_0 = v_0 = u$, *and* $u_i = v_j$ *for some* $i, j > 0$, *then* $u_{i+k} = v_{j+k}$ *for all* $0 \le k \le \ell - i = m - j$.

*Proof.* Let $P(e) = (u_0, u_1, \ldots, u_\ell)$ be the leftmost path from $u_0 u_1$ to $A$. Suppose there are $0 \le i < j \le \ell$ such that $u_i = u_j$, and choose such $i,j$ for which $|j - i|$ is minimum. Then $C = (u_i, u_{i+1}, \ldots, u_j)$ is an oriented cycle with $j - i$ vertices; let $V' \subset V(G)$ be the vertices lying on or to the right of this cycle. Since $\overrightarrow{E}$ is minimal, $C$ is necessarily a clockwise cycle, so $\mathrm{v}(c) \notin V'$. Also, neither $A$ nor $B$ are in any directed cycles, and it follows that $\{A, B, \mathrm{v}(c)\} \cap V' = \emptyset$. Since $\overrightarrow{E}$ is a 3-orientation it follows that for all $x \in V'$, $\deg^+_{\overrightarrow{E}}(x) = 3$. Furthermore, for all $x \in V' \setminus \{u_i\}$, since $P(e)$ is a leftmost path, all out-neighbours of $x$ are elements of $V'$. Writing $G'$ for the sub-map of $G$ induced by $V$, it follows that $|E(G')| \ge 3|V' \setminus \{u_i\}|$. But $G'$ is a simple planar map, and $C$ is a face of $G'$ of degree $j - i \ge 3$. It follows by Euler's formula that $|E(G')| \le 3|V'| - 3 - (j-i) \le 3|V' \setminus \{u_i\}| - 3$, a contradiction.

The proof that two leftmost paths merge if they meet after their starting point follows the same lines and is left to the reader. □

The next proposition provides a key connection between the corner labelling $\lambda_{\mathrm{T}}$ and the lengths of leftmost paths.

**Proposition 7.4.** *For any edge $e = \{u, w\} \in E(G)$ with $uw \in \overrightarrow{E}$ and $u \neq B$, $\lambda_{\mathrm{T}}(\kappa^{\ell}(u, w)) = |P(e)|$.*

*Proof.* First, a simple counting argument shows that if $\{x, y_1\}$ is an inner edge of $T$, and $\{x, y_2\}$ is the first stem following $\{x, y_1\}$ in clockwise order around $x$, then writing $c$ for the corner of $T$ incident to $y_2$ we have $\lambda_{\mathrm{T}}(\kappa^{r}(x, y_1)) = \lambda_{\mathrm{T}}(c)$. Recall the definition of the *successor* function $s$ from (5) and the equivalent definition from Section 5.2. Since $\{x, y_2\}$ is a stem, in $G$, $y_2$ is identified with $s(c)$, and by definition $\lambda_{\mathrm{T}}(s(c)) = \lambda_{\mathrm{T}}(c) - 1$.

Next, recall the definition of the labelling $\lambda^* = \lambda^*_{\mathrm{T}} : \mathcal{C}(G) \to \mathbb{Z}^{\geq 0}$ from the end of Section 5.2. It follows from that definition that for any oriented edge $xy \in \overrightarrow{E}$, $\lambda^*(\kappa^r(y, x)) = \lambda^*(\kappa^{\ell}(x, y)) - 1$. In other words, the label on the left decreases by exactly one when following any oriented edge.

Now write $P(e) = (u_0, u_1, \ldots, u_\ell)$. Since there are no edges oriented away from $P(e)$ leaving $P(e)$ to the left, it follows from the two preceding paragraphs that for $0 < i < \ell$ we have $\lambda^*(\kappa^r(u_{i+1}, u_i)) = \lambda^*(\kappa_{\ell}(u_i, u_{i+1})) - 1 = \lambda^*(\kappa^r(u_i, u_{i-1})) - 1$, so

$$\lambda^*(\kappa^r(u_\ell, u_{\ell-1})) = \lambda^*(\kappa^{\ell}(u_0, u_1)) - \ell = \lambda^*(\kappa^{\ell}(u, w)) - \ell\,.$$

. Finally, $\lambda^*(\kappa^r(u_\ell, u_{\ell-1})) = 1$ by definition since $u_\ell = A$ and $u_{\ell-1} \neq \mathrm{v}(\xi)$. We thus obtain $\lambda^*(\kappa^{\ell}(u, w)) = \ell + 1 = |P(e)|$. ☐

**Corollary 7.5.** *For all $u \in V(G)$, $d_G(u, A) \leq Y_{\mathrm{T}}(u) - 1$.*

*Proof.* Recall the convention that $Y_{\mathrm{T}}(B) = 2$ and $Y_{\mathrm{T}}(A) = 1$; since also $Y_{\mathrm{T}}(\mathrm{v}(\xi)) = 2$, it suffices to prove the result for $u \in V(G) \setminus \{A, B, \mathrm{v}(\xi)\}$. For such $u$, if $\{u, w\}$ is the first stem incident to $u$ in clockwise order around $u$ starting from $\{u, p_T(u)\}$, then $Y_{\mathrm{T}}(u) = \lambda(\kappa^{\ell}(u, w))$. The claim then follows from Proposition 7.4. ☐

7.2. **Bounding distances between two points using modified leftmost paths.** In this section we use arguments similar to those of the preceding section, this time to prove deterministic upper bounds on pairwise distances in $G$. Fix $u, v \in V(G)$ with $u \preceq_{\mathrm{lex,T}} v$. Let $c_u$ be the first corner $c$ of $T$ (with respect to $\preceq_{\mathrm{ctr,T}}$) for which $v(c) = u$, and define $c_v$ likewise. Then set

$$\check{Y}_{\mathrm{T}}(u, v) = \min\{Y_{\mathrm{T}}(w) : \exists c \in \mathcal{C}(T), c_u \preceq_{\mathrm{ctr}} c \preceq_{\mathrm{ctr}} c_v, w = \mathrm{v}(c)\}\,,$$
$$\check{Y}_{\mathrm{T}}(v, u) = \min\{Y_{\mathrm{T}}(w) : \exists c \in \mathcal{C}(T), c_v \preceq_{\mathrm{ctr}} c \text{ or } c \preceq_{\mathrm{ctr}} c_u, w = \mathrm{v}(c)\}\,.$$

**Proposition 7.6.** *For all $u, v \in V(G)$,*

$$d_G(u, v) \leq Y_{\mathrm{T}}(u) + Y_{\mathrm{T}}(v) - 2\max\{\check{Y}_{\mathrm{T}}(u, v), \check{Y}_{\mathrm{T}}(v, u)\} + 2.$$

Before proving the proposition, we establish some preliminary results. Given an oriented edge $e = u_0u_1$ with $\{u_0, u_1\} \in E(T)$, the *modified leftmost path* from $e$ to $A$ is the unique (not necessarily oriented) path $Q(e) = (u_0, u_1, \ldots, u_\ell)$ in $G$ with $u_\ell = A$ and such that for each $1 \leq i \leq \ell - 1$, $u_iu_{i+1}$ is the first edge (considering the edges incident to $u_i$ in clockwise order starting from $\{u_{i-1}, u_i\}$) which is either an outgoing edge (with respect to the orientation $\overrightarrow{E}$) incident to $u_i$ or an *inner edge of* $T$. Equivalently, it is the leftmost oriented path, with the modified orientation obtained by viewing edges of $E(T)$ as unoriented (or as oriented in both directions).

We view $Q(e)$ as an oriented path from $e$ to $A$ (though the edge orientations given by the path need not agree with $\overrightarrow{E}$); we may thus speak of the left and right side of $Q(e)$.

**Fact 7.7.** *For $1 \leq i \leq \ell - 1$, $\lambda^*(\kappa^{\ell}(u_i, u_{i+1})) = \lambda^*(\kappa^{\ell}(u_{i-1}, u_i)) - 1$. In other words, the labels along the left of a modified leftmost path decrease by one along each edge.*

*Proof.* First, by the definitions of $\lambda$ and $\lambda^*$, for any edge $\{u_{i-1}, u_i\}$ of a modified leftmost path, $\lambda^*(\kappa^r(u_i, u_{i-1})) = \lambda^*(\kappa^\ell(u_{i-1}, u_i)) - 1$. Moreover, from the definition of a modified leftmost path, there is no stem incident to $u_i$ in $T$ that lies strictly between $\{u_{i-1}, u_i\}$ and $\{u_i, u_{i+1}\}$ (in clockwise order around $u_i$ starting from $\{u_{i-1}, u_i\}$). Hence $\kappa^\ell(u_i, u_{i+1}) = \kappa^r(u_i, u_{i-1})$ in $T$ (see the proof of Proposition 7.4 for more details). The result follows. □

Given $\{u, v\} \in E(G)$, if $\{u, v\} \notin E(T)$ and $\{u, v\} \neq \{A, B\}$ then by symmetry we may assume there is an edge $\{u, b\} \in E(T)$ such that $v = \mathrm{v}(s(b))$. In this case, by a slight abuse of notation we write $\kappa^\ell(u, v) = \kappa^\ell(u, b)$.

In the statement and proof of the next fact, write $L = \lambda^*(\kappa^\ell(u_0, u_1))$ and let $M = \min\{\lambda(\xi) : \xi \in \mathcal{C}(T), \kappa^\ell(u_0, u_1) \preceq_{\mathrm{ctr}} \xi\}$, where we view $\{u_0, u_1\}$ as an edge of $E(T)$. By the discussion on Page 15, $M \in \{2, 3\}$ and $M = 3$ precisely if $c' \preceq_{\mathrm{ctr}} \kappa^\ell(u_0, u_1)$, where $c'$ is the unique element of $\mathcal{C}(T) \setminus \{c\}$ for which $(T, c')$ is balanced.

Let $c_e^*(0) = \kappa^\ell(u_0, u_1)$, and for $1 \leq j \leq L - M$ let $c_e^*(j)$ be the first corner following $c_e^*(0)$ in $T$ for which $\lambda(c) = L - j$. For $1 \leq j \leq L - M$, $c_e^*(j)$ is necessarily an inner corner of $T$. Then, for $0 \leq j \leq L - M$ let $v_e^*(j) = \mathrm{v_T}(c_e^*(j))$.

**Fact 7.8.** *For all $0 \leq j \leq L - M$, $c_e^*(j) = \kappa^\ell(u_j, u_{j+1})$, so $v_e^*(j) \in Q(e)$.*

Before giving the proof, observe that this property need not hold for a regular leftmost path; this is the reason we require modified leftmost paths.

*Proof.* For $j = 0$ this holds by definition; we now fix $j \geq 1$ and argue by induction. The definition of $\lambda$ yields that $c_e^*(j) \preceq_{\mathrm{ctr,T}} c_e^*(j+1)$, for any $0 \leq j < L - 2$. We consider two cases. First, suppose $\{u_{j-1}, u_j\}$ is an inner edge of $T$. Let $w \in V(T)$ be such that $\kappa^r(u_j, u_{j-1}) = (\{u_{j-1}, u_j\}, \{u_j, w\})$. If $w$ is an inner vertex then $w = u_{j+1}$. Likewise, if $w$ is a blossom then $\mathrm{v}(s(w)) = u_{j+1}$. In either case, $\kappa^\ell(u_j, u_{j+1}) = \kappa^r(u_j, u_{j-1})$ in $T$. Hence $\kappa^\ell(u_j, u_{j+1})$ is the corner immediately following $\kappa^\ell(u_{j-1}, u_j)$ in the contour exploration of $T$. By Fact 7.7, $\lambda^*(\kappa^\ell(u_j, u_{j+1})) = \lambda^*(\kappa^\ell(u_{j-1}, u_j)) - 1$ , and $c_e^*(j-1) = \kappa^\ell(u_{j-1}, u_j)$ by the inductive hypothesis. It follows that $c_e^*(j) = \kappa^\ell(u_j, u_{j+1})$.

Second, suppose $\{u_{j-1}, u_j\}$ is not an inner edge. By definition, there is no edge in $T$ incident to $u_j$ and lying strictly between $\{u_{j-1}, u_j\}$ and $\{u_j, u_{j+1}\}$ in clockwise order around $u_j$. Hence, in $T$, $s(u_{j-1}) = \kappa^\ell(u_j, u_{j+1})$. In this case the result follows by the definition of $s(u_{j-1})$ and by induction. □

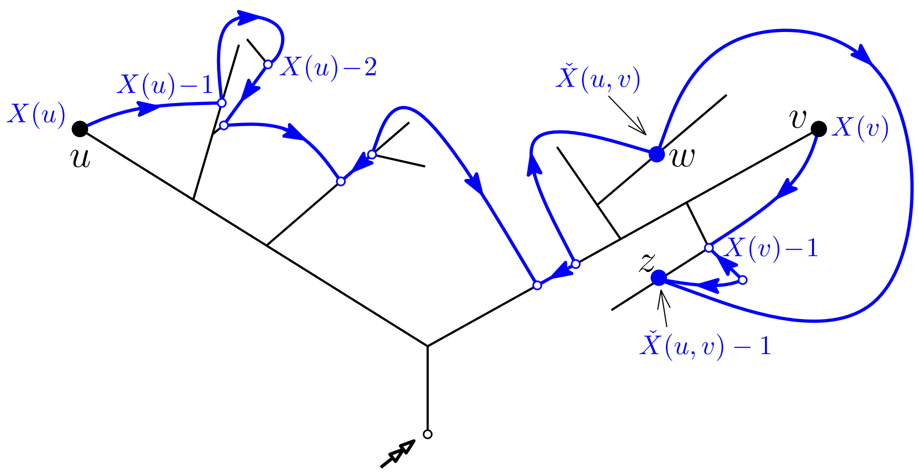



FIGURE 8. Path between $u$ and $v$ formed by concatenating sections of two modified leftmost paths. Arrows indicate orientation in $\overrightarrow{E}$ . Straight arrows along the path are edges of $E(T)$; curved arrows are edges of $E(G) \setminus E(T)$.

*Proof of Proposition 7.6.* Fix $u, v \in V(T)$ write $s_u$ and $s_v$ for the lexicographically first stem edges incident to $u$ and $v$, respectively, and $e_u$ and $e_v$ for the corresponding edges in $G$. Write $Q(e_u) = (u_0, u_1, \ldots, u_{\lambda^*(\kappa^\ell(u_0,u_1))-1})$ with $u_0 = u$ and $e_u = \{u_0, u_1\}$, and likewise write $Q(e_v) = (v_0, v_1, \ldots, v_{\lambda^*(\kappa^\ell(v_0,v_1))-1})$. Observe that $\lambda^*(\kappa^\ell(u_0, u_1)) = Y_{\mathrm{T}}(u)$ and $\lambda^*(\kappa^\ell(v_0, v_1)) = Y_T(v)$.

We assume for simplicity that $\check{Y}_{\mathrm{T}}(u, v) > 3$ (when $\check{Y}_{\mathrm{T}}(u, v) \le 3$ there is a minor case analysis involving the presence of vertices $A$ and $B$ in $Q(e_u)$ and $Q(e_v)$; the details are straightforward and we omit them). By Fact 7.8 and the definition of $\check{Y}_{\mathrm{T}}(u, v)$, necessarily $v^*_{e_u}(Y_T(u) - \check{Y}(u, v) + 1) = v^*_{e_v}(Y_T(v) - \check{Y}(u, v) + 1)$. Let $P$ be the concatenation of the subpath of $Q(e_u)$ from $u = u_0$ to $u_{Y_T(u)-\check{Y}(u,v)+1}$ with the subpath of $Q(e_v)$ from $v_{Y_T(v)-\check{Y}(u,v)+1}$ to $v_0 = v$ (see Figure 8 for an illustration). Then $P$ connects $u$ and $v$ in $G$, so $d_G(u, v) \le |P| - 1 = Y_T(u) + Y_T(v) - 2\check{Y}(u, v) + 2$.

A symmetric argument proves the existence of a path $P'$ in $G$ between $v$ and $u$ of length $Y_T(u) + Y_T(v) - 2\check{Y}(v, u) + 2$; this gives the desired bound. □

7.3. **Winding numbers and distance lower bounds.** It turns out that the lower bound on $d_G(u, A)$ given by Corollary 7.2 can be improved by considering winding numbers around $u$; we now remind the reader of their definition.

Consider a closed curve $\gamma : [0, 1] \to \mathbb{R}^2 \setminus \{0\}$, and parametrize $\gamma$ in polar coordinates as $((r(t), \theta(t)), 0 \le t \le 1)$ so that $\theta$ is a continuous function. We define the *winding number of* $\gamma$ around zero to be $(\theta(1) - \theta(0))/(2\pi)$. Next, fix a reference point $r \in \mathbb{S}^2$. For $x \in \mathbb{S}^2 \setminus \{r\}$ and a closed curve $\gamma : [0, 1] \to \mathbb{S}^2 \setminus \{r, x\}$, let $\varphi : \mathbb{S}^2 \setminus \{r\}$ to $\mathbb{R}^2$ be a homeomorphism with $\varphi(x) = 0$, and define the winding number $\mathrm{wind}_r(x, \gamma)$ of $\gamma$ around $x$ to be the winding number of $\varphi \circ \gamma : [0, 1] \to \mathbb{R}^2$ around zero. It is straightforward that this definition does not depend on the choice of $\varphi(x)$.

In what follows, it is useful to imagine having chosen a particular representative from the equivalence class of $G$, or in other words a particular planar embedding (it is straightforward to verify that the coming arguments do not depend on which embedding is chosen). Let $r$ be any point in the interior of the face of $G$ incident to $c$.

**Definition 7.9.** *Fix an oriented edge $e = uw \in \overrightarrow{E}$ and a simple path $Q = (v_0, v_1, \ldots, v_m)$ from $u$ to $A$. Define the* winding number $w(Q, e) = w_{\mathrm{G}}(Q, e)$ of $Q$ around $e$ *as follows. Write $P(e) = (u_0, u_1, \ldots, u_\ell)$. Note that $u_0 = v_0 = u$, $u_1 = w$ and $u_\ell = v_m = A$. Form a cycle $C = (v_0, v_1, \ldots, v_m = u_\ell, u_{\ell-1}, \ldots, u_0)$. Then fix a point $x$ in the interior of the face incident to $\kappa^r(u, w)$, and let $w(Q, e) = \mathrm{wind}_r(x, C)$.*

In the preceding definition, we conflate $C$ with its image in $\mathbb{S}^2$ under the embedding of $G$ (and likewise with $x$); it is straightforward to verify that $w(Q, e)$ does not depend on the choice of such embeddings.

**Proposition 7.10.** *For all $e = uw \in \overrightarrow{E}$, if $Q$ is a simple path from $u$ to $A$ then $|Q| \ge |P(e)| + 2(w(Q, e) - 2)$.*

*Proof.* Write $P(e) = (u_0, u_1, \ldots, u_\ell)$. Let $R = (w_0, w_1, \ldots, w_k)$ be a simple path meeting $P(e)$ only at $w_0$ and $w_k$, with $w_0 = u_i$, $w_k = u_j$ for some $0 \le i < j \le \ell$. If $j < \ell$ then let $\hat{c} = \kappa^r(u_j, u_{j+1})$ and if $j = \ell$ (so $u_j = A$) then let $\hat{c}$ be the corner of the root face of $(G, c)$ incident to $A$.

We say $R$ *leaves* $P(e)$ *from the right* if $i > 0$ and the corner $\kappa^r_G(u_i, u_{i+1})$ precedes $\kappa^r_G(u_i, w_1)$ in clockwise order around $u_i$ starting from $\kappa^r_G(u_i, u_{i-1})$. Otherwise say that $R$ leaves $P(e)$ from the left; in particular, if $i = 0$ then $R$ leaves from the left by convention. Likewise, $R$ *returns to* $P(e)$ *from the right* if $\hat{c}$ precedes $\kappa^r(u_j, w_{k-1})$ in clockwise order around $u_j$ starting from $\kappa^r(u_j, u_{j-1})$; otherwise say that $R$ returns to $P(e)$ from the left.

The key to the proof is the following set of inequalities. Note that $k = |R| - 1$.

(1) If $R$ leaves $P(e)$ from the right and returns from the left then $k \geq j-i-2$.
(2) If $R$ leaves $P(e)$ from the left and returns from the left then $k \geq j-i$.
(3) If $R$ leaves $P(e)$ from the left and returns from the right then $k \geq j-i-1+2(\mathbb{1}i>0+\mathbf{1}_{[j<\ell]})$.
(4) If $R$ leaves $P(e)$ from the right and returns from the right then $k \geq j-i-1+2\mathbf{1}_{[j<\ell]}$.

For later use, we say $R$ has *type 1* if $R$ leaves $P(e)$ from the right and returns from the left, and define types $2,3$, and $4$ accordingly. Let $C=(w_0,\ldots,w_k=u_j,\ldots,u_i)$ be the cycle contained in the union of $R$ and $P(e)$. We provide the details of the bounds from (1) and (3), as (2) and (4) are respectively similar.

Note that although $C$ does not respect the orientation of edges given by $\overrightarrow{E}$, it is nonetheless an oriented cycle, so it makes sense to speak of the right- and left-hand sides of $C$. For (1), let $V'$ be the set of vertices on or to the right of $C$, and let $G'$ be the submap of $G$ induced by $V'$. All faces of $G'$ have degree three except $C$, which has degree $k+j-i$. By Euler's formula it follows that $|E(G')|=3|V'|-3-(k+j-i)$.

For $i<m<j$, since $P(e)$ is a leftmost path, $|\{x\in V' : u_m x\in \overrightarrow{E}\}|=1$. Also, since $R$ returns from the left, we must have $w_{k-1}u_j\in\overrightarrow{E}$ (or else $w_{k-1}=u_{j+1}$, which contradicts that $P(e)$ meets $R$ only at its endpoints), so $|\{x\in V' : u_j x\in\overrightarrow{E}\}|=0$. Since $\overrightarrow{E}$ is a 3-orientation, it follows that $E(G')\leq 3|V'|-2(j-i)-1$, which combined with the equality of the preceding paragraph yields that $k\geq j-i-2$.

For (3) let $V'$ be the set of vertices on or to the left of $C$. Euler's formula again yields $|E(G')|=3|V'|-3-(k+j-i)$. For $x\in V'$ not lying on $C$, we have $x\notin\{A,B,\mathrm{v}(c)\}$, so since $\overrightarrow{E}$ is a 3-orientation, $|\{y\in V' : xy\in\overrightarrow{E}\}|=3$. For $i<m<j-1$ we have $m<\ell-1$, so $u_m$ is not on the root face; since $R$ returns from the right, it follows that $|\{y\in V' : u_m y\in\overrightarrow{E}\}|=3$. Lastly, $|\{x\in V' : u_{j-1}x\in\overrightarrow{E}\}|\geq 1$ since $u_{j-1},u_j$ lies on $C$, and likewise $|\{x\in V' : u_i x\in\overrightarrow{E}\}|\geq 1$. The edges of $R$ are disjoint from the sets of edges counted above, so

$$|E(G')|\geq 3|V'\setminus\{w_1,\ldots,w_k,u_i,u_{j-1}\}|+2+k=3|V'|-2k-4.$$

Combined with the equality given by Euler's formula this yields $k\geq (j-i)-1$. Next, since $P(e)$ is leftmost, $u_m\notin\{A,B,\mathrm{v}(c)\}$ for $m<\ell-1$, which is straightforwardly seen. Thus, if $j<\ell$ then since $\overrightarrow{E}$ is a 3-orientation, we in fact have $|\{x\in V' : u_{j-1}x\in\overrightarrow{E}\}|=3$, and the same counting argument yields that $k\geq (j-i)+1$. Similarly, if $i>0$ then $|\{x\in V' : u_i x\in\overrightarrow{E}\}|=3$ and again $k\geq (j-i)+1$. Finally, if $0<i<j<\ell$ then the same argument yields $k\geq (j-i)+3$.

To conclude, subdivide the path $Q$ into edge-disjoint sub-paths $R_1,\ldots,R_t$, each of which is either a sub-path of $P(e)$ or else meets $P(e)$ only at its endpoints. We assume $R_1,\ldots,R_t$ are ordered so that $Q$ is the concatenation of $R_1,\ldots,R_t$, so in particular, $u=u_0$ is the first vertex of $R_1$, $A=u_\ell$ is the last vertex of $R_t$, and for $1\leq s<t$ the last vertex of $R_s$ is the first vertex of $R_{s+1}$.

For $1\leq i\leq 4$, let $n_i$ be the number of sub-paths of type $i$ among $\{R_1,\ldots,R_t\}$. Since $R_t$ is the only sub-path that intersects the root face, and $R_1$ is the only sub-path which may contain $u=u_0$, the above inequalities and a telescoping sum give

$$|Q|=1+\sum_{s=1}^{t}(|R_i|-1)\geq |P(e)|-2n_1+3n_3+n_4-2(\mathbf{1}_{[R_1\text{ has type }3]}+\mathbf{1}_{[R_t\text{ has type 3 or 4}]}).$$

In particular, we obtain the bound $|Q|\geq |P(e)|+2(n_3-n_1-2)$. Finally, sub-paths that leave from the right and return from the left correspond to clockwise windings of $C$ around $u$, and subpaths that leave from the left and return from the right correspond to counterclockwise windings of $C$ around $u$. It follows that $n_3-n_1$ is precisely the winding number $w(Q,e)$; this completes the proof. ☐

In what follows, if $C$ is an oriented cycle in $G$ then we write $V^l(C)$ (resp. $V^r(C)$) for the sets of vertices lying on or to the left (resp. on or to the right) of $C$, and note that $V^l(C) \cap V^r(C) = V(C)$.

**Proposition 7.11.** *For all $e = uw \in \overrightarrow{E}$, if $Q$ is a shortest path from $u$ to $A$ and $w(Q,e) < 0$ then there is a cycle $C$ in $G$ such that $G[V^l(C)]$ and $G[V^r(C)]$ each have diameter at least $\lfloor -w(Q,e)/2 \rfloor - 2$, and such that $\max_{y \in V^h(C)} Y_{\mathrm{T}}(y) - \min_{y \in V^h(C)} Y_{\mathrm{T}}(y) \geq \lfloor -w(Q,e)/2 \rfloor$ for $h \in \{l, r\}$.*

*Proof.* We write $Q = (u_0, u_1, \ldots, u_\ell)$, and partition $Q$ into edge-disjoint sub-paths $R_1, \ldots, R_t$ as at the end of the proof of Proposition 7.10. For $1 \leq s \leq t$ and $1 \leq i \leq 4$, let $n_i(s)$ be the number of sub-paths of type $i$ among $\{R_1, \ldots, R_s\}$. If $u_0 u_1 \neq e$ then by definition $R_1$ leaves $P(e)$ from the left, so $n_1(1) = 0$.

Let $m = \lfloor -w(Q,e)/2 \rfloor$, and let $s$ be minimal so that $n_1(s) - n_3(s) = m$; necessarily, $R_s$ has type 1. Also, $s \geq m+1$, and since $n_1(t) - n_3(t) = -w(Q,e) \geq 2m$ we also have $m \leq t - m$. Write $R_s = (w_0, w_1, \ldots, w_k)$, with $w_0 = u_i$, $w_1 = u_j$ for distinct $i, j \in \{1, \ldots, \ell\}$. By reversing $R_s$ if necessary, we may assume $i < j$,[8] and write $C = (w_0, w_1, \ldots, w_k, u_{j-1}, \ldots, u_i = w_0)$. Since $m + 1 \leq s \leq t - m$, the concatenation of $R_1, \ldots, R_{s-1}$ has length at least $m$ and so does the concatenation of $R_{s+1}, \ldots, R_t$. Since $Q$ is a shortest path from $u$ to $A$, it follows that for $0 \leq i \leq k$, $d_G(u, w_i) \geq m + i + 1$ and $d_G(A, w_i) \geq m + (k - i)$.

Fix $0 < a < j - i$ and let $S$ be a shortest path from $u$ to $u_{i+a}$. The concatenation of $S$, $(u_{i+a}, \ldots, u_j)$, and $R_{s+1}, \ldots, R_t$ has $d_G(u, u_{i+a}) + (j-i-a) + d_G(u_j, A)$ edges. On the other hand, by the inequality in (1) from the proof of Proposition 7.10, we have $k \geq j - i - 2$, so $Q$ has at least $d_G(u, u_i) + j - i - 2 + d_G(u_j, A)$ edges. Since $Q$ is a shortest path, it follows that

$$d_G(u, u_{i+a}) \geq d_G(w, u_i) + a - 2 \geq m + a - 2 \geq m - 1.$$

A similar argument shows that for all $0 < a < j - i$, $d_G(A, u_{i+a}) \geq m - 1$. Finally, one of $G[V^l(C)]$ or $G[V^r(C)]$ contains $R_1, \ldots, R_s$, and the other contains $R_s, \ldots, R_t$. Therefore, each of $G[V^l(C)]$ and $G[V^r(C)]$ contains at least $m$ vertices of $P(e)$; since vertex labels strictly decrease along $P(e)$, the final claim of the proposition follows. ☐

**Proposition 7.12.** *For all $e = uw \in \overrightarrow{E}$, if $Q$ is a shortest path from $u$ to $A$ and $w(Q,e) < -2$ then there is an oriented cycle $C$ in $G$ of length at most $6(|Q|-1)/(-w(Q,e)-2)$ such that $G[V^l(C)]$ and $G[V^r(C)]$ each have diameter at least $\lfloor -w(Q,e)/3 \rfloor - 2$ and such that $\max_{y \in V^h(C)} Y_{\mathrm{T}}(y) - \min_{y \in V^h(C)} Y_{\mathrm{T}}(y) \geq \lfloor -w(Q,e)/3 \rfloor$ for $h \in \{l, r\}$.*

*Proof.* The proof is very similar to that of Proposition 7.11, so we omit most details. Partition $Q$ into $R_1, \ldots, R_t$ and define $n_i(s)$, $1 \leq s \leq t$, $1 \leq i \leq 4$ as before. Let $m = \lfloor -w(Q,e)/3 \rfloor$. There are at least $m$ values of $s$ such that $R_s$ has type 1 and $m + 1 \leq n_1(s) - n_3(s) \leq 2m$; among these, let $s^\star$ minimize $|R_{s^\star}|$. Then

$$d_G(u, A) \geq m \cdot (|R_{s^\star}| - 1) \geq \frac{-(w(Q,e)+2)}{3} \cdot (|R_{s^\star}| - 1).$$

The sub-path of $P(e)$ joining the endpoints of $R_{s^\star}$ has at most two more edges than $R_{s^\star}$, so the cycle formed by this sub-path of $P(e)$ and $R_{s^\star}$ has at most $2|R_{s^\star}| \leq 6(d_G(w, A) + 1)/(-w(Q,e)-2) = 6(|Q|-1)/(-w(Q,e)-2)$ vertices. The remainder of the proof closely follows that of Proposition 7.11. ☐

[8] It is not hard to prove that there is always some shortest path $Q$ for which the ordered sequence of intersections with $P(e)$ respect the orientation of $P(e)$, so that there is no need to reverse $R_s$ to ensure $i < j$. However, we do not require such a property for the current proof.

## 8. Labels approximate distances for random triangulations

Fix $n \in \mathbb{N}$, and let $(T, \xi, \hat{\xi})$ be uniformly distributed in $\mathcal{T}_n^\bullet$. (We will later take $n \to \infty$, but suppress the dependence of $(T, \xi, \hat{\xi})$ on $n$ for readability.) As in Section 7, let $(G, c, \hat{c}) = \chi_n^\bullet(T, \xi, \hat{\xi})$ and let $(T', \xi', D) = \phi_n(\psi_n(T, \xi, \hat{\xi}))$, and write $\mathrm{G} = (G, c)$ and $\mathrm{T} = (T, \xi)$. By (4), $(G, c, \hat{c})$ is uniformly distributed in $\triangle_n^\bullet$ and $(T', \xi', D')$ is uniformly distributed in $\mathcal{T}_n^{\mathrm{vl}}$. Again define $Y_{\mathrm{T}}$ as in Section 5.3 and again extend $Y_{\mathrm{T}}$ to $V(G)$ by taking $Y_{\mathrm{T}}(A) = 1$ and $Y_{\mathrm{T}}(B) = 2$.

Using Corollary 7.5 and Proposition 7.10, we now show that with high probability, the labelling $Y_{\mathrm{T}} : V(G) \to \mathbb{Z}^{\geq 0}$ gives distances to $A$ up to a uniform $o(n^{1/4})$ correction.

**Theorem 8.1.** *For all $\epsilon > 0$,*

$$\lim_{n\to\infty} \mathbf{P}\left\{\exists\, u \in V(G) : d_G(u, A) \notin [Y_{\mathrm{T}}(u) - \epsilon n^{1/4}, Y_{\mathrm{T}}(u) - 1]\right\} = 0\,.$$

The upper bound $d_G(u, A) \leq Y_{\mathrm{T}}(u) - 1$ holds deterministically by Corollary 7.5. To prove the lower bound (in probability), we begin by stating a lemma whose proof, postponed to the end of the section, is based on soft convergence arguments and the continuity of the Brownian snake. Recall the definition of the contour exploration $(r_{\mathrm{T}}(j), 0 \leq j \leq 2n-2)$. Given $0 \leq i \leq 2n-2 = 2|V(T)| - 2$ and $\Delta > 0$, let

$$\begin{aligned} g_{\mathrm{T}}(i, \Delta) &= \sup\{j < i : |Y_{\mathrm{T}}(r_{\mathrm{T}}(j)) - Y_{\mathrm{T}}(r_{\mathrm{T}}(i))| \geq \Delta \text{ or } j = 0\} \\ d_{\mathrm{T}}(i, \Delta) &= \inf\{j > i : |Y_{\mathrm{T}}(r_{\mathrm{T}}(j)) - Y_{\mathrm{T}}(r_{\mathrm{T}}(i))| \geq \Delta \text{ or } j = 2n-2\}\,. \end{aligned}$$

Then let $N(i, \Delta) = \{v \in V(T) : \exists\, g_{\mathrm{T}}(i, \delta) \leq j \leq d_{\mathrm{T}}(i, \Delta), r_{\mathrm{T}}(j) = v\}$ be the set of vertices of $T$ visited by the contour exploration between times $g_{\mathrm{T}}(i, \Delta)$ and $d_{\mathrm{T}}(i, \Delta)$.

**Lemma 8.2.** *For all $\epsilon > 0$ and $\beta > 0$, there exist $\alpha > 0$ and $n_0 \in \mathbb{N}$ such that for $n \geq n_0$,*

$$\mathbf{P}\left\{\inf\left\{|N(i, \beta n^{1/4})| : 0 \leq i \leq 2n-2\right\} \geq \alpha n\right\} \geq 1 - \epsilon\,.$$

*Proof of Theorem 8.1.* As mentioned, we need only prove the lower bound. It suffices to show that for all $\epsilon > 0$,

$$\limsup_{n\to\infty} \mathbf{P}\left\{\exists\, e = uv \in \overrightarrow{E} : d_G(u, A) < Y_{\mathrm{T}}(u) - 6(\epsilon n^{1/4} + 2)\right\} \leq 4\epsilon$$

(we have done a little anticipatory selection of constants in the preceding formula). Write $\mathrm{diam}(G)$ for greatest distance between any two vertices of $G$. By Corollary 7.5 , $\mathrm{diam}(G) \leq 2\max_{u\in V(G)}(Y_{\mathrm{T}}(u) - 1) = 2(\max_{u\in V(T)} Y_{\mathrm{T}}(u) - \min_{u\in V(T)} Y_{\mathrm{T}}(u)) + 2$, so by Fact 5.10, $\mathrm{diam}(G) \leq \max_{u\in V(T')} X_{(T',\xi',D)}(u) - \min_{u\in V(T')} X_{(T',\xi',D)}(u) + 8$. Finally,

$$\max_{u\in V(T')} X_{(T',\xi',D)}(u) - \min_{u\in V(T')} X_{(T',\xi',D)}(u) = \max_{x\in[0,1]} Z_{(T',\xi',D)}(x) - \min_{x\in[0,1]} Z_{(T',\xi',D)}(x)\,,$$

and Proposition 6.1 implies that $(\max_{x\in[0,1]} Z_{(T',\xi',D)}(x) - \min_{x\in[0,1]} Z_{(T',\xi',D)}(x))n^{-1/4}$ converges in distribution as $n = |V(T')| \to \infty$, to an almost surely finite random variable. It follows that there is $c = c(\epsilon) > 0$ such that $\mathbf{P}\left\{\mathrm{diam}(G) \geq cn^{1/4}\right\} < \epsilon$. Choose such $c$, and let $B$ be the event that $G$ contains a cycle $C$ of length at most $2c/\epsilon$ such that with $V^l(C)$ and $V^r(C)$ as defined earlier, for $h \in \{l, r\}$ we have

$$\max_{y\in V^h(C)} Y_{\mathrm{T}}(y) - \min_{y\in V^h(C)} Y_{\mathrm{T}}(y) \geq \epsilon n^{1/4}.$$

Next, suppose there exists $e = uv \in \overrightarrow{E}$ for which $d_G(u, A) < Y_{\mathrm{T}}(u) - 6(\epsilon n^{1/4} + 2)$. Fix such an edge $e$, and any shortest path $Q$ from $u$ to $A$; by Proposition 7.10 we have $w(Q, e) \leq -3\epsilon n^{1/4} - 2$. It follows from Proposition 7.12 that either $\mathrm{diam}(G) \geq cn^{1/4}$ or else $B$ occurs. It thus suffices to show that

$$\mathbf{P}\left\{B, \mathrm{diam}(G) \leq cn^{1/4}\right\} \leq 3\epsilon. \tag{15}$$

Suppose $B$ occurs, let $C$ be as in the definition of $B$, and let $F$ be the subgraph of $T$ induced by $V(T) \setminus V(C)$. Then $F$ is a forest, and each component of $F$ is contained within $G[V^l(C)]$ or $G[V^r(C)]$ since $T$ is a subgraph of $G$. Also, for $\{u, w\} \in E(G)$ we have $|Y_{\mathrm{T}}(u) - Y_{\mathrm{T}}(w)| \leq 1$. It follows that, for $h \in \{l, r\}$, if $G[V^h(C)]$ contains $k$ components of $F$ then one such component $T_h$ must have

$$\max_{y \in V(T_h)} Y_{\mathrm{T}}(y) - \min_{y \in V(T_h)} Y_{\mathrm{T}}(y) > \epsilon n^{1/4}/k - 1.$$

But $F$ has at most $|E(C)| \leq 2(|Q| - 1)/(\epsilon n^{1/4})$ connected components. When $\mathrm{diam}(G) \leq cn^{1/4}$ we have $2(|Q| - 1)/(\epsilon n^{1/4}) \leq 2c/\epsilon$, so for each $h \in \{l, r\}$, some component $T_h$ of $F$ contained in $G[V^h(C)]$ must have

$$\max_{y \in V(T_h)} Y_{\mathrm{T}}(y) - \min_{y \in V(T_h)} Y_{\mathrm{T}}(y) \geq \frac{\epsilon^2 n^{1/4}}{2c} - 1.$$

Using again that labels of adjacent vertices differ by at most one, if $\mathrm{diam}(G) \leq cn^{1/4}$ then for $h \in \{l, r\}$ there is $v_h \in V^h(C)$ such that

$$\min_{v \in V(C)} |Y_{\mathrm{T}}(v_h) - Y_{\mathrm{T}}(v)| \geq \frac{\epsilon^2 n^{1/4}}{4c} - \frac{1}{2} - \frac{2c}{\epsilon}.$$

Now for $h \in \{l, r\}$ let $j_h = j_h(\mathrm{T}) = \inf\{0 \leq i \leq 2n - 2 : r_{\mathrm{T}}(i) = v_h\}$. Also, fix any $\beta \in (0, \epsilon^2/2c)$. By Lemma 8.2 there is $\alpha > 0$ such that for $n$ sufficiently large,

$$\mathbf{P}\left\{\min(N(j_l, \beta n^{1/4}), N(j_r, \beta n^{1/4})) \leq \alpha n\right\} \leq \epsilon.$$

For $n$ large enough that $\epsilon^2 n^{1/4}/(4c) - 1/2 - 2c/\epsilon > \beta n^{1/4}$, for $h \in \{l, r\}$ we also have $N(j_h, \beta n^{1/4}) \subset V^h(C)$, and it follows that for $n$ sufficiently large

$$\begin{aligned} &\mathbf{P}\left\{B, \mathrm{diam}(G) \leq cn^{1/4}\right\} \\ \leq &2\epsilon + \mathbf{P}\left\{\exists\ C \text{ a cycle in } G, |C| \leq \frac{2c}{\epsilon},\ \min(|V^l(C)|, |V^r(C)|) \geq \alpha n\right\}. \end{aligned} \tag{16}$$

The event in the last probability is that $G$ contains a separating cycle of length at most $2c/\epsilon$ that separates $G$ into two subtriangulations, each of size at least $\alpha n$. The number $t_{n,m}$ of simple triangulations of an $(m + 2)$-gon with $n$ inner vertices has been computed in [8], and has the asymptotic form $t_{n,m} \sim A_m \alpha^n n^{-5/2}$, where $A_m$ and $\alpha$ are explicit constants. (Observe that, in this notation, the number of rooted simple triangulations with $n$ vertices is equal to $t_{n-3,1}$.) For $K \in \mathbb{N}$ and $\alpha > 0$, denote by $\Gamma_K(\alpha)$ the event that a random simple triangulation with $n$ vertices admits a separating cycle $\gamma_n$ of length at most $K$ that separates $G_n$ into two components each of size at least $\alpha n$. Then

$$\mathbf{P}\left\{\Gamma_K(\alpha)\right\} \sim (t_{n-3,1})^{-1} \sum_{k=1}^{K-2} \int_{\alpha}^{1-\alpha} t_{\lfloor un \rfloor, k} t_{\lfloor (1-u)n \rfloor, k} du \sim A_{K,\alpha} n^{-5/2}, \tag{17}$$

where $A_{K,\alpha}$ depends only on $\alpha$ and $K$. The event within the last probability in (16) is contained within the event $\Gamma_{\lceil 2c/\epsilon \rceil}(\alpha)$, so for $n$ sufficiently large its probability is at most $\epsilon$. In view of (15), this completes the proof. □

*Proof of Lemma 8.2.* Fix $\epsilon > 0$ and $\beta > 0$ as in the statement of the lemma. List the elements of $V(T_n')$ according to lexicographic order in $\mathrm{T}_n$ as $v_n(1), \ldots, v_n(n)$, and for $1 \leq m \leq n$ let $i_n(m) = \inf\{i \geq 0 : r_{\mathrm{T}_n}(i) = v_n(m)\}$.

By considering the height process, a straightforward argument (almost identical that given for equations (12) and (13) of [19]) shows that

$$\sup_{0 \leq t \leq 1} \left| \frac{i_n(\lfloor tn \rfloor)}{2n - 2} - t \right| \xrightarrow{\mathrm{d}} 0.$$

from which it follows that for any $\delta > 0$,

$$\mathbf{P}\left\{\frac{\inf\left\{|N(i,\beta n^{1/4})| : 0 \le i \le 2n-2\right\} + \delta n}{\inf\{d_{\mathrm{T}_n}(i,\beta n^{1/4}) - g_{\mathrm{T}_n}(i,\beta n^{1/4}) : 0 \le i \le 2n-2\}} < \frac{1}{2}\right\} \to 0.$$

In particular, given $\alpha > 0$, for $n$ large, if $d_{\mathrm{T}_n}(i,\beta n^{1/4}) - g_{\mathrm{T}_n}(i,\beta n^{1/4}) > \alpha n$ for all $0 \le i \le 2n-2$ then with high probability $\inf\left\{|N(i,\beta n^{1/4})| : 0 \le i \le 2n-2\right\} > \alpha n/3$. It therefore suffices to prove there exists $\alpha > 0$ such that for all $n$ sufficiently large,

$$\mathbf{P}\left\{\inf\{d_{\mathrm{T}_n}(i,\beta n^{1/4}) - g_{\mathrm{T}_n}(i,\beta n^{1/4}) : 0 \le i \le 2n-2\} \ge \alpha n\right\} > 1-\epsilon\,. \tag{18}$$

By Proposition 6.1 and Skorohod's embedding theorem, we now work in a space in which

$$\left((3n)^{-1/2}C_{\mathrm{T}_n}(t), (4n/3)^{-1/4}Z_{\mathrm{T}_n}(t)\right)_{0\le t\le 1} \overset{\text{a.s.}}{\to} (\mathbf{e}(t), Z(t))_{0\le t\le 1}\,. \tag{19}$$

Let $A = A(Z) = \inf\{|x-y| : x,y \in [0,1], |Z(x)-Z(y)| > \beta/(2\cdot(4/3)^{1/4})\}$, or let $A(Z) = 1$ if the set in the preceding infimum is empty. When $(d_{\mathrm{T}_n}(i,\beta n^{1/4}) - g_{\mathrm{T}_n}(i,\beta n^{1/4}))/(2n-2) < 1$ either $d_{\mathrm{T}_n}(i,\beta n^{1/4}) \ne 0$ or $g_{\mathrm{T}_n}(i,\beta^{1/4}) \ne 2j-2$, so either $Z_{\mathrm{T}_n}(d_{\mathrm{T}_n}(i,\beta n^{1/4})/(2n-2)) - Z_{\mathrm{T}_n}(i/(2n-2)) > \beta n^{1/4}$ or $Z_{\mathrm{T}_n}(i/(2n-2)) - Z_{\mathrm{T}_n}(g_{\mathrm{T}_n}(i,\beta n^{1/4})/(2n-2)) > \beta n^{1/4}$. By (19), it follows that a.s.

$$(2n-2)^{-1}\cdot\inf\{d_{\mathrm{T}_n}(i,\beta n^{1/4}) - g_{\mathrm{T}_n}(i,\beta n^{1/4}) : 0 \le i \le 2n-2\} > A$$

for all $n$ sufficiently large. Finally, since $Z$ is a.s. uniformly continuous on $[0,1]$, almost surely $A > 0$, and (18) follows immediately. ☐

## 9. The proof of Theorem 1.1 for triangulations

Fix $n \in \mathbb{N}$ and $(T,\xi,\hat{\xi}) \in \mathcal{T}_n^{\bullet}$. Let $(M,\zeta,\hat{\zeta}) = \chi_n^{\bullet}(T,\xi,\hat{\xi}) \in \triangle_{n+2}^{\bullet}$, and let $(T',\xi',D) = \phi_n(\psi_n(T,\xi,\hat{\xi})) \in \mathcal{T}_n^{\mathrm{vl}}$. Then let $\mathrm{M} = (M,\hat{\zeta})$, let $\mathrm{T} = (T',\xi')$, let $R = V(T') = V(M) \setminus \{A,B\}$, let $X = X_{(T',\xi',D)}$ be as in Section 2.3. Finally, note that $\mathrm{v}_T(\xi) = \mathrm{v}_M(\zeta) \in R$ and that $\mathrm{v}_T(\hat{\xi}) = \mathrm{v}_M(\hat{\zeta}) \in R$. Then set $\mathrm{P} = \mathrm{P}(T,\xi,\hat{\xi}) = (\mathrm{M},\mathrm{T},R,X,\mathrm{v}_T(\xi))$, and let

$$\mathcal{P}_n = \{\mathrm{P}(T,\xi,\hat{\xi}) : (T,\xi,\hat{\xi}) \in \mathcal{T}_n^{\bullet}\}\,.$$

The triple $(T,\xi,\hat{\xi})$ may be recovered from $(T',\xi',D)$ and the vertex $\mathrm{v}_T(\xi)$, and $(T',\xi',D)$ may be recovered from $(T',\xi')$ and $X$; it follows that P is a bijection between $\mathcal{T}_n^{\bullet}$ and $\mathcal{P}_n$.

Now let $(T_n,\xi_n,\hat{\xi}_n)$ be a uniformly random element of $\mathcal{T}_n^{\bullet}$ and let $\mathrm{P}_n = \mathrm{P}(T_n,\xi_n,\hat{\xi}_n)$, so that $\mathcal{P}_n$ is a uniformly random element of $\mathcal{P}_n$. For later use, let $(M_n,\zeta_n,\hat{\zeta}_n) = \chi_n^{\bullet}(T_n,\xi_n,\hat{\xi}_n)$, and let $(T_n',\xi_n',D_n) = \phi_n(\psi_n(T_n,\xi_n,\hat{\xi}_n))$. Then set $\mathrm{M}_n = (M_n,\hat{\zeta}_n)$, set $\mathrm{T}_n = (T_n',\xi_n')$, set $R_n = V(M_n) \setminus \{A,B\}$, set $X_n = X_{(T_n',\xi_n',D_n')}$, and set $u_n = \mathrm{v}_{T_n}(\xi_n)$, so that $\mathrm{P}_n = (\mathrm{M}_n,\mathrm{T}_n,R_n,X_n,u_n)$.

To prove Theorem 1.1 for triangulations, we verify that $\mathcal{P}_n$ is a CS family, with sequences $a_n = (3n)^{-1/2}$ and $b_n = (4n/3)^{-1/4}$. Assuming this, to conclude note that, by Corollary 5.8, $(M_n,\zeta_n)$ is a uniformly random element of $\triangle_{n+2}^{\circ}$. Since $b_{n+2}/b_n \to 1$ as $n \to \infty$, the result then follows from Theorem 4.1.

By Proposition 5.5 and Corollaries 5.8 and 5.9, $(T_n',\xi_n',D_n)$ has law $\mathrm{LGW}(\mu,\nu,n)$, where the law of $\mu$ is given by (6) and $\nu = (\nu_k, k \ge 1)$ is as in Corollary 5.9. Condition **1.** then holds by Proposition 6.1. Condition **2.**(i) is immediate from the construction as $\mathrm{M}_n$ contains only two vertices ($A$ and $B$) that are not elements of $R_n$. Next, for $u,v \in V(M_n)$, by Proposition 7.6 and Fact 5.10,

$$d_{M_n}(u,v) \le X_{\mathrm{T}_n}(u) + X_{\mathrm{T}_n}(v) - 2\max\{\check{X}_{\mathrm{T}_n}(u,v), \check{X}_{\mathrm{T}_n}(v,u)\} + 14,$$

where the additive constant 14 arises from the 2 in Proposition 7.6 and four times the additive error of 3 from Fact 5.10. By the definition of $\check{X}_{\mathrm{T}}(u,v)$, if $i$ and $j$ are such that $u = r_{\mathrm{T}_n(R_n)}(i)$ and $v = r_{\mathrm{T}_n(R_n)}(j)$ then $\check{X}_{\mathrm{T}_n}(u,v) \le \check{Z}_{\mathrm{P}_n}(i/m,j/m) + 2$ and $\check{X}_{\mathrm{T}_n}(v,u) \le$

$\check{Z}_{\mathrm{P}_n}(i/m, j/m) + 2$ (these additive factors of 2 arise from the possibility that $i$ and $j$ not the first times $u$ and $v$ are visited by the contour exploration). It follows that for all $i, j \in [2|R_n| - 2]$,

$$d_{M_n}(u, v) \leq Z_{\mathrm{P}_n}(i/m) + Z_{\mathrm{P}_n}(j/m) - 2\max\left(\check{Z}_{\mathrm{P}_n}(i/m, j/m), \check{Z}_{\mathrm{P}_n}(j/m, i/m)\right) + 18\,,$$

which verifies **3.**(i). Condition **3.**(ii) follows from Theorem 8.1 since $u_n$ and $A$ are always adjacent in M. It remains to establish **2.**(ii).

Since $\mathrm{v}_{M_n}(\hat{\zeta}_n) = \mathrm{v}_{T_n}(\xi'_n) = r_{\mathrm{T}'_n(R_n)}(0)$, $X_n(\mathrm{v}(\xi'_n)) = 0$. By (3) it follows that

$$b_n \cdot |d_{M_n}(\mathrm{v}_{M_n}(\hat{\zeta}_n), u_n) + \check{Z}_n(0, 1)| \xrightarrow{\mathrm{d}} 0\,,$$

so by **1.**,

$$b_n d_{M_n}(\mathrm{v}_{M_n}(\hat{\zeta}_n), u_n) \xrightarrow{\mathrm{d}} -\check{Z}(0, 1) \stackrel{\mathrm{d}}{=} Z(V) - \check{Z}(0, 1)\,, \tag{20}$$

where $V \stackrel{\mathrm{d}}{=} \mathrm{Uniform}[0, 1]$ is independent of $Z$; the last equality in distribution is from (2). Now let $V_n$ be a uniformly random element of $R_n$. Arguing from (3) and **1.** as above, we obtain

$$b_n d_{M_n}(V_n, u_n) \xrightarrow{\mathrm{d}} Z(V) - \check{Z}(0, 1). \tag{21}$$

Next, recall that $(M_n, \zeta_n)$ is uniformly random in $\triangle^{\circ}_{n+2}$. It follows that, conditional on $M_n$, $\zeta_n$ is a uniformly random element of $\mathcal{C}(M_n)$; let $c_n$ be another uniformly random element of $\mathcal{C}(M_n)$, independent of $\zeta_n$ and of $V_n$. Since $u_n = \mathrm{v}_{M_n}(\zeta_n)$, it follows that

$$d_{M_n}(V_n, u_n) \stackrel{\mathrm{d}}{=} d_{M_n}(V_n, \mathrm{v}_{M_n}(c_n)). \tag{22}$$

Let $\overrightarrow{E}$ be the minimal 3-orientation associated to $\mathrm{M}_n$. Writing $c_n = (\{x_n, y_n\}, \{y_n, z_n\})$, let $\tilde{\mathrm{v}}_{M_n}(c_n) = y_n$ if $y_n z_n \in \overrightarrow{E}$, and $\tilde{\mathrm{v}}_{M_n}(c_n) = z_n$ otherwise. Note that $\tilde{\mathrm{v}}_{M_n}(c_n)$ is either equal to or incident to $\mathrm{v}_{M_n}(c_n)$, so $|d_{M_n}(V_n, \mathrm{v}_{M_n}(c_n)) - d_{M_n}(V_n, \tilde{\mathrm{v}}_{M_n}(c_n))| \leq 1$. Further, since $c_n$ is a uniformly random corner of $M_n$, $\{y_n, z_n\}$ is a uniformly random edge of $M_n$, so for all $v \in V(M_n)$, $\mathbf{P}\{\tilde{\mathrm{v}}_{M_n}(c_n) = v\}$ is proportional to the outdegree of $v$ in $\overrightarrow{E}$. Since all inner vertices of $M_n$ have outdegree 3 in $\overrightarrow{E}$, and $c_n$ is independent of $V_n$, we may couple $\tilde{\mathrm{v}}_{M_n}(c_n)$ with a uniformly random element $U_n$ of $R_n$, independent of $V_n$, such that $\mathbf{P}\{U_n \neq \tilde{\mathrm{v}}_{M_n}(c_n)\} \to 0$ as $n \to \infty$. Furthermore, $d_{M_n}(\mathrm{v}_{M_n}(c_n), U_n) \leq 1$ on $\{U_n = \tilde{\mathrm{v}}_{M_n}(c_n)\}$, so $b_n d_{M_n}(\mathrm{v}_{M_n}(c_n), U_n) \to 0$ in probability, as $n \to \infty$. It then follows from (21) and (22) that

$$b_n d_{M_n}(V_n, U_n) \xrightarrow{\mathrm{d}} Z(V) - \check{Z}(0, 1)$$

With (20), this establishes **2.**(ii) and completes the proof. □

## 10. The proof of Theorem 1.1 for quadrangulations

The results on which the proof for simple triangulations rely all have nearly exact analogues for simple quadrangulations, which makes the proof for quadrangulations quite straightforward. In this section, we state the required results, with an emphasis on the details that differ between the two cases.

10.1. **Simple quadrangulations and blossoming trees.** The counterpart of the bijection between simple triangulations and 2-blossoming trees is a bijection between simple quadrangulations and 1-blossoming trees, due to Fusy [14]. In this section, by "blossoming trees" we mean 1-blossoming trees, and write $\mathcal{T}_{\square,n}$ for the set of blossoming trees with $n$ inner vertices. Fix a blossoming tree $T$. Given a stem $\{b, u\}$ with $b \in \mathcal{B}(T)$, if $bu$ is followed by *three* inner edges in a clockwise contour exploration of $T$ – $uv$, $vw$ and $wz$, say – then the *local closure* of $\{b, u\}$ consists in removing the blossom $b$ (from both $V(T)$ and $\mathcal{B}$) and its stem, and adding a new edge $\{u, z\}$.

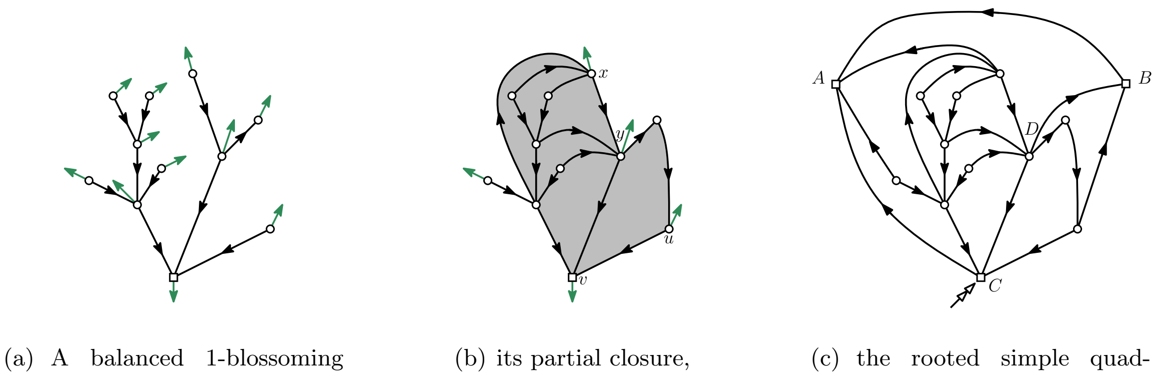
After all local closures have been performed, all unclosed blossoms are incident to a single face $f$. A simple counting argument shows that there exist exactly two edges $\{u, v\}$ and $\{x, y\}$ of $f$ such that $u, v, x$ and $y$ are each incident to one unclosed stem; between any two other consecutive unclosed stems, there are two edges of $f$. Assume by symmetry that $f$ lies to the left of both $uv$ and $xy$, and write $\xi_C = \kappa^r(v, u)$, $\xi_D = \kappa^r(y, x)$, $C = \mathrm{v}(\xi_C)$ and $D = \mathrm{v}(\xi_D)$ (see Fig.9(b)).

Given $\xi \in \mathcal{C}(T)$, the planted blossoming tree $(T, \xi)$ is *balanced* if $\xi = \xi_C$ or $\xi = \xi_D$. Suppose $\xi \in \{\xi_C, \xi_D\}$ and write $v = \mathrm{v}(\xi)$. Let $S_{CD}$ (resp. $S_{DC}$) be the set of non-blossom vertices $u$ incident to an unclosed blossom in the partial closure, such that in the planted tree $(T, C)$ (resp. $(T, D)$) we have $C \preceq_{\mathrm{ctr}} v \prec_{\mathrm{ctr}} D$ (resp. $D \preceq_{\mathrm{ctr}} v \prec_{\mathrm{ctr}} C$).

To finish the construction, remove the remaining blossoms and their stems. Add two additional vertices $A$ and $B$ within the outer face, and an edge between $A$ (resp. $B$) and each of the vertices of $S_{CD}$ (resp. of $S_{DC}$). In the resulting map, define a corner $c$ by $c = (\{C, B\}, \{C, A\})$ if $v = C$ or $c = (\{D, A\}, \{D, B\})$ if $v = D$. Finally, add an edge between $A$ and $B$ in such a way that, after its addition, $c$ lies on the same face as $A, B$, and $v$ (see Fig.9(c)). Write $\chi_\square(\mathrm{T})$ for the resulting map.

Fix a planted planar quadrangulation $(Q, \xi)$, and view $(Q, \xi)$ as embedded in $\mathbb{R}^2$ so that the face containing $\xi$ is the unique unbounded face. A 2-*orientation* of a $(Q, \xi)$ is an orientation for which $\alpha(v) = 2$ for each vertex $v$ not incident to the root face and, listing the vertices of the unbounded face in clockwise order as $v, A, B, w$ with $v = \mathrm{v}(\xi)$, we have $\alpha(A) = 0$, $\alpha(B) = \alpha(v) = 1$ and $\alpha(w) = 2$. Write $\overrightarrow{E}_\square$ for the resulting quadrangulation. Ossona de Mendez [30] showed that a quadrangulation admits a 2-orientation if and only if it is simple, and in this case admits a unique *minimal* 2-orientation.

**Proposition 10.1** ([14]). *The closure operation $\chi_{\square,n}$ is a bijection between the set $\mathcal{T}^\circ_{\square,n}$ of balanced 1-blossoming trees with $n$ inner vertices and the set $\square^\circ_{n+2}$ of planted quadrangulations with $n+2$ vertices. Furthermore, for $\mathrm{T} \in \mathcal{T}_{\square,n}$, $\chi_{\square,n}(\mathrm{T})$ is naturally endowed with its minimal 2-orientation by viewing stems of $\mathrm{T}$ as oriented toward blossoms, and all other edges as oriented toward the root.*




(a) A balanced 1-blossoming tree,

(b) its partial closure,

(c) the rooted simple quadrangulation obtained, endowed with its unique minimal 2-orientation

FIGURE 9. The closure of a balanced 1-blossoming tree into a simple quadrangulation.

10.2. **Sampling simple quadrangulations.** Given a blossoming tree $\mathrm{T}_\square = (T, \xi)$, define $\lambda_\square := \lambda_{\square,\mathrm{T}_\square} : \mathcal{C}(T) \to \mathbb{Z}$ as follows. Let $(\xi_{\mathrm{T}_\square}(i), 0 \le i \le 2|V(T)|-2)$ be the contour ordering from Section 2.2, with $\xi_0 = \xi$. Let $\lambda_{\square,\mathrm{T}_\square}(\xi_0) = 2$ and, for $0 \le i < 2|V(T)| - 3$, set

$$\lambda_{\square,\mathrm{T}_\square}(\xi_{\mathrm{T}_\square}(i+1)) = \begin{cases} \lambda_{\square,\mathrm{T}_\square}(\xi_{\mathrm{T}_\square}(i)) - 1 & \text{if } \mathrm{v}(\xi_i) \notin \mathcal{B}, \mathrm{v}(\xi_{\mathrm{T}_\square}(i+1)) \notin \mathcal{B}, \\ \lambda_{\square,\mathrm{T}_\square}(\xi_{\mathrm{T}_\square}(i)) & \text{if } \mathrm{v}(\xi_i) \notin \mathcal{B}, \mathrm{v}(\xi_{\mathrm{T}_\square}(i+1)) \in \mathcal{B}, \\ \lambda_{\square,\mathrm{T}_\square}(\xi_{\mathrm{T}_\square}(i)) + 2 & \text{if } \mathrm{v}(\xi_i) \in \mathcal{B}, \mathrm{v}(\xi_{\mathrm{T}_\square}(i+1)) \notin \mathcal{B}, \end{cases}$$

As opposed to Section 5.2, here the label increases by 2 after each stem.

It is not hard to see that $\mathrm{T}_\square = (T, \xi)$ is balanced if and only if $\xi$ is incident to one stem and $\lambda_{\square,\mathrm{T}_\square}(c) \ge 2$ for all $c \in \mathcal{C}(T)$. With the same definition of successors for corners, and the same construction as in Section 5.2, this labelling yields another description of the bijection from Section 10.1.

Let $\mathcal{T}^{\mathrm{vl}}_{\square,n}$ be the set of triples $(T, \xi', d)$ where $(T, \xi')$ is a planted plane tree and $d = (d_e, e \in E(T))$ is a $\pm 1$ labeling of $E(T)$ such that for all $v \in V(T)$, listing the edges from $v$ to its children in lexicographic order as $e_1, \ldots, e_k$, the sequence $d_{e_1}, \ldots, d_{e_k}$ is non-decreasing.

Let $X_\square \overset{\mathrm{d}}{=} \mathrm{Geometric}(2/3)$, and let $B_\square$ have law given by

$$\mathbf{P}\{B_\square = c\} = \frac{(c+1)\mathbf{P}\{X_\square = c\}}{\mathbf{E}(X_\square + 1)}, \text{ for } c \in \mathbb{N}. \tag{23}$$

The following is the analogue of Corollary 5.9 for quadrangulations.

**Proposition 10.2.** *Let $(T', \xi')$ be a Galton–Watson tree with branching factor $B_\square$ conditioned to have $n$ vertices. Conditional on $(T', \xi')$, independently for each $v \in V(T')$, list the children of $v$ in clockwise order as $c(v, 1), \ldots, c(v, k)$ and let $(D_{\{v,c(v,j)\}}, 1 \le j \le k$ be a random vector with law $\nu_{\square,k}$, where $\nu_{\square,k}$ is the uniform law over non-decreasing vectors $(d_1, \ldots, d_k) \in \{-1, 1\}^k$. Finally, let $D = (D_e, e \in E(T'))$. Then $(T', \xi', D)$ is uniformly distributed in $\mathcal{T}^{\mathrm{vl}}_{\square,n}$ and the closure $\chi_{\square,n}(T', \xi', D)$ is uniformly distributed in $\square^{\circ}_{n+2}$.*

The proof of Theorem 6.1 extends immediately to this setting and we obtain the following convergence (see Appendix A for the computation of the constants).

**Proposition 10.3.** *For $n \in \mathbb{N}$ let $\mathrm{T}_n = (T_n, \xi_n, D_n)$ be a uniformly random element of $\mathcal{T}^{\mathrm{vl}}_{\square,n}$. Then as $n \to \infty$,*

$$\left( \frac{3}{4n^{1/2}} C_{\mathrm{T}_n}(t), \left(\frac{3}{8n}\right)^{1/4} Z_{\mathrm{T}_n}(t) \right)_{0 \le t \le 1} \overset{\mathrm{d}}{\to} (\mathbf{e}(t), Z(t))_{0 \le t \le 1}, \tag{24}$$

*for the topology of uniform convergence on $C([0,1], \mathbb{R})^2$.*

10.3. **Labels and distance in simple quadrangulations.** We next state analogues of the results of Sections 7 and 8 for quadrangulations. Fix $n \in \mathbb{N}$ and $(T, \xi) \in \mathcal{T}^{\circ}_{\square,n}$, let $(Q, c) = \chi_{\square,n}(T, \xi)$ be endowed with its minimal 2-orientation $\overrightarrow{E}_\square$ and let $(T', \xi', D) \in \mathcal{T}^{\mathrm{vl}}_{\square,n}$ be the validly-labelled tree associated to $(T, \xi)$. Finally, write $\mathrm{Q} = (Q, c)$ and $\mathrm{T}_\square = (T, \xi)$.

The definition of leftmost paths for simple quadrangulations is an obvious modification of that for triangulations. Together with the fact that (with $Y_T$ defined as before) for $\{u, w\} \in E(Q)$, $|Y_T(u) - Y_T(w)| \le 3$, we obtain the following facts. The lemma is a counterpart of Lemma 7.1 and Corollary 7.5; the proposition is a counterpart of Proposition 7.6, and uses an identical definition for $\check{Y}_{\mathrm{T}_\square}(u, v)$.

**Lemma 10.4.** *For all $u \in V(Q)$, $Y_{\mathrm{T}_\square}(u)/3 \le d_Q(u, A) \le Y_{\mathrm{T}_\square}(u) - 1$.*

**Proposition 10.5.** *For all $u, v \in V(Q)$,*

$$d_Q(u, v) \le Y_{\mathrm{T}_\square}(u) + Y_{\mathrm{T}_\square}(v) - 2\max\{\check{Y}_{\mathrm{T}_\square}(u, v), \check{Y}_{\mathrm{T}_\square}(v, u)\} + 2.$$

The winding number introduced in Definition 7.9 is used in the following analogue of Proposition 7.10.

**Proposition 10.6.** *For all* $e = uw \in \overrightarrow{E}_{\square}$*, if* $Q$ *is a simple path from* $e$ *to* $A$ *then* $|Q| \geq |P(e)| + 2(w(Q,e) - 1)$.

*Proof.* The proof of Proposition 7.10 extends readily to the case of quadrangulations. Keeping the same notation, the following inequalities (whose proofs are left to the reader) allow one to conclude along the same lines.

(1) If $R$ leaves $P(e)$ from the right and returns from the left then $k \geq j - i - 2$.
(2) If $R$ leaves $P(e)$ from the left and returns from the left then $k \geq j - i$.
(3) If $R$ leaves $P(e)$ from the left and returns from the right then $k \geq j - i + 2(\mathbf{1}_{[i>0]} + \mathbf{1}_{[j<\ell]})$.
(4) If $R$ leaves $P(e)$ from the right and returns from the right then $k \geq j - i + 2\mathbf{1}_{[j<\ell]}$. □

Combining Lemma 10.4 and Proposition 10.6, we obtain that with probability tending to one, distances to $A$ in $Q$ are given by labels in $\mathrm{T}_{\square}$ up to a $o(n^{1/4})$ perturbation.

**Theorem 10.7.** *For all* $\epsilon > 0$*,*

$$\lim_{n\to\infty} \mathbf{P}\left\{\exists\ u \in V(Q) : d_Q(u,A) \notin [Y_{\mathrm{T}_{\square}}(u) - \epsilon n^{1/4}, Y_{\mathrm{T}_{\square}}(u) - 1]\right\} = 0\,.$$

*Proof.* The only element of the proof of Theorem 8.1 that cannot be directly applied here is the approximation of $\mathbf{P}\{\Gamma_K(\alpha)\}$ given in (25) that relies on the number $t_{n,m}$ of simple triangulations of an $(m+2)$-gon. This has an easy fix: for $\alpha > 0$, write $\Gamma_{\square,K}(\alpha)$ for the event that a uniformly random simple quadrangulation $Q_n$ with $n$ faces admits a separating cycle of length at most $K$, separating $Q_n$ into two components each of size at least $\alpha n$. An explicit expression for the number $q_{n,m}$ of simple quadrangulations of a $2m$-gon with $n$ inner vertices is derived in [9], and has the asymptotic form $q_{n,m} \sim A_m \alpha^n n^{-5/2}$, where $A_m$ and $\alpha$ are explicit constants. (Observe that, in this notation, the number of rooted simple quadrangulations with $n$ vertices is equal to $q_{n-4,2}$.). Then

$$\mathbf{P}\{\Gamma_K(\alpha)\} \sim (q_{n-4,2})^{-1} \sum_{k=0}^{K} \int_{\alpha}^{1-\alpha} t_{\lfloor un \rfloor,k} t_{\lfloor (1-u)n \rfloor,k} du \sim A_{\square,K,\alpha} n^{-5/2}, \tag{25}$$

where $A_{\square,K,\alpha}$ depends only on $\alpha$ and $K$. □

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## List of notation and terminology

## References


[1] Marie Albenque and Dominique Poulalhon. Generic method for bijections between blossoming trees and planar maps. arXiv:1305.1312 [math.CO], 2013. 3

[2] David Aldous. The continuum random tree. II. An overview. In *Stochastic analysis (Durham, 1990)*, volume 167 of *London Math. Soc. Lecture Note Ser.*, pages 23–70. Cambridge Univ. Press, Cambridge, 1991. 19

[3] Jan Ambjørn, Bergfinnur Durhuus, and Thordur Jonsson. *Quantum geometry*. Cambridge Monographs on Mathematical Physics. Cambridge Univ. Press, 1997. 2

[4] Cyril Banderier, Philippe Flajolet, Gilles Schaeffer, and Michèle Soria. Random maps, coalescing saddles, singularity analysis, and Airy phenomena. *Random Structures Algorithms*, 19 (3-4):194–246, 2001. 41

[5] Johel Beltran and Jean-François Le Gall. Quadrangulations with no pendant vertices. Bernoulli, to appear, 2013+. 2

[6] Olivier Bernardi and Éric Fusy. A bijection for triangulations, quadrangulations, pentagulations, etc. *J. Combin. Theory Ser. A*, 119(1):218–244, 2012. 3

[7] Jérémie Bouttier and Emmanuel Guitter. Distance statistics in quadrangulations with no multiple edges and the geometry of minbus. *J. Phys. A*, 43(20):205207, 31, 2010. 41, 42

[8] William G Brown. Enumeration of triangulations of the disk. *Proc. London Math. Soc*, 14(3): 746–768, 1964. 33

[9] William G Brown. Enumeration of quadrangular dissections of the disk. *Canad. J. Math*, 17 (3):302–317, 1965. 38

[10] Dmitri Burago, Yuri Burago, and Sergei Ivanov. *A course in metric geometry*, volume 33 of *Graduate Studies in Mathematics*. American Mathematical Society, 2001. 9

[11] Philippe Chassaing and Gilles Schaeffer. Random planar lattices and integrated superBrownian excursion. *Probab. Theory Related Fields*, 128(2):161–212, 2004. 3

[12] Bertrand Duplantier and Scott Sheffield. Liouville quantum gravity and KPZ. *Invent. Math.*, 185:333–393, 2011. 2

[13] Steven N Evans and Anita Winter. Subtree prune and regraft: a reversible real tree-valued markov process. *Ann. Probab.*, 34(3):918–961, 2006. 10

[14] Éric Fusy. *Combinatoire des cartes planaires et applications algorithmiques*. PhD thesis, École Polytechnique, 2010. 35, 36

[15] Zhicheng Gao and Nicholas C Wormald. Sharp concentration of the number of submaps in random planar triangulations. *Combinatorica*, 23(3):467–486, 2003.

[16] Christophe Garban. Quantum gravity and the KPZ formula. In *Séminaire Bourbaki*, volume 64, 2011-2012. 2

[17] Svante Janson and Jean-François Marckert. Convergence of discrete snakes. *J. Theoret. Probab.*, 18(3):615–645, 2005. 19

[18] Jean-François Le Gall. *Spatial branching processes, random snakes and partial differential equations*. Birkhauser Basel, 1999. 4

[19] Jean-François Le Gall. Random trees and applications. *Probab. Surv.*, 2:245–311, 2005. 17, 33, 42

[20] Jean-François Le Gall. The topological structure of scaling limits of large planar maps. *Invent. Math.*, 169(3):621–670, 2007. 2, 43, 44

[21] Jean-François Le Gall. Geodesics in large planar maps and in the brownian map. *Acta Math.*, 205(2):287–360, 2010. 12

[22] Jean-François Le Gall. Uniqueness and universality of the brownian map. Ann. Probab., to appear, 2013+. 2, 4, 5, 41, 43, 45

[23] Jean-François Le Gall and Frédéric Paulin. Scaling limits of bipartite planar maps are homeomorphic to the 2-sphere. *Geom. Funct. Anal.*, 18(3):893–918, 2008. 2

[24] Jean-François Marckert. The lineage process in Galton–Watson trees and globally centered discrete snakes. *Ann. Appl. Probab.*, 18(1):209–244, 2008. 20

[25] Jean-François Marckert and Grégory Miermont. Invariance principles for random bipartite planar maps. *Ann. Probab.*, 35(5):1642–1705, 2007. 20

[26] Jean-François Marckert and Abdelkader Mokkadem. Limit of normalized quadrangulations: the Brownian map. *Ann. Probab.*, 34(6):2144–2202, 2006. 4

[27] Grégory Miermont. On the sphericity of scaling limits of random planar quadrangulations. *Electron. Commun. Probab*, 13:248–257, 2008. 2
[28] Grégory Miermont. Tessellations of random maps of arbitrary genus (mosaıques sur des cartes aléatoires en genre arbitraire). *Ann. Sci. Éc. Norm. Supér.*, 42(5):725–781, 2009. 9, 10, 12
[29] Grégory Miermont. The Brownian map is the scaling limit of uniform random plane quadrangulations. Acta Math., to appear, 2013+. 2, 4
[30] Patrice Ossona de Mendez. *Orientations bipolaires*. PhD thesis, École des Hautes Etudes en Sciences Sociales, Paris, 1994. 36
[31] Dominique Poulalhon and Gilles Schaeffer. Optimal coding and sampling of triangulations. *Algorithmica*, 46(3):505–527, 2006. 6, 13
[32] Gilles Schaeffer. *Conjugaison d'arbres et cartes combinatoires aléatoires*. PhD thesis, Université Bordeaux I, 1998. 12, 13
[33] Walter Schnyder. Planar graphs and poset dimension. *Order*, 5:323–343, 1989. 5
[34] Kenneth Stephenson. *Introduction to circle packing*. Cambridge Univ. Press, Cambridge, 2005. 3

## Appendix A. Notes about constants

In this section we briefly derive the forms of the constant coefficients arising in Theorem 1.1 and Proposition 6.1.

For simple triangulations, we work with a critical Galton–Watson tree with a branching factor $B$ uniquely specified by the following facts.

(1) Criticality: $\mathbf{E}B = 1$.
(2) There exists $p \in (0, 1)$ such that if $G$ is Geometric($p$) then the law of $B$ is given by setting, for each $c \in \mathbb{N}$,

$$\mathbf{P}\{B = c\} = \frac{\binom{c+2}{2}\mathbf{P}\{G = c\}}{\mathbf{E}\binom{G+2}{2}}.$$

From these conditions, a straightforward calculation shows that $p = 3/4$, and another easy computation yields that $\mathbf{E}\left[B^2\right] = 7/3$ so $\mathbf{Var}\{B\} = 4/3$. In the notation of Section 6.1, this yields $\sigma_\mu = 2/3^{1/2}$.

Next, the displacement $D$ between a node in our tree and a uniformly selected child is equal to one of $\{-1, 0, 1\}$, each with equal probability; it follows that $\mathbf{E}\left[D^2\right] = 2/3$. Symmetrize as in Section 6.1, then let $\nu_k$ be the law of the displacement vector for a vertex with $k$ children. Again using the notation of Section 6.1, it follows that $\sigma^2_{\hat{\nu}_k(i)} = 2/3$ for all $1 \le i \le k$, so $\sigma^2_{\hat{\nu}} = \sigma^2_\nu = 2/3$ and $(\sigma_\mu/2)^{1/2}/\sigma_\nu = (3/4)^{1/4}$. Together with Theorem 4.1, this explains the values of constants relating to triangulations.

We remark that the scaling required for convergence of triangulations in Theorem 1.1 agrees with the intuition described in [7], Section 4.1. It differs by a factor $8^{1/4}$ from the scaling for general triangulations that arises in Theorem 1.1 of [22], which can be understood as follows. First, in [22], the index $n$ denotes the number of faces rather than the number of vertices, which accounts for a factor $2^{1/4}$. The size of the simple core of a loopless triangulation with $m$ vertices is typically $\sim m/2$ (see Table 4 of [4]); this explains another factor $2^{1/4}$. Finally, the loopless core of a simple triangulation with $m$ vertices is again typically of order $\sim m/2$ (this is not proved in [4] but may be handled using the same technology); this explains the final factor $2^{1/4}$. The latter factor does not arise in considering quadrangulations, which can not contain loops; this may be viewed as explaining the different form of the constant for triangulations versus those of bipartite maps in Theorem 1.1 of [22].

For simple quadrangulations, we work with a critical Galton–Watson tree with branching factor $B$ uniquely specified by the following facts.

(1) Criticality: $\mathbf{E}B = 1$.

(2) There exists $p \in (0,1)$ such that if $G$ is Geometric($p$) then the law of $B$ is given by setting, for each $c \in \mathbb{N}$,

$$\mathbf{P}\{B = c\} = \frac{(c+1)\mathbf{P}\{G = c\}}{\mathbf{E}[G+1]} .$$

From these calculations, a straightforward calculation shows that $p = 2/3$, and another easy computation then yields that $\mathbf{E}\left[B^2\right] = \frac{5}{2}$, so $\mathbf{Var}\{B\} = 3/2$. Next, the displacement $D$ between a node $v$ and a uniformly selected child is equal to $-1$ or to 1, each with equal probability, so has $\mathbf{E}[D] = 0$ and $\mathbf{Var}\{D\} = 1$. Using Theorem 4.1 as above then yields the scaling for quadrangulations in Theorem 1.1, and agrees with the two-point calculation for simple quadrangulations by Bouttier and Guitter [7].

## Appendix B. Convergence for Chassaing–Schaeffer families

Throughout the section we let $\mathcal{P} = (\mathcal{P}_n, n \in \mathbb{N})$ be a CS family (with associated sequences $(a_n, n \in \mathbb{N})$ and $(b_n, n \in \mathbb{N})$ of constants), let $\mathrm{P}_n = (\mathrm{M}_n, \mathrm{T}_n, R_n, X_n, u_n)$ be a uniformly random element of $\mathcal{P}_n$, and write $\mathrm{M}_n = (M_n, \zeta_n)$ and $\mathrm{T}_n = (T_n, \xi_n)$. We also write $r_n = r_{\mathrm{T}_n(R_n)}$, $C_n = C_{\mathrm{P}_n}$, and $Z_n = Z_{\mathrm{P}_n}$ for succinctness.

Next, for $n \geq 1$, list the vertices of $R_n$ according to their lexicographic order in $\mathrm{T}_n(R_n)$ as $v_n(1), \ldots, v_n(|R_n|)$, and given $1 \leq m \leq |R_n|$, let $i_n(m) = \inf\{i : r_n(i) = v_n(m)\}$ be the index at which $v_n(m)$ first appears in the contour exploration.

**Lemma B.1.** *As $n \to \infty$, we have*

$$\sup_{0 \leq t \leq 1} \left| \frac{i_n(\lfloor t \cdot |R_n| \rfloor)}{2|R_n| - 2} - t \right| \xrightarrow{\mathrm{d}} 0 .$$

*Proof.* As in Lemma B.1, a straightforward argument using the height process (following (12) and (13) of [19]) shows that when $|R_n| \geq 2$, deterministically

$$\sup_{0 \leq t \leq 1} \left| \frac{i_n(\lfloor t \cdot |R_n| \rfloor)}{2|R_n| - 2} - t \right| \leq \frac{\max\{|[\![\mathrm{v}_{T_n}(\xi_n), v]\!] \cap R_n| : v \in R_n\} + 4}{2|R_n| - 2} ,$$

where we abuse notation by writing $[\![\mathrm{v}_{T_n}(\xi_n), v]\!]$ for the set of vertices of the simple path between $\mathrm{v}_{T_n}(\xi_n)$ and $v$ in $T_n$. Since $|R_n| \to \infty$ it thus suffices to show that $\max\{|[\![\mathrm{v}_{T_n}(\xi_n), v]\!] \cap R_n| : v \in R_n\}/(2|R_n| - 2) \xrightarrow{\mathrm{d}} 0$. To see this, let $U$ and $V$ be independent Uniform$[0,1]$ random variables. If the latter convergence fails to hold then for infinitely many $n$, with uniformly positive probability, a single path from the root in $\mathrm{T}_n$ contains a macroscopic proportion of the elements of $R_n$. It follows easily that

$$\limsup_{n \to \infty} \mathbf{P}\left\{ U < V, C_n(U) = \min_{U \leq x \leq V} C_n(x) \right\} > 0 .$$

On the other hand, $\mathbf{P}\{U < V, \mathbf{e}(U) = \min_{U \leq x \leq V} \mathbf{e}(x)\} = 0$, so the preceding equation implies that $\mathbf{e}$ is not the distributional limit of any rescaling of $C_n$. Thus **1.** does not hold, a contradiction. □

*Proof of Theorem 4.1.* We claim that it suffices to establish

$$(R_n, b_n d_{M_n}, \mu_n) \xrightarrow{\mathrm{d}} (S, d, \mu). \tag{26}$$

for $d_{\mathrm{GHP}}$. (In the above, by $d_{M_n}$ we really mean the distance on $R_n$ induced by $d_{M_n}$. This slight notational abuse should cause no confusion.) Indeed, suppose the latter convergence holds. By Skorohod's representation theorem, we may work in a space in which the convergence (26) is almost sure. Fix $\epsilon > 0$, and let $E_n$ be the event that $\max_{v \in V(M_n)} b_n \cdot d_{M_n}(v, R_n) \leq \epsilon/2$ and $d^2_{\mathrm{GH}}(R_n, b_n d_{M_n}, \mathrm{v}_{M_n}(\zeta_n), u_n), (S, d, \rho, u)) \leq \epsilon/2$. Now let $\mathcal{R}^0_n = \{(x,y) \in R_n \times V(M_n); b_n d_{M_n}(x,y) \leq \epsilon/2\}$; then $\mathcal{R}^0_n$ has distortion at most $\epsilon$. Furthermore, $(\mathrm{v}_{M_n}(\zeta_n), \mathrm{v}_{M_n}(\zeta_n)) \in \mathcal{R}^0_n$ and $(u_n, u_n) \in \mathcal{R}^0_n$. Let $\nu_n$ be the probability measure on

$R_n \times V(M_n)$ whose restriction to $\{(v,v) : v \in R_n\}$ is the uniform probability measure. Then $\nu_n$ is a coupling of $\mu_n$ (as a measure on $R_n$) and $\mu_n$ (as a measure on $V(M_n)$), and $\nu_n(\mathcal{R}_n^0) = 1$.

On $E_n$ we have that $\mathcal{R}_n^0$ is a correspondence, so on $E_n$,

$$d_{\mathrm{GHP}}((V(M_n), b_n d_{M_n}, \mu_n), (R_n, b_n d_{M_n}, \mu_n)) \le \epsilon/2,$$

and by the triangle inequality it follows that on $E_n$,

$$d_{\mathrm{GHP}}((V(M_n), b_n d_{M_n}, \mu_n), (S, d, \mu)) \le \epsilon\,.$$

Finally, in this space, since $\mathcal{P}$ is a CS family, $\mathbf{P}\{E_n\} \to 1$ as $n \to \infty$, and it follows that $(V(M_n), b_n d_{M_n}, \mu_n) \stackrel{\mathrm{d}}{\to} (S, d, \mu)$ for $d_{\mathrm{GHP}}$. We thus turn our attention to proving (26).

The first part of our argument is based on that of [20], Proposition 3.2; the second part is based on an argument from Section 8.3 of [22]. Define a function $d_n : [0,1]^2 \to [0,\infty)$ as follows. Let $m = m_n = 2|R_n| - 2$, and for $i, j \in [m]$, let $d_n(i/m, j/m) = d_{M_n}(r_n(i), r_n(j))$. Then extend $d_n$ to $[0,1]$ by "bilinear interpolation": if $(x,y) = ((i+\alpha)/m, (j+\beta)/m)$ for $0 \le \alpha < 1$, $0 \le \beta < 1$ then let

$$\begin{aligned} d_n(x,y) &= \alpha\beta d_n((i+1)/m, (j+1)/m) + \alpha(1-\beta) d_n((i+1)/m, j/m) \\ &\quad + (1-\alpha)\beta d_n(i/m, (j+1)/m) + (1-\alpha)(1-\beta) d_n(i/m, j/m)\,. \end{aligned}$$

Using **1.**, we now work in a space in which

$$(a_n C_n, b_n Z_n) \stackrel{\text{a.s.}}{\to} (\mathbf{e}, Z). \tag{27}$$

We will show that in such a space, additionally

$$b_n d_n \stackrel{\text{a.s.}}{\to} d^*\,, \tag{28}$$

for the topology of uniform convergence on $C([0,1]^2)$. Assume (28) holds, and for $n \in \mathbb{N}$, consider the correspondence $\mathcal{R}_n$ between $(S, d)$ and $(R_n, b_n d_{M_n})$ given by letting $[s] \in [0,1]/\{d^* = 0\} = S$ correspond to $r_n(i)$ if and only if $\lceil s \cdot m \rceil = i$, for $0 \le i \le m$. [9] By (28), the distortion of $\mathcal{R}_n$ tends to zero.

Let $\mu_n^-$ be the uniform probability measure on $R_n \setminus \{\mathrm{v}(\zeta_n)\}$. Define a coupling between $\mu_n^-$ and $\mu$ as follows. Fix $s \in [0,1]$. Let $f_1(s) = [s] \in S$. If $s = i/m$ then let $f_2(s) = r_n(i)$. If $s \in (i/m, (i+1)/m)$ and $\{r_n(i), r_n(i+1)\} = \{u, p(u)\} \in E(T_n)$ then let $f_2(s) = u$. Finally, let $f = (f_1, f_2) : [0,1] \to S \times R_n$, let $\lambda$ denote one-dimensional Lebesgue measure on $[0,1]$, and let $\nu = f_* \lambda$. Write $\pi$ and $\pi'$ for the projection maps from $S \times R_n$ to $S$ and to $R_n$, respectively. We clearly have $\pi_* \nu = \mu$. Also, for each edge $e \in E(T)$, there are precisely two indices $i, j \in \{0, 1, \ldots, 2|R_n| - 2\}$ for which $\{r_n(i), r_n(i+1)\} = \{u, p(u)\}$; it follows that $\pi'_* \nu = \mu_n^-$.

For any pair $([s], r_n(i)) \in \mathcal{R}_n$, either $f_2(s) = r_n(i)$ or $f_2(s) = p(r_n(i))$, the two possibilities due to the two directions in which the edge $\{p(r_n(i)), r_n(i)\}$ is traversed during the contour exploration. We thus let

$$\mathcal{R}_n^+ = \{([s], w) : ([s], w) \in \mathcal{R}_n \text{ or } ([s], p(w)) \in \mathcal{R}_n\}\,.$$

Since $\mathcal{R}_n$ was a correspondence, $\mathcal{R}_n^+$ is again a correspondence, and $\nu(\mathcal{R}_n^+) = 1$. Finally, $\mathrm{dis}(\mathcal{R}_n^+) \le \mathrm{dis}(\mathcal{R}_n) + 2b_n$ so $\mathrm{dis}(\mathcal{R}_n^+) \to 0$ as $n \to \infty$. It follows by definition that

$$(V(M_n), b_n d_{M_n}, \mu_n^-) \stackrel{\mathrm{d}}{\to} (S, d, \mu)$$

for $d_{\mathrm{GHP}}$. Finally, the Prokhorov distance between $\mu_n^-$ and $\mu_n$ is $1/|R_n|$, which tends to zero as $n \to \infty$. We may therefore replace $\mu_n^-$ by $\mu_n$ and the preceding convergence still holds, which establishes (26) and so proves the theorem. It thus remains to prove (28) .

Define a function $d_n^\circ : [0,1]^2 \to [0,\infty)$ as follows: for $x, y \in \{i/m, 0 \le i \le m\}$, let

$$d_n^\circ(x,y) = Z_n(x) + Z_n(y) - 2\max\left(\check{Z}_n(x,y), \check{Z}_n(y,x)\right)\,.$$

[9] A similar technique is used at the end of Section 8 of [22].

Then extend $d_n^\circ$ to $[0,1]^2$ by bilinear interpolation as with $d_n$. Recalling that for integer $i \le m$, $Z_n(i/m) = X_n(r_n(i))$, it follows straightforwardly from **1.** that for all $\epsilon, \delta > 0$,

$$\limsup_{n\to\infty} \mathbf{P}\left\{\sup_{|x-y|\le\delta} b_n d_n^\circ(x,y) \ge \epsilon\right\} \le \mathbf{P}\left\{\sup_{|x-y|\le\delta} (Z(x)+Z(y)-2\max(\check{Z}(x,y),\check{Z}(y,x))) \ge \epsilon\right\}$$

(the derivation of this inequality is spelled out in a little more detail in [20], Section 3). Since $Z$ is almost surely continuous, it follows that for any $\eta > 0$ and $k \in \mathbb{N}$, there exist $\delta_k > 0$ and $n_k \in \mathbb{N}$ such that for all $n \ge n_k$,

$$\mathbf{P}\left\{\sup_{|x-y|\le\delta_k} b_n d_n^\circ(x,y) \ge 2^{-(k+1)}\right\} \le \frac{\eta}{2^{k+1}}\,.$$

Next, by **3**(i), after increasing $n_k$ if necessary, for $n \ge n_k$,

$$\mathbf{P}\left\{\sup_{x,y\in[0,1]} b_n(d_n(x,y) - d_n^\circ(x,y)) \ge 2^{-(k+1)}\right\} \le \frac{\eta}{2^{k+1}}\,. \tag{29}$$

By decreasing $\delta_k$ if necessary, we may also ensure that for $n < n_k$,

$$\mathbf{P}\left\{\sup_{|x-y|\le\delta_k} b_n \max(d_n(x,y), d_n^\circ(x,y)) \le 2^{-(k+1)}\right\} = 1\,.$$

Combining the three preceding estimates yields that for all $n \ge 1$,

$$\mathbf{P}\left\{\sup_{|x-y|\le\delta_k} b_n d_n(x,y) \ge 2^{-k}\right\} \le \frac{\eta}{2^k}\,,$$

so

$$\mathbf{P}\left\{\forall n,k,\ \sup_{|x-y|\le b_n\delta_k} d_n(x,y) < 2^{-k}\right\} \ge 1-\eta\,.$$

In other words, with $(\delta_k)_{k\ge 0}$ as above, with probability at least $1-\eta$, for all $n$, $b_n d_n$ belongs to the compact

$$K = \{f \in C([0,1]^2,\mathbb{R}) : f(0,0) = 0, \forall k,\ \sup_{|x-y|\le\delta_k} f(x,y) \le 2^{-k}\}\,,$$

which implies that $\{b_n d_n, n \in \mathbb{N}\}$ is tight in $C([0,1]^2,\mathbb{R})$ . For the remainder of the proof, we let $\tilde{d} \in C([0,1]^2,\mathbb{R})$ be any almost sure subsequential limit of $b_n d_n$; we suppress the subsequence from the notation for readability.

Recall that we work in a space where (27) holds. In such a space, it follows from the continuity of $Z$ that $b_n d_n^\circ \stackrel{\text{a.s.}}{\to} d_Z$, where $d_Z : [0,1]^2 \to \mathbb{R}$ is as defined in Section 1.1. By (29), it follows that for any $\eta > 0$ and $p \ge 1$,

$$\limsup_{n\to\infty} \mathbf{P}\left\{\sup_{x,y\in[0,1]} (b_n d_n(x,y) - d_Z(x,y)) \ge 2^{-p}\right\} \le \eta 2^{-p}\,,$$

so a.s. $\tilde{d} \le d_Z$.

We next claim that a.s. $\tilde{d}(x,y) = 0$ for all $x \ne y \in [0,1]$ for which $x \sim_{\mathbf{e}} y$. To see this, suppose that $x \sim_{\mathbf{e}} y$ for some $x, y \in [0,1]$, and assume by symmetry that $x < y$. Continuing to write $m = m_n = 2|R_n| - 2$, (28) implies there exist random integer sequences $(x_n, n \in \mathbb{N})$ and $(y_n, n \in \mathbb{N})$ such that $x_n/m_n \stackrel{\text{a.s.}}{\to} x$, $y_n/m_n \stackrel{\text{a.s.}}{\to} y$, and

$$C_n(x_n/m_n) = C_n(y_n/m_n) = \min\{C_n(z) : x_n \le m_n \cdot z \le y_n\}\,.$$

It follows that $r_n(x_n) = r_n(y_n)$,or equivalently that $i_n(x_n) = i_n(y_n)$, so

$$d_n(x_n/m_n, y_n/m_n) = d_{M_n}(r_n(i_n(x_n)), r_n(i_n(y_n))) = 0\,.$$

Since $b_n d_n \xrightarrow{\text{a.s.}} \tilde{d}$ (along a subsequence) and $x_n/m_n \xrightarrow{\text{a.s.}} x$, $y_n/m_n \xrightarrow{\text{a.s.}} y$, this implies that

$$0 = d_n(x_n/m_n, y_n/m_n) \xrightarrow{\text{a.s.}} \tilde{d}(x, y)\,,$$

so $\tilde{d}(x, y) \overset{\text{a.s.}}{=} 0$ as claimed.

Since, almost surely, $\tilde{d} = 0$ on $\{\{x, y\} : x \sim_{\mathbf{e}} y\}$, and $\tilde{d} \le d_Z$, we must have $\tilde{d} \le d^*$ since $d^*$ is the largest pseudo-metric on $[0, 1]$ satisfying these constraints. We now show that in fact, almost surely $\tilde{d} = d^*$.

Let $U, V$ be independent and uniform on $[0, 1]$. By **1.** we have

$$b_n X_n(r_n(I_n)) = b_n \check{Z}_n(0, 1) \xrightarrow{\text{d}} \check{Z}(0, 1) \overset{\text{d}}{=} -d^*(U, V)\,,$$

the last identity by (2) (which is Corollary 7.3 of [22]). Since $\mathrm{v}_{M_n}(\zeta_n) = r_n(0)$ and $X_n(r_n(0)) = 0$, by (3) we also have

$$\lim_{n\to\infty} \mathbf{P}\left\{b_n \cdot |d_{M_n}(\mathrm{v}_{M_n}(\zeta_n), u_n) + X_n(r_n(I_n))| > \epsilon\right\} = 0\,,$$

so since $X_n(r_n(I_n)) = \check{Z}_n(0, 1)$, we obtain

$$d_{M_n}(\mathrm{v}_{M_n}(\zeta_n), u_n) \xrightarrow{\text{d}} d^*(U, V)\,.$$

Since the Prokhorov metric topologizes weak convergence, by **2.**(ii) it follows that for $U_n$ and $V_n$ two independent random elements of $R_n$, then

$$b_n d_{M_n}(U_n, V_n) \xrightarrow{\text{d}} d^*(U, V)\,.$$

Now let $1 \le J_n, K_n \le |R_n|$ be such that $v_n(J_n) = U_n$ and $v_n(K_n) = V_n$. The preceding convergence implies $b_n d_n(J_n, K_n) \xrightarrow{\text{d}} d^*(U, V)$. Lemma B.1 implies that $(J_n, K_n) \xrightarrow{\text{d}} (U, V)$, so the tightness of the collection $(b_n d_n, n \ge 1)$ then yields

$$b_n d_n(U, V) \xrightarrow{\text{d}} d^*(U, V)\,.$$

Finally, along the subsequence where $b_n d_n \xrightarrow{\text{a.s.}} \tilde{d}$, we also have $b_n d_n(U, V) \xrightarrow{\text{d}} \tilde{d}(U, V)$, so it must be that $\tilde{d}(U, V) \overset{\text{d}}{=} d^*(U, V)$. Since a.s. $\tilde{d} \le d^*$, it must therefore hold that $\tilde{d} \overset{\text{a.s.}}{=} d^*$.

We have now shown that in the space where (27) holds, any subsequential limit $\tilde{d}$ of $b_n d_n$ must satisfy $\tilde{d} \overset{\text{a.s.}}{=} d^*$; this implies that in fact, in this space we have $b_n d_n \xrightarrow{\text{a.s.}} \tilde{d}$, which establishes (28) and so completes the proof. ☐

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