Convexity in partial cubes: the hull number

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Abstract. We prove that the combinatorial optimization problem of determining the hull number of a partial cube is NP-complete. This makes partial cubes the minimal graph class for which NP-completeness of this problem is known and improves some earlier results in the literature.
On the other hand we provide a polynomial-time algorithm to determine the hull number of planar partial cube quadrangulations.
Instances of the hull number problem for partial cubes described include poset dimension and hitting sets for interiors of curves in the plane.
To obtain the above results, we investigate convexity in partial cubes and characterize these graphs in terms of their lattice of convex subgraphs, improving a theorem of Handa. Furthermore we provide a topological representation theorem for planar partial cubes, generalizing a result of Fukuda and Handa about rank 3 oriented matroids.

1 Introduction

The object of this paper is the study of convexity and particularly of the hull number problem on different classes of partial cubes. Our contribution is twofold. First, we establish that the hull number problem is NP-complete for partial cubes, second, we emphasize reformulations of the hull number problem for certain classes of partial cubes leading to interesting problems in geometry, poset theory and plane topology.

Denote by Q^d the hypercube graph of dimension d. A graph G is called a *partial cube* if there is an injective mapping $\phi : V(G) \to V(Q^d)$ such that $d_G(v,w) = d_{Q^d}(\phi(v),\phi(w))$ for all $v,w \in V(G)$, where, d_G and d_{Q^d} denote the graph distance in G and Q^d, respectively. It implies in particular that for each pair of vertices of G, at least one shortest path between them in Q^d belongs also to G. In other words $\phi(G)$, seen as an induced subgraph of Q^d, is an *isometric embedding* of G in Q^d.

Partial cubes were introduced by Graham and Pollak in [24] in the study of interconnection networks and continue to find strong applications; they form for instance the central graph class in media theory (see the recent book [18]) and frequently appear in chemical graph theory e.g. [17]. Partial cubes form a generalization of several important graph classes, thus have also many applications in different fields of mathematics. Indeed, they "present one of the central and most studied classes of graphs in all of the metric graph theory", citing [30].

This article discusses some examples of such families of graphs including Hasse diagrams of upper locally distributive lattices or equivalently antimatroids [20] (Section 2.2), region graphs of halfspaces and hyperplanes (Section 3), and tope graphs of oriented matroids [11] (Section 5). These families contain many graphs defined on sets of combinatorial objects: flip-graphs of strongly connected and acyclic orientations of digraphs [12], linear extension graphs of posets [32], integer tensions of digraphs [20], configurations of chip-firing games [20], to name a few.

Convexity for graphs is the natural counterpart of Euclidean convexity and is defined as follows; a subgraph G' of G is said to be *convex* if all shortest paths in G between vertices of G' actually belong to G'. The *convex hull* of a subset V' of vertices – denoted $\mathrm{conv}(V')$ – is defined as the smallest convex subgraph containing V'. Since the intersection of convex subgraphs is clearly convex, the convex hull of V' is the intersection of all the convex subgraphs that contain V'.

A subset of vertices V' of G is a *hull set* if and only if $\mathrm{conv}(V') = G$. The *hull number* or *geodesic hull number* of G, denoted by $hn(G)$, is the size of a smallest hull set. It was introduced in [19], and since then has been the object of numerous papers. Most of the results on the hull number are about computing good bounds for specific graph classes, see e.g. [9, 28, 7, 6, 16, 8]. Only recently, in [15] the focus was set on computational aspects of the hull number and it was proved that determining the hull number of a graph is NP-complete. This was strengthened to bipartite graphs in [1]. On the other hand, polynomial-time algorithms have been obtained for unit-interval graphs, cographs and split graphs [15], cactus graphs and P_4-sparse graphs [1], distance hereditary graphs and chordal graphs [29]. Moreover, in [2], a fixed parameter tractable algorithm to compute the hull number of any graph class was obtained. Here, the parameter is the size of a vertex cover.

Let us end this introduction with an overview of the results and the organization of this paper. Section 2 is devoted to properties of convexity in partial cubes and besides providing tools for the other sections, its purpose is to convince the reader that convex subgraphs of partial cubes behave nicely. First a characterization of partial cubes in terms of their convex subgraphs is given. In particular, convex subgraphs of partial cubes behave somewhat like polytopes in Euclidean space. Namely, they satisfy an analogue of the Representation Theorem of Polytopes [35]. We then prove that for any vertex v in a partial cube G, the set of convex subgraphs of G containing v ordered by inclusion forms an upper locally distributive lattice. This property leads to a new characterization of partial cube, strengthening a theorem of Handa [26].

In Section 3 the problem of determining the hull number of a partial cube is proved to be NP-complete, improving earlier results of [15] and [1]. Our proof implies a stronger result by showing that determining the hull number of a region graph of an arrangement of halfspaces and hyperplanes is also NP-complete.

In Section 4 the relation between the hull number problem for linear extension graphs and the dimension problem of posets is discussed. We present a quasi-polynomial-time algorithm to compute the dimension of a poset given its linear extension graph and conjecture that the problem is polynomial-time solvable.

Section 5 is devoted to planar partial cubes. We provide a new characterization, which is a topological representation theorem generalizing work of Fukuda and Handa on rank 3 oriented matroids [22]. This characterization is then exploited to obtain a polynomial-time algorithm that computes the hull number of planar partial cube quadrangulations.

2 Convexity in partial cubes

2.1 Partial cubes and cut-partitions

All graphs studied in this article are supposed to be connected, simple and undirected. We use the classic graph terminology of [5]. Given a graph G a *cut* $C \subseteq E$ is an inclusion-minimal set of edges whose removal disconnects G. The removal of a cut C leaves exactly two connected components called its *sides*, denoted by C^+ and C^-. For $V' \subset V$, a cut C *separates* V' if both $C^+ \cap V'$ and $C^- \cap V'$ are not empty. A *cut-partition* of G is a set $\mathcal{C}$ of cuts partitioning E. For a cut $C \in \mathcal{C}$ and $V' \subseteq V$ define $C(V')$ as G if C separates V' and otherwise as the side of C containing V'.

Observation 1. A graph G is bipartite if and only if G has a cut-partition.

The equivalence classes of the *Djoković-Winkler relation* of a partial cube [14, 33] can be interpreted as the cuts of a cut-partition. Reformulating some properties of these equivalence classes as well as some results from [10, 3] the following new characterization of partial cubes in terms of cut partitions can be obtained.

Theorem 2. *A connected graph G is a partial cube if and only if G admits a cut-partition $\mathcal{C}$ satisfying one of the following equivalent conditions:*

(i) *for all $u, v \in V$, there is a shortest path between them using no $C \in \mathcal{C}$ twice*
(ii) *no shortest path in G uses any $C \in \mathcal{C}$ twice*
(iii) *for all $V' \subseteq V : \operatorname{conv}(V') = \bigcap_{C \in \mathcal{C}} C(V')$*
(iv) *for all $v, w \in V : \operatorname{conv}(v, w) = \bigcap_{C \in \mathcal{C}} C(v, w)$*

Note that *(iii)* resembles the *Representation Theorem for Polytopes*, see [35]; where the role of points is taken by vertices and the halfspaces are mimicked by the sides of the cuts in the cut-partition. Thanks to *(iii)*, the hull number problem has now a very useful interpretation as a hitting set problem:

Corollary 3. *Let $\mathcal{C}$ be a cut-partition that satisfies Theorem 2 then V' is a hull set if and only if on both sides of C there is a vertex of V', for all $C \in \mathcal{C}$.*

2.2 Partial cubes and upper locally distributive lattices

In this subsection we present another indication for how nice partial cubes behave with respect to convexity. Generalizing a theorem of Handa [26] we characterize partial cubes in terms of their lattice of convex subgraphs, see Fig.1.

A partially ordered set or poset $\mathcal{L} = (X, \leq)$ is a *lattice*, if each pair of elements $x, y \in \mathcal{L}$ admits both a unique largest element smaller than both of them called their *meet* and denoted $x \wedge y$, and a unique smallest element larger than both of them called their *join* and denoted $x \vee y$. Since both these operations are associative, we can define $\bigvee M := x_1 \vee \ldots \vee x_k$ and $\bigwedge M := x_1 \wedge \ldots \wedge x_k$ for $M = \{x_1, \ldots, x_k\} \subseteq \mathcal{L}$. Furthermore define $\bigvee \emptyset$ and $\bigwedge \emptyset$ as respectively the minimal and maximal element of $\mathcal{L}$.

An element is called *meet-reducible* if it can be written as the meet of elements all different from itself and is called *meet-irreducible* otherwise. For $\mathcal{L} = (X, \leq)$ and $x, y \in X$, one says that y *covers* x and writes $x \prec y$ if and only if $x < y$ and there is no $z \in X$ such that $x < z < y$. The *Hasse diagram* of $\mathcal{L}$ is then the directed graph on the elements of X with an arc (x, y) if $x \prec y$. The classical convention is to represent a Hasse diagram as undirected graph but with a drawing in the plane such that the orientation of edges can be recovered by directing them in upward direction. It is easy to see that an element x is a meet-irreducible if and only if there is exactly one edge in the Hasse diagram leaving x in upward direction. (Note that the maximum of $\mathcal{L}$ is indeed meet-reducible since it can be written $\bigwedge \emptyset$.)

A lattice is called *upper locally distributive* or ULD if each of its elements admits a unique minimal representation as the meet of meet irreducibles. In other words, for every $x \in \mathcal{L}$ there is a unique inclusion-minimal set $\{m_1, \ldots, m_k\} \subseteq \mathcal{L}$ of meet-irreducibles such that $x = m_1 \wedge \ldots \wedge m_k$.

ULDs were first defined by Dilworth [13] and have thereafter often reappeared, see [31] for an overview until the mid 80s. In particular, the Hasse diagram of a ULD is a partial cube, see e.g. [20]. The following theorem sheds light on the special role played by ULDs among partial cubes with respect to convexity.

Theorem 4. *A graph G is a partial cube if and only if for every vertex v the inclusion order of convex subgraphs containing v forms a ULD whose Hasse diagram contains G as an isometric subgraph.*

3 NP-completeness of hull number in partial cubes

The section is devoted to the proof of the following result:

Theorem 5. *Given a partial cube G and an integer k it is NP-complete to decide whether* $\mathrm{hn}(G) \leq k$.

Proof. Observe first that by Corollary 3, computing the convex hull of a set of vertices in a partial cube is doable in polynomial-time. It is also doable in polynomial-time in general graphs, see e.g. [15]. To prove the NP-completeness, we exhibit a reduction from the following problem, known to be NP-complete [23]:

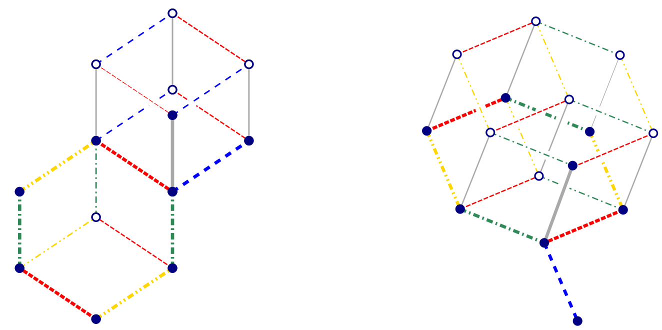


Fig. 1. Two ULDs obtained from the same partial cube (thick edges) by fixing a different vertex.

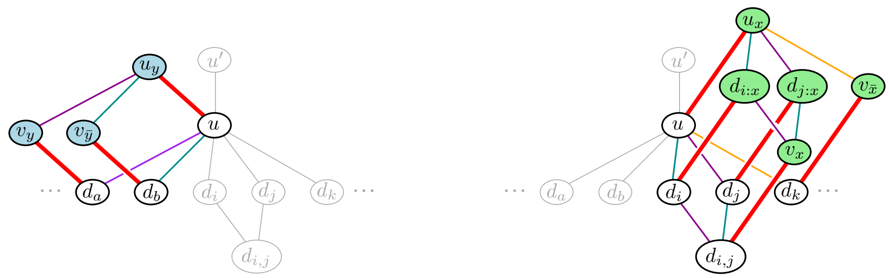
SAT-AM3:
Instance: A formula F in Conjunctive Normal Form on m clauses $D_1, \ldots, D_m$, each consisting of at most three literals on variables $x_1, \ldots, x_n$. Each variable *appears in at most three clauses.*
Question: Is F satisfiable?

Given an instance F of SAT-AM3, we construct a partial cube G_F such that F is satisfiable if and only if $\mathrm{hn}(G_F) \leq n + 1$.

Given F we start with two vertices u and u' connected by an edge. For each $1 \leq i \leq m$, introduce a vertex d_i and link it to u. If two clauses, say D_i and D_j, share a literal, add a new vertex $d_{i,j}$ and connect it to both d_i and d_j.

Now for each variable x, introduce a copy G_x of the subgraph induced by u and the vertices corresponding to clauses that contain x (including vertices of the form $d_{\{i,j\}}$ in case x appears in the same literal in D_i and D_j). Connect G_x to the rest of the graph by introducing a matching M_x connecting each original vertex with its copy in G_x and call G_F the graph obtained. Assume without loss of generality that each Boolean variable x used in F appears at least once non-negated (denoted by x with a slight abuse of notations) and once negated (denoted by $\bar{x}$). Then, each literal appears at most twice in F and the two possible options for G_x are displayed on Fig.2. Label the vertices of G_x according to that figure.

Observe first that G_F is a partial cube. Define a cut partition of G_F into $n+m+1$ cuts as follows. One cut consists of the edge (u, u'). The cut associated to a clause D_i contains the edge $\{u, d_i\}$, any edge of the form $\{d_{\{i,j\}}, d_j\}$ and all the copies of such edges that belong to one of the G_x. Let us call this cut C_i. Finally, the cut associated to a variable x is equal to M_x. This cut partition satisfies Theorem 2(i). Indeed, for a cut C denote respectively $\partial^+ C$ and $\partial^- C$ the vertices in C^+ and C^- incident to edges of C. Theorem 2(i) is in fact equivalent to say that, for each cut $C \in \mathcal{C}$, between any pair of vertices of $\partial^+ C$ or $\partial^- C$, there exists a shortest path that contains no edge of C. A case by case analysis of the different cuts in G_F concludes the proof.



(a) The variable y appears twice in F,   (b) the variable x three times.

Fig. 2. General structure of the graph G_F, with the two possible examples of gadgets associated to a variable. Red edges correspond to the cut M_y on (a) and M_x on (b).

Assume F is satisfiable and let S be a satisfying assignment of variables. Let H be the union of $\{u'\}$ and the subset of vertices of G_F corresponding to S. More formally, for each variable x, H contains the vertex v_x if x is set to true in S or the vertex $v_{\bar{x}}$ otherwise. Let us prove that H is a hull set. Since u belongs to any path between u' and any other vertex, u belongs to $\text{conv}(H)$. Moreover, for each variable x, the vertex u_x lies on a shortest path both between v_x and u' and between $v_{\bar{x}}$ and u', hence all the vertices u_x belong to $\text{conv}(H)$. Next, for each literal ℓ and for each clause D_i that contains ℓ, there exists a shortest path between u' and v_ℓ that contains d_i. Then, since S is a satisfying assignment of F, each clause vertex belongs to $\text{conv}(H)$. It follows that $\text{conv}(H)$ also contains all vertices $d_{i,j}$.

To conclude, it is now enough to prove that for all $\ell \notin S$, the vertex v_ℓ also belongs to $\text{conv}(H)$. In the case where ℓ appears in only one clause D_i, then v_ℓ belongs to a shortest path between d_i and u_ℓ. In the other case, v_ℓ belongs to a shortest $(u_\ell, d_{i,j})$-path. Thus, $\text{conv}(H) = G$.

Assume now that there exists a hull set H, with $|H| \leq n+1$. By Corollary 3, the set H necessarily contains u' and at least one vertex of G_x for each variable x. This implies that $|H| = n+1$ and therefore for all variables x, H contains exactly one vertex h_x in G_x. Since any vertex of G_x lies either on a shortest (u', v_x)-path or $(u', v_{\bar{x}})$-path, we can assume without loss of generality that h_x is either equal to v_x or to $v_{\bar{x}}$. Hence, H defines a truth assignment S for F. Now let C_i be the cut associated to the clause D_i and let C_i^+ be the side of C_i that contains d_i. Observe that if v_x belongs to C_i^+, then x appears in D_i. By Corollary 3, H intersects C_i^+, hence there exists a literal ℓ such that v_ℓ belongs to H. Thus, H encodes a satisfying truth-assignment of F.

□

The gadget in the proof of Theorem 5 is a relatively special partial cube and the statement can thus be strengthened. For a polyhedron P and a set $\mathcal{H}$ of hyperplanes in $\mathbb{R}^d$, the *region graph* of $P \setminus \mathcal{H}$ is the graph whose vertices are the connected components of $P \setminus \mathcal{H}$ and where two vertices are joined by an edge if their respective components are separated by exactly one hyperplane of $\mathcal{H}$. The proof of Theorem 5 can be adapted to obtain:

Corollary 6. *Let $P \subset \mathbb{R}^d$ be a polyhedron and $\mathcal{H}$ a set of hyperplanes. It is NP-complete to compute the hull number of the region graph of $P \setminus \mathcal{H}$.*

4 The hull number of a linear extension graph

Given a poset $(P, \leq_P)$, a *linear extension* L of P is a total order $\leq_L$ on the elements of P compatible with $\leq_P$, i.e., $x \leq_P y$ implies $x \leq_L y$. The set of vertices of the *linear extension graph* $G_L(P)$ of P is the set of all linear extensions of P and there is an edge between L and L' if and only if L and L' differ by a neighboring transposition, i.e., by reversing the order of two consecutive elements.

Let us see that property (i) of Theorem 2 holds for $G_L(P)$. Each incomparable pair $x \parallel y$ of $(P, \leq_P)$ corresponds to a cut of $G_L(P)$ consisting of the edges where x and y are reversed. The set of these cuts is clearly a cut-partition of $G_L(P)$. Observe then that the distance between two linear extensions L and L' in $G_L(P)$ is equal to the number of pairs that are ordered differently in L and L', i.e., no pair $x \parallel y$ is reversed twice on a shortest path. Hence $G_L(P)$ is a partial cube.

A *realizer* of a poset is a set S of linear extensions such that their intersection is P. In other words, for every incomparable pair $x \parallel y$ in P, there exist $L, L' \in S$ such that $x <_L y$ and $x >_{L'} y$. It is equivalent to say that, for each cut C of the cut-partition of $G_L(P)$, the sets $C^+ \cap S$ and $C^- \cap S$ are not empty. By Corollary 3, it yields a one-to-one correspondence between realizers of P and hull sets of $G_L(P)$. In particular the size of a minimum realizer – called the *dimension* of the poset and denoted $\dim(P)$ – is equal to the hull number of $G_L(P)$. The dimension is a fundamental parameter in poset combinatorics, see e.g. [32]. In particular, for every *fixed* $k \geq 3$, it is NP-complete to decide if a given poset has dimension at least k, see [34]. But if instead of the poset its linear extension graph is considered to be the input of the problem, then we get:

Proposition 7. *The hull number of a linear extension graph (of size n) can be determined in time $O(n^{c \log n})$, i.e., the dimension of a poset P can be computed in quasi-polynomial-time in $G_L(P)$.*

Proof. An antichain in a poset is a set of mutually incomparable elements of P and the width $\omega(P)$ of P is the size of the largest antichain of P, see [32]. It is a classic result that $\dim(P) \leq \omega(P)$. Since any permutation of an antichain appears in at least one linear extension, $\omega(P)! \leq n$ and therefore $\dim(P) \leq \log(n)$. Thus, to determine the hull-number of $G_L(P)$ it suffices to compute the convex hull of all subsets of at most $\log(n)$ vertices. Since the convex hull can be computed in polynomial-time, we get the claimed upper bound. □

In fact, since the number of linear extensions of a poset is generally exponential in the size of the poset, it seems reasonable to conjecture:

Conjecture 8. The dimension of a poset given its linear extension graph can be determined in polynomial-time.

5 Planar partial cubes and rank 3 oriented matroids

Oriented matroids have many equivalent definitions. We refer to [4] for a thorough introduction to oriented matroids and their plenty applications. Here, we will not state a formal definition. It suffices to know that the topes of an oriented matroid $\mathcal{M}$ on n elements are a subset of $\{1,-1\}^n$ satisfying several axioms. Moreover, the topes determine $\mathcal{M}$ uniquely. Joining two topes by an edge if they differ by the sign of exactly one entry yields the *tope graph* of $\mathcal{M}$.

From the axioms of oriented matroids it follows that the tope graph G of an oriented matroid is an *antipodal partial cube*, i.e., G is a partial cube such that for every $u \in G$ there is a $\overline{u} \in G$ with $\mathrm{conv}(u, \overline{u}) = G$, see [4]. In particular we have $hn(G) = 2$. But, not all antipodal partial cubes can be represented as the tope graphs of oriented matroids, see [26] and finding a general graph theoretical characterization is still a big problem in oriented matroid theory. The exception is for tope graphs of oriented matroids of rank at most 3 which admit a characterization as planar antipodal partial cubes, see [22]. We need a few definitions to state this characterization precisely.

A *Jordan curve* is a simple closed curve in the plane. For an arrangement $\mathcal{S}$ of Jordan curves and $S \in \mathcal{S}$, $\mathbb{R}^2 \setminus S$, the complement of S has two components: one is bounded and is called its *interior*, the other one, unbounded, is called its *exterior*. The closure of the interior of the exterior of S are denoted respectively S^+ and S^-. The *region graph* of an arrangement $\mathcal{S}$ of Jordan curves is the graph whose vertices are the connected components of the complement of $\mathcal{S}$ in the plane and where two vertices are neighbors if their corresponding components are separated by exactly one element of $\mathcal{S}$. Using the Topological Representation Theorem for Oriented Matroids [4] the characterization of tope graphs of oriented matroids of rank at most 3 may be rephrased as:

Theorem 9 ([22]). *A graph G is an antipodal planar partial cube if and only if G is the region graph of an arrangement $\mathcal{S}$ of Jordan curves such that for every $S, S' \in \mathcal{S}$ we have $|S \cap S'| = 2$ and for $S, S', S'' \in \mathcal{S}$ either $|S \cap S' \cap S''| = 2$ or $|S^+ \cap S' \cap S''| = |S^- \cap S' \cap S''| = 1$.*

Given a Jordan curve S and a point $p \in \mathbb{R}^2 \setminus S$ denote by $S(p)$ the closure of the side of S not containing p. An arrangement $\mathcal{S}$ of Jordan curves is called *non-separating*, if for any region $p \in \mathbb{R}^2 \setminus \mathcal{S}$ and subset $\mathcal{S}' \subseteq \mathcal{S}$ the set $\mathbb{R}^2 \setminus \bigcup_{S \in \mathcal{S}'} S(p)$ is connected. Two important properties of non-separating arrangements are summarized in the following:

Observation 10. Let $\mathcal{S}$ a non-separating arrangement. Then the interiors of $\mathcal{S}$ form a family of *pseudo-discs*, i.e., different curves $S, S' \in \mathcal{S}$ intersect in at most two points. Moreover, $\mathcal{S}$ has the *topological Helly property*, i.e., if the interiors of $S_1, S_2, S_3 \in \mathcal{S}$ mutually intersect, then $S_1^+ \cap S_2^+ \cap S_3^+ \neq \emptyset$.

In Fig. 3 we show how violating the pseudo-disc or the topological Helly property violates the property of being non-separating.

Non-separating arrangements of Jordan curves yield a generalization of Theorem 9. The construction of the proof is exemplified in Fig.4.

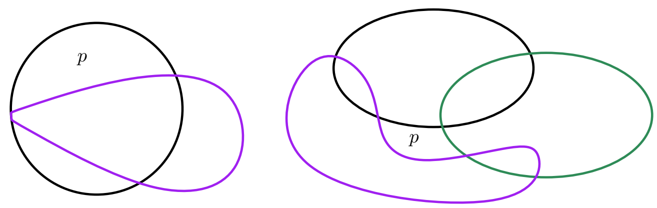


Fig. 3. Illustration of Observation 10. Left: Two curves intersecting in more than 2 points. Right: Three interiors of curves intersecting mutually but not having a common point. In both cases p is a point proving that the arrangement is not non-separating.

Theorem 11. *A graph G is a planar partial cube if and only if G is the region graph of a non-separating arrangement $\mathcal{S}$ of Jordan curves.*

Proof. Let G be a planar partial cube with cut-partition $\mathcal{C}$. We consider G with a fixed embedding and denote by G^* the planar dual. By planar duality each cut $C \in \mathcal{C}$ yields a simple cycle S_C in G^*. The set of these cycles, seen as Jordan curves defines $\mathcal{S}$. Since $(G^*)^* = G$ the region graph of $\mathcal{S}$ is isomorphic to G. Note that picking $p \in \mathbb{R}^2 \setminus S$ and looking at all the $S(p)$ is a special choice of $\sigma \in \{+1,-1\}^{\mathcal{S}}$ and looking at all $S^{\sigma(S)}$. But for every $\mathcal{S}' \subseteq \mathcal{S}$ and $\sigma \in \{+1,-1\}^{\mathcal{S}'}$ the set $\mathbb{R}^2 \setminus \bigcup_{S\in\mathcal{S}'} S^{\sigma(S)}$ hosts a convex subgraph of G namely $\bigcap_{S\in\mathcal{S}'} C_S^{-\sigma(S)}$. In particular, the region graph of $\mathcal{S}$ induced on $\mathbb{R}^2 \setminus \bigcup_{S\in\mathcal{S}'} S^{\sigma(S)}$ is connected and therefore $\mathbb{R}^2 \setminus \bigcup_{S\in\mathcal{S}'} S^{\sigma(S)}$ is connected.

Conversely, let $\mathcal{S}$ be a non-separating set of Jordan curves and suppose its region graph G is not a partial cube. In particular the cut-partition $\mathcal{C}$ of G arising by dualizing $\mathcal{S}$ does not satisfy Theorem 2 (i). That means there are regions R, T such that every curve S contributing to the boundary of R contains R and T on the same side, i.e., for any $p \in R \cup T$ and such S we have $R, T \subseteq S(p)$. Let $\mathcal{S}'$ be the union of these curves. The union $\bigcup_{S\in\mathcal{S}'} S(p)$ separates R and T, i.e., $\mathbb{R}^2 \setminus \bigcup_{S\in\mathcal{S}'} S(p)$ is not connected. $\square$

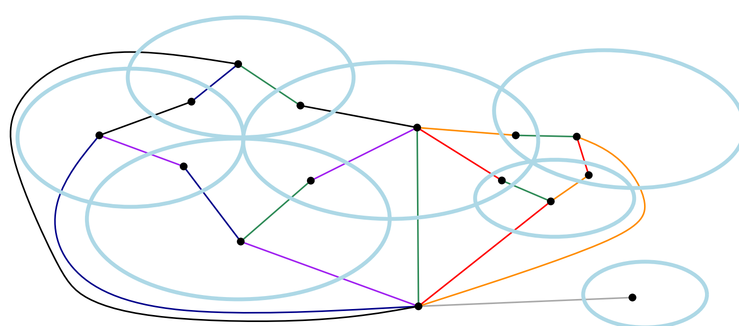


Fig. 4. A (non-simple) non-separating set of Jordan curves and its region graph.

A set of Jordan curves is *simple* if no point of the plane is contained in more than two curves. In the following, we always assume that a set $\mathcal{S}$ of Jordan curves is encoded by a planar embedding of its 1-skeleton.

Lemma 12. *A minimal hitting set for open interiors of a non-separating simple set $\mathcal{S}$ of Jordan curves can be computed in polynomial-time.*

Proof. We first prove that if all *closed* interiors of a given subset $\mathcal{S}' \subseteq \mathcal{S}$ intersect pairwise, then they have a non-empty common intersection. By Observation 10, the interiors of every triple $S_1, S_2, S_3 \in \mathcal{S}'$ have a common point. We can apply the classical *Topological Helly Theorem* [27], i.e., for any family of pseudo-discs in which every triple has a point in common, all members have a point in common.

Now, since $\mathcal{S}$ is simple the intersection of mutually intersecting interiors actually has to contain a region and not only a point.

Next, the intersection graph of open interiors of $\mathcal{S}$ is chordal: assume that there is a chordless cycle witnessed by $S_1^+, \ldots, S_k^+$ for $k \geq 4$. Since $\mathcal{S}$ is non-separating $S_1^+ \cap \ldots \cap S_k^+$ must be non-empty. But, if there is no edge between two vertices of the intersection graph, the corresponding open interiors must be disjoint. The set $S_1^+ \cap \ldots \cap S_k^+$ is therefore reduced to a single point, contradicting simplicity.

Chordal graphs form a subset of perfect graphs and hence by [25] their minimum clique-cover number – that is the least integer k for which the graph admits a partition of its vertices into k cliques – can be computed in polynomial time. Given a clique cover of the intersection graph of $\mathcal{S}$, cliques can be assumed to be maximal. Since the intersection of several interiors actually has to contain a region, each maximal clique corresponds to one region of the region graph. Picking one point in the interior of each of those regions yields a hitting set for open interiors. $\square$

Theorem 13. *A minimal hitting-set for open interiors and exteriors of a non-separating simple set $\mathcal{S}$ of Jordan curves can be computed in polynomial-time.*

Proof. Viewing $\mathcal{S}$ now as embedded on the sphere, any choice of a region v as the unbounded region yields a different arrangement $\mathcal{S}_v$ of Jordan curves in the plane. Denote the size of a minimum hitting set of the interiors of $\mathcal{S}_v$ by h_v.

Let us now prove that there exist some regions u, v such that $h_u < h_v$ if and only if there is a hitting set of size h_v of exteriors and interiors of $\mathcal{S}$. Let $h_u < h_v$ for some regions u, v. Extending the hitting set witnessing h_u by the unbounded region in $\mathcal{S}_u$ yields a hitting set of exteriors and interiors of size at most h_v. Conversely let H be a hitting set of size h_v of exteriors and interiors of $\mathcal{S}$ and let $u \in H$. Now, because all sides hit by u now are unbounded, $H \setminus u$ hits all bounded sides of $\mathcal{S}_u$ and therefore $h_u < h_v$.

It follows that a minimum hitting set of exteriors and interiors of $\mathcal{S}$ is of size $min_{v \in V} h_v + 1$. Since by Lemma 12 every h_v can be computed in polynomial-time and $|V|$ is linear in the size of the input, we are done. $\square$

Combining Corollary 3 and Theorems 11 and 13, we get:

Corollary 14. *The hull number of a plane quadrangulation that is a partial cube can be determined in polynomial-time.*

Notice that in [21], it was shown that the hitting set problem restricted to open interiors of (simple) sets of unit squares in the plane remains NP-complete and that the gadget used in that proof is indeed not non-separating.

We conclude this paper with a conjecture. Combined with Theorem 13, it would give a polynomial-time algorithm for the hull number of planar partial cubes.

Conjecture 15. A minimum hitting set for open interiors of a non-separating set of Jordan curves can be found in polynomial-time.

Acknowledgments. The authors thank Stefan Felsner, Matjaž Kovše, and Bartosz Walczak for fruitful discussions. M.A. would also like to thank Stefan Felsner for his invitation in the Discrete Maths group at the Technical University of Berlin, where this work was initiated. M.A. acknowledges the support of the ERC under the agreement "ERC StG 208471 - ExploreMap" and of the ANR under the agreement "ANR 12-JS02-001-01". K.K. was supported by TEOMATRO (ANR-10-BLAN 0207) and DFG grant FE-340/8-1 as part of ESF project GraDR EUROGIGA.

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