ON SYMMETRIC QUADRANGULATIONS AND TRIANGULATIONS

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Abstract. This article presents new enumerative results related to symmetric planar maps. In the first part a new way of enumerating rooted simple quadrangulations and rooted simple triangulations is presented, based on the description of two different quotient operations on symmetric simple quadrangulations and triangulations. In the second part, based on results of Bouttier, Di Francesco and Guitter and on quotient and substitution operations, the series of three families of symmetric quadrangular and triangular dissections of polygons are computed, with control on the distance from the central vertex to the outer boundary.

Introduction

Enumeration of families of plane maps, that is, plane embeddings of graphs, has received a lot of attention since the 60's; several methods can be applied: the recursive method introduced by Tutte [15], the random matrix method introduced by Brézin et al [4], and the bijective method introduced by Cori and Vauquelin [7] and Schaeffer [12]. In the first part of this note, we show another method for the enumeration of rooted simple quadrangulations and triangulations based on quotienting symmetric simple versions of them. Historically, the enumeration of symmetric maps of order k (*i.e.*, such that a rotation of order k fixes the map) was reduced to the enumeration of rooted maps via a quotient argument, a method used by Liskovets [10]. We proceed in the reverse way, namely we use two quotient operations on symmetric simple quadrangulations and triangulations to build in each case an algebraico-differential equation (Equations (3) and (8)) satisfied by the generating series of rooted corresponding simple maps, which can be explicitly solved to obtain the formulas for the number of rooted simple quadrangulations (due to Tutte [15] and bijectively proved by Schaeffer [12]) and of rooted simple triangulations (due to Tutte [14] and bijectively proved by Poulalhon and Schaeffer [11]). One quotient operation is classical and is decribed in Section 1; the other quotient operation is new and, as described in Section 2, relies deeply on the existence and properties of α-orientations; the new equations for generating series of simple quadrangulations and triangulations are derived and solved in Section 3.

The results in the second part are expressions of the series of several families of symmetric quadrangular and triangular dissections with control on the distance from the central vertex to the outer boundary. We recall that symmetric dissections have been counted according to the number of inner faces by Brown [5, 6] using the recursive method (Liskovet's quotient method [10] can also be applied, reducing the enumeration to rooted quadrangular dissections). Our approach, developed in Sections 4 and 5, relies on the quotient method and substitution operations combined with results by Bouttier *et al.* [1, 3], which express the series of quadrangulations or triangulations with a marked vertex and marked edge at prescribed distance from each other. Our expressions illustrate again the property that the

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series expression of a "well behaved" map family $\mathcal{M}$ refined by a distance parameter d is typically expressed in terms of the dth power of an algebraic series of singularity type $z^{1/4}$ (implying that asymptotically the distance parameter d on a random map of size n in $\mathcal{M}$ converges in the scale $n^{1/4}$ as a random variable).

1. Plane maps, symmetry and classical quotient

1.1. **Triangulations, quadrangulations and dissections.** A *plane map* is a connected graph embedded in the plane up to continuous deformation; the unique unbounded face of a plane map is called the *outer face*, the other ones are called *inner faces*. Vertices and edges are also called outer if they belong to the outer face and inner otherwise. A map is said to be *rooted* if an edge of the outer face is marked and oriented so as to have the outer face on its left. This edge is the *root edge*, and its origin is the *root vertex*. A map is *pointed* if one of its *inner* vertices is marked. For any map M, we denote by $\mathcal{V}(M)$, $\mathcal{F}(M)$, $\mathcal{E}(M)$ its sets of vertices, faces and edges, and by $v(M)$, $f(M)$, $e(M)$ their cardinalities.

Triangulations and quadrangulations are respectively maps with all faces of degree 3 or 4, and to avoid the degenerated cases, the outer face is required to be a simple cycle. For $k \geqslant 1$ and $d \geqslant 3$, a *d-angular dissection of a k-gon* or *d-angular k-dissection* is a map whose outer face contour is a simple cycle of length k, and with all inner faces of one and the same degree d. A dissection is said to be *triangular* if d equals 3, *quadrangular* if d equals 4. Observe that quadrangular k-dissections can only exist for even k.

A map is said to be *simple* if it has no multiple edges; a d-angular k-dissection is called *irreducible* if the interior of every cycle of length at most d is a face.

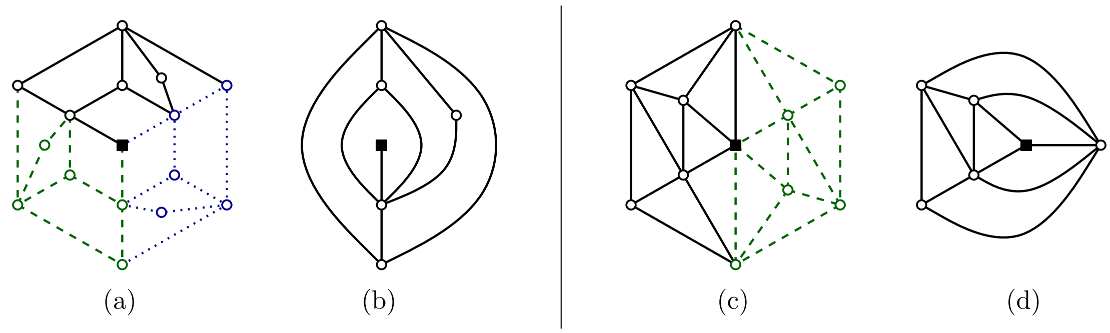
1.2. **Symmetric maps and classical quotient.** For $k \geqslant 2$, a dissection D is said to be *k-symmetric* if its plane embedding (conveniently deformed) is invariant by a $2\pi/k$-rotation centered at a vertex – called the *center* of D.

As observed by Liskovets [10], any two semi-infinite straight lines starting from the center and forming an angle of $2\pi/k$ delimit a sector of D. When keeping only this sector and pasting these two lines together, we obtain a plane map, called the *k-quotient map* of D. In other words, the $2\pi/k$-rotation defines equivalence relations on the sets $\mathcal{V}(D)$ and $\mathcal{E}(D)$, and the quotient map of D is the map in which equivalent vertices and equivalent edges are identified. Figure 1 shows the example of two symmetric dissections of an hexagon and their quotients. Denote by $o(D)$ the degree of the outer face of a dissection. The following lemma is straightforward:

Lemma 1.1. *For $k \geqslant 2$, let D be a k-symmetric dissection, and E its k-quotient; we have:*

$$v(D) - 1 = k\,(v(E) - 1), \quad e(D) = k\,e(E), \quad f(D) - 1 = k\,(f(E) - 1), \quad o(D) = k\,o(E).$$

The quotient operation clearly preserves the degrees of vertices and faces, hence quotients of d-angular dissections are d-angular dissections. More precisely, as any k-symmetric dissection is implicitly pointed (at the center), its k-quotient is a pointed dissection. We define the *radial distance* $r(D)$ of a pointed dissection D as the distance between its marked vertex and the outer face boundary (for instance, the example of Figure 1(b) has radial distance 2 while that of Figure 1(d) has radial distance 1). We also $\ell(D)$ the length of a shortest cycle strictly enclosing u. The following lemma summarizes the relations between distances in k-symmetric dissections and their k-quotients:



(a) (b) (c) (d)

FIGURE 1. Examples of symmetric 6-dissections. A 3-symmetric quadrangular dissection (a). Its 3-quotient (b) is a pointed quadrangular 2-dissection. A 2-symmetric triangular dissection (c). Its 2-quotient (d) is a pointed triangulation.

Lemma 1.2. *For $k \geqslant 2$, let D be a k-symmetric dissection, and E its k-quotient; we have:*

$$r(D) = r(E) \quad \textit{and} \quad \ell(D) = k\,\ell(E).$$

Proof. Let P be a pointed dissection, with pointed vertex u. A labelling-function is a function $\lambda : \mathcal{V}(P) \to \mathbb{Z}$ such that $\lambda(u) = 0$ and for all adjacent vertices v and v', $|\lambda(v) - \lambda(v')| \leqslant 1$. It is easy to check that the function δ_P giving the distance from u, called distance-labelling, is the unique labelling-function such that each vertex $v \neq u$ has a neighbour of smaller label. With this characterization, it is straightforward that δ_E is the quotient of δ_D; in particular, the radial distance is the same in D as in E.

The second statement is equivalent to the following one: one among the cycles of minimal length stricly enclosing the center is itself symmetric. Indeed, let C be a cycle of minimal length $\ell(D)$ strictly enclosing the center that is minimal for inclusion, and suppose that C is asymmetric. Let $\tilde{C}$ be its image by the π/k-rotation; as they both enclose the center, C and $\tilde{C}$ are intersecting cycles and hence define two other cycles enclosing the center with total length $2\ell(D)$, hence each of length $\ell(D)$; one of these two cycles is included in C and $\tilde{C}$, contradiction. □

1.3. **Symmetric simple quadrangulations and triangulations.** A pointed dissection is called *quasi-simple* if the pointed vertex lies strictly in the interior of every 1-cycle or 2-cycle; in particular each vertex can carry at most one loop, otherwise the two loops would form a 2-cycle that does not contain the pointed vertex. Unlike that of simple dissections, the family of quasi-simple dissections is stable under quotient:

Lemma 1.3. *For $k \geqslant 2$, let D be a k-symmetric dissection and E its k-quotient – both canonically pointed. Then D is quasi-simple if and only if E is quasi-simple.*

Proof. This just follows from the fact that any 2-cycle of D not enclosing the pointed vertex yields a 2-cycle of E not enclosing the pointed vertex, and reciprocally. □

Note also that, according to Lemma 1.2, no cycle enclosing the center of a k-symmetric dissection D can be of length smaller than k. In particular, D has no loop, and no 2-cycle enclosing the center if $k > 2$. Hence if $k > 2$, k-symmetric quasi-simple dissections are indeed simple.

Symmetric simple quadrangulations. Lemma 1.1 implies that quadrangulations, having outer degree 4, may only be 2- or 4-symmetric. Moreover, their quotients are quadrangular dissections, hence they are bipartite, which prevents them of containing loops, in particular the outer face has length at least 2. Hence quadrangulations may only be 2-symmetric (which we simply call *symmetric*), and quotients of symmetric quadrangulations are pointed quadrangular 2-dissections. This implies also that quasi-simple symmetric quadrangulations are indeed simple. Hence the following proposition is a direct consequence of Lemma 1.3:

Proposition 1.4. *The 2-quotient is a one-to-one correspondence between symmetric simple quadrangulations with* $2n$ *inner faces and quasi-simple pointed 2-dissections with* n *inner faces.*

Symmetric simple triangulations. Similarly, triangulations may only be 3-symmetric (which we simply call *symmetric*), and the 3-quotient of a symmetric triangulation is a pointed triangular 1-dissection. Note that a quasi-simple symmetric triangulation is simple (indeed, by Lemma 1.2, a 3-symmetric dissection D satisfies $\ell(D) \geqslant 3$). Hence:

Proposition 1.5. *The 3-quotient is a one-to-one correspondence between symmetric simple triangulations with* $3n$ *inner faces and quasi-simple pointed triangular 1-dissections with* n *triangular faces, where* n *is any positive odd integer.*

2. Another quotient for symmetric simple quadrangulations and triangulations

In this section, we only consider simple quadrangulations and triangulations, and show that a specific quotient can be defined for these families, using their characterization in terms of d-orientations.

2.1. The minimal 2- or 3-orientation.

An *orientation* of a plane map is the choice of an orientation for each of its *inner* edges. An orientation without counterclockwise cycle is said to be *minimal*. A d-orientation of a map is an orientation such that the outdegree of each inner vertex is equal to d, while outer vertices have outdegree equal to zero.

An important property of simple quadrangulations and triangulations is that they can be characterized in terms of d-orientations:

Proposition 2.1 ([13],[8],[9])**.** *A quadrangulation is simple if and only if it admits a 2-orientation. A triangulation is simple if and only if it admits a 3-orientation. In both cases, there exists a unique minimal such orientation.*

Figure 2(a) and 3(a) show two examples of a simple quadrangulation and a simple triangulation respectively endowed with their minimal 2- or 3-orientation.

Now let M be a k-symmetric map endowed with a d-orientation O; then the image of O by the $2\pi/k$-rotation is clearly a d-orientation, and it is minimal if O is minimal. Hence:

Lemma 2.2. *The unique minimal 2-orientation (*resp. *3-orientation) of a symmetric simple quadrangulation (*resp. *triangulation) is itself symmetric.*

2.2. Leftmost paths.

Let M be a plane map endowed with an orientation. For each inner vertex u of M, a *leftmost path starting at* u is a maximal oriented path P starting at u such that for any triple v, v', v'' of successive vertices along P, (v', v'') is the first outgoing edge after (v, v') in clockwise order around v'.

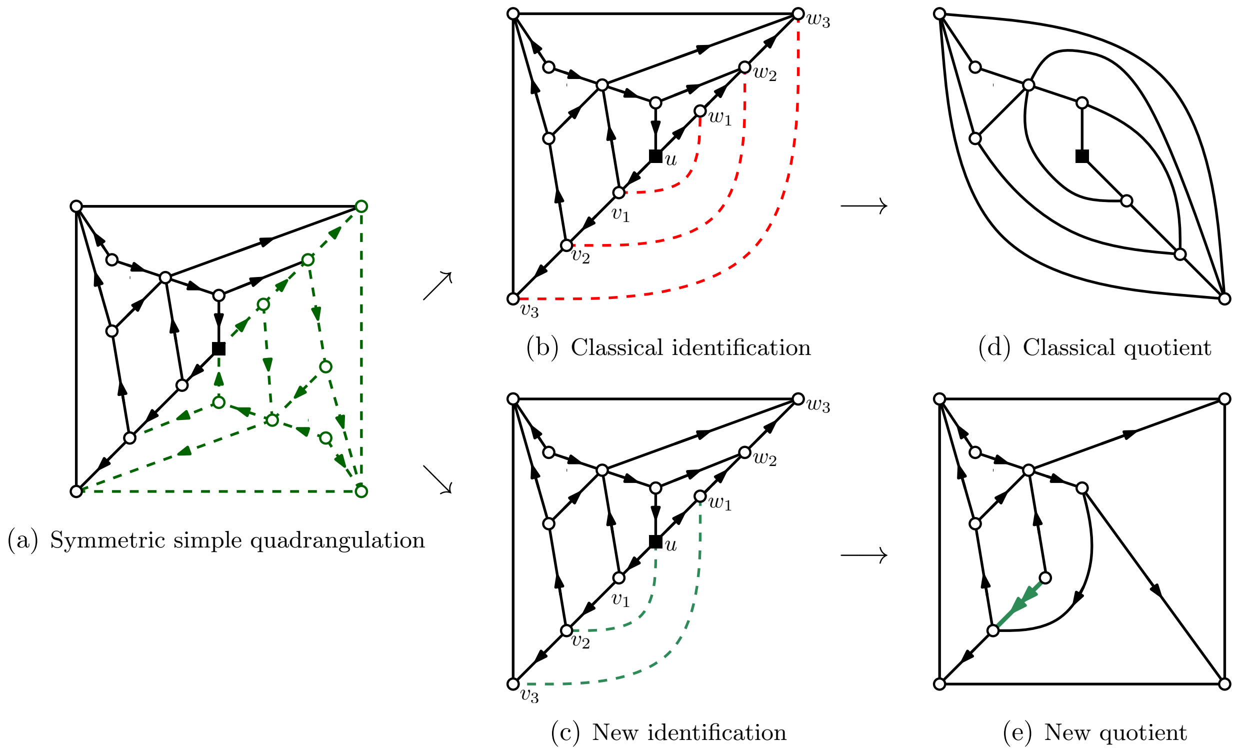


FIGURE 2. Classical 2-quotient (b), (d) and the new quotient (c), (e) of a symmetric simple quadrangulation endowed with its minimal 2-orientation (a).

Lemma 2.3. *If M is a simple quadrangulation or triangulation endowed with its minimal 2- or 3- orientation, P is necessarily a simple path ending at one outer vertex of M.*

Proof. Assume by contradiction that P is self intersecting and let $v_0, v_1, \ldots, v_l = v$ be a sequence of vertices of P explored successively in this order and forming a cycle. Since the orientation O of M is chosen to be minimal, this cycle is necessarily clockwise. Hence, since P is a left-most path, the outgoing edges of each vertex of the cycle lie in the interior region of the cycle (except eventually the other outgoing edges of v_0, if v_0 is the starting point of the path), hence denoting by R the enclosed region, $e(R)$ is at least $2v(R) - 1$ (*resp.* $3v(R) - 2$). Euler relation leads to a contradiction: if the length of the cycle is ℓ, Euler formula implies that $2v(R) = e(R) + 2 + \ell/2$ (*resp.* $3v(R) = e(R) + 2 + \ell$). Note that this is in accordance with the well-known property that inner edges of a simple triangulation (*resp.* quadrangulation) can be partitionned into three (*resp.* two) spanning trees (see [13]). □

In the next two subsections, we use leftmost paths starting at the center to decompose symmetric simple quadrangulations and triangulations in 2 (*resp.* 3) sectors, and show how this particular sector can be sticked together again in a non-classical way to obtain a simple quadrangulation or triangulation.

2.3. **A new way of quotienting symmetric simple quadrangulations.** Let Q be a symmetric simple quadrangulation, u the central vertex, e_1, e_2 its two outgoing edges, and $P_1 = (u = v_0, v_1, \ldots, v_p)$, $P_2 = (u = w_0, w_1, \ldots, w_p)$ the leftmost paths of e_1 and e_2 respectively. Clearly P_1 and P_2 map to one another by the π-rotation. Hence P_1 and P_2 cannot meet except at their starting point u, otherwise they would meet twice, which would imply the existence of an oriented cycle without outgoing edges, leading to the same contradiction as in Lemma 2.3.

Let us cut Q along $P_1 \cup P_2$ to split Q into two isomorphic dissections, see Figure 2(c), and define Q_1 as the one with clockwise contour $u, v_1, v_2, \ldots, w_1$. If Q_1 is a quadrangulation, we set $\Phi(Q) := Q_1$ and mark the edge (u, v_1). Otherwise, for any $i \leqslant p-2$, we identify in Q_1 vertices v_{i+2} with w_i, and merge corresponding edges; this defines the map $\Phi(Q)$, in which we then mark the edge (v_1, v_2).

Concerning orientations, the identification of v_{i+2} with w_i creates an orientation conflict only when merging (u, v_1) with (v_1, v_2). We choose to orient the merged edge from v_1 to v_2. With this convention, $\Phi(Q)$ is naturally endowed with its minimal 2-orientation and the leftmost path of the marked edge (v_1, v_2) is $(v_1, v_2, \ldots, v_p)$ (to justify that the 2-orientation of $\Phi(Q)$ thus obtained is minimal, one just has to observe that no oriented path goes from some vertex w_i to some vertex v_j, hence when doing the identifications of vertices no counterclockwise circuit is created). It is then easy to describe the inverse mapping and to obtain:

Theorem 2.4. *The mapping Φ is a one-to-one correspondence between symmetric simple quadrangulations with $2n$ inner faces and simple quadrangulations with n inner faces and a marked edge.*

2.4. **A new way of quotienting symmetric simple triangulations.** Let T be a simple symmetric triangulation endowed with its minimal 3-orientation. Let u be its central vertex, e_1, e_2 and e_3 its three outgoing edges, and $P_1 = (u = v_0, v_1, \ldots, v_p)$, $P_2 = (u = w_0, w_1, \ldots, w_p)$ and $P_3(u = x_0, x_1, \ldots, x_p)$ the leftmost paths of e_1, e_2 and e_3 respectively. Clearly P_1, P_2 and P_3 maps respectively onto P_2, P_3 and P_1 by the $2\pi/3$-rotation centered at u. Hence they cannot meet except at their starting point u. Let us cut T along $P_1 \cup P_2 \cup P_3$ to split T into three isomorphic dissections, see Figure 3(c), and define T_1 as the one with clockwise contour $u, v_1, v_2, \ldots, w_1$. If T_1 is a triangulation, we set $\Phi_{\triangle}(T) := T_1$ and mark the edge (u, v_1). Otherwise, for any $i \leqslant p-2$, we identify in T_1 vertices v_{i+2} with w_i, and merge corresponding edges; this defines the map $\Phi_{\triangle}(T)$, in which we then mark the edge (v_1, v_2).

Concerning orientations, the identification of v_{i+2} with w_i creates an orientation conflict only when merging (u, v_1) with (v_1, v_2). We choose to orient the merged edge from v_1 to v_2. With this convention, $\Phi(Q)$ is naturally endowed with its minimal 3-orientation and the leftmost path of the marked edge (v_1, v_2) is $(v_1, v_2, \ldots, v_p)$. It is then easy to describe the inverse mapping and to obtain:

Theorem 2.5. *The mapping $\Phi_{\triangle}$ is a one-to-one correspondence between symmetric simple triangulations with $3n$ inner faces and simple triangulations with n inner faces and one marked edge, for any positive odd integer n.*

3. Simple quadrangulations and triangulations via symmetric ones

3.1. **Getting a functional equation for simple quadrangulations.** Proposition 1.4 and Theorem 2.4 describe two bijections between symmetric simple quadrangulations and two other families that are hence also in one-to-one correspondence: quasi-simple pointed 2-dissections and simple quadrangulations with a marked edge. Because 2-dissections are bipartite, each pointed 2-dissection corresponds to two different rooted pointed 2-dissections.

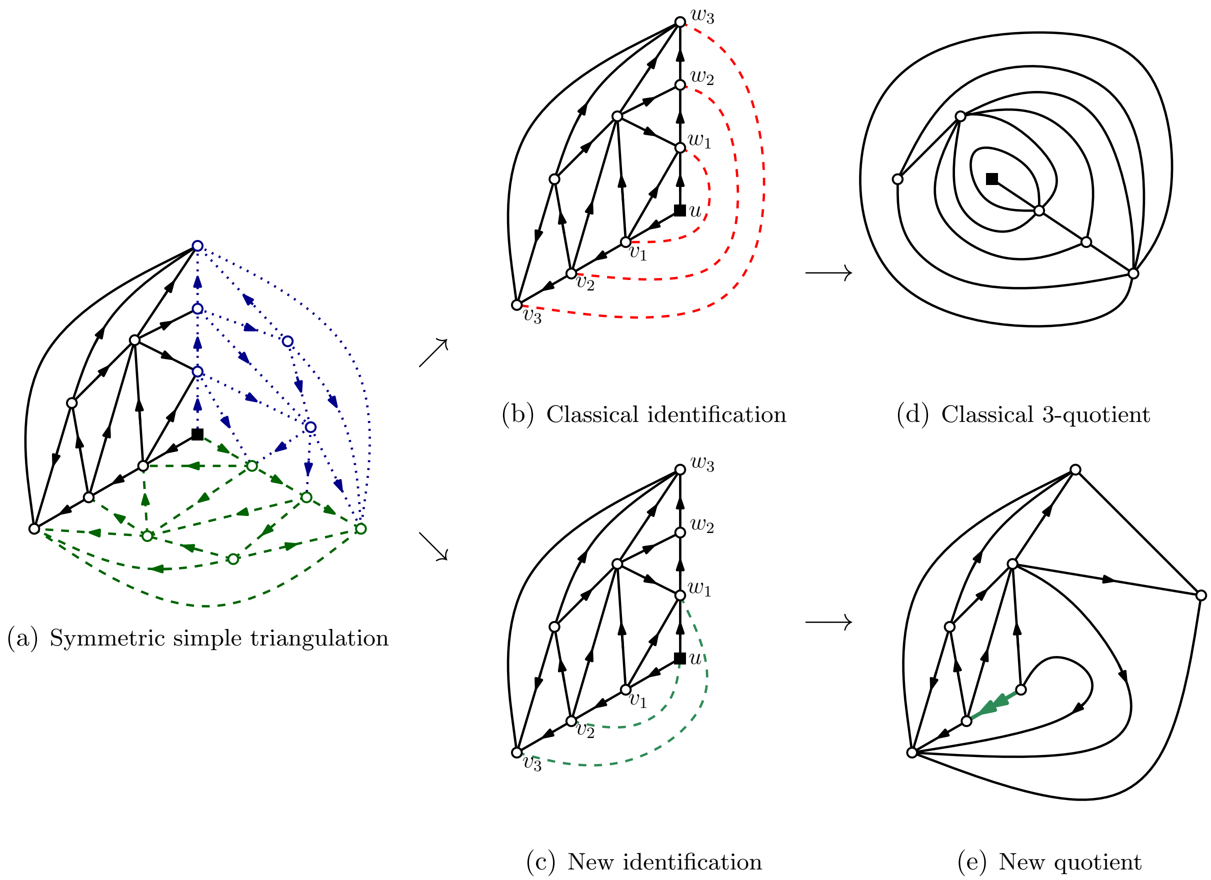


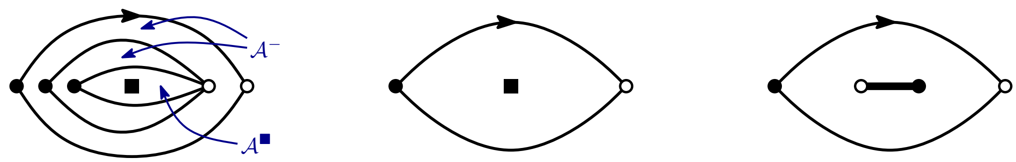
FIGURE 3. Classical 3-quotient (b), (d) and the new quotient (c), (e) of a symmetric simple triangulation endowed with its minimal 3-orientation (a).

Similarly, any edge of a simple quadrangulation has an implicit orientation given by its minimal 2-orientation, hence a simple quadrangulation with a marked edge corresponds to two distincts quadrangulations with a marked oriented edge, that can be seen as rooted simple quadrangulations with a marked face (possibly the outer one). Hence we obtain:

Corollary 3.1. *Rooted simple quadrangulations with n inner faces and a marked face are in one-to-one correspondence with rooted quasi-simple pointed 2-dissections with n inner faces.*

This correspondence allows us to get a functional equation for the generating series of the family $\mathcal{Q}$ of non-degenerated (meaning, with at least 2 faces) rooted simple quadrangulations. Let $q(x) = \sum_{n\geqslant 2} q_n x^n$ be the series of $\mathcal{Q}$ according to the number of faces (including the outer one). The generating series of rooted simple quadrangulations with a marked face is then equal to $q'(x)$. We want to express also the family $\mathcal{D}_4$ of rooted quasi-simple 2-dissections in terms of $\mathcal{Q}$.

Observe that for any $D \in \mathcal{D}_4$, its separating 2-cycles are nested and therefore ordered from innermost to outermost. This yields a decomposition of D_4 as a sequence of components. Let $\mathcal{A}^{\blacksquare}$ (*resp.* $\mathcal{A}^{-}$) be the family of rooted 2-dissections with a marked inner vertex (*resp.* with a marked inner edge) and where the unique 2-cycle is the outer face contour. We can cut D along the nested 2-cycles to obtain a pointed 2-dissection in $\mathcal{A}^{\blacksquare}$ and a sequence of maps in



(a) Nested 2-cycles (b) Element of $\mathcal{A}^{\blacksquare}$ (c) Element of $\mathcal{A}^-$

FIGURE 4. The decomposition of a quasi-simple quadrangular 2-dissection.

$\mathcal{A}^-$ (see Figure 4). Denoting respectively by $d(x)$, $a^{\blacksquare}(x)$, $a^-(x)$ the generating series of $\mathcal{D}$, $\mathcal{A}^{\blacksquare}$, and $\mathcal{A}^-$ according to the number of quadrangular faces, this gives:

$$d(x) = \frac{a^{\blacksquare}(x)}{1 - a^-(x)}. \tag{1}$$

Deleting the non-root outer edge of a rooted 2-dissection gives a rooted quadrangulation (possibly degenerated). Taking into account the marked inner vertex or edge, respectively chosen among the n inner vertices and $2n-1$ inner edges, we get:

$$\begin{cases} a^{\blacksquare}(x) \;=\; 2x + \sum_{n\geqslant 2} n q_n x^n \;=\; 2x + xq'(x) \\ a^-(x) \;=\; 2x + \sum_{n\geqslant 2}(2n-1) q_n x^n \;=\; 2x + 2xq'(x) - q(x) \end{cases} \tag{2}$$

Putting together Equations (1) and (2), Corollary 3.1 implies:

Proposition 3.2. *The series $q(x)$ satisfies the following equation:*

$$x \cdot [2q'(x)^2 + 3q'(x) + 2] = q'(x) \cdot [1 + q(x)]. \tag{3}$$

From this equation, written as $q' = x(2 + 2q'^2 + 3q')/(1+q)$, one readily extracts the development of $q(x)$ incrementally:

$$q(x) = x^2 + 2x^3 + 6x^4 + 22x^5 + 91x^6 + 408x^7 + 1938x^8 + \dots$$

As shown next, the exact expression of the coefficients (first obtained by Tutte from the recursive method [15] and subsequently recovered by Schaeffer [12] using a bijection with ternary trees) can also be recovered from (3):

Corollary 3.3. *For $n \geqslant 1$, the number of rooted simple quadrangulations with n faces is equal to:*

$$\frac{4(3n)!}{n!(2n+2)!}.$$

Equivalently, the series $q(x)$ is expressed as $q(x) = x[\alpha(x) - 2][1 - \alpha(x)]$, where $\alpha \equiv \alpha(x)$ is the series of rooted ternary trees, specified by $\alpha = 1 + x\alpha^3$.

Proof. Equation (3) above admits a unique power series solution that is equal to 0 at 0, hence it suffices to check that $f \equiv f(x) := x[\alpha(x) - 2][1 - \alpha(x)]$ is indeed solution of (3); note that x and $f(x)$ have rational expressions in terms of α, hence this also holds for $f'(x)$, since $f'(x) = \frac{\mathrm{d}f}{\mathrm{d}\alpha} / \frac{\mathrm{d}x}{\mathrm{d}\alpha}$. We have:

$$x = \frac{\alpha - 1}{\alpha^3}, \qquad f(x) = \frac{(\alpha-1)^2(2-\alpha)}{\alpha^3}, \quad \text{and} \quad f'(x) = 2\alpha - 2.$$

Plugging these expressions in Equation (3), the two handsides coincide, which concludes the proof.

As Alin Bostan showed us, Equation (3) can however be solved directly, without guessing the solution. Let $r \equiv r(x) = q'(x)$, Equation (3) rewrites:

$$1 + q = x \cdot \frac{2r^2 + 3r + 2}{r},$$

hence, taking the derivative:

$$r = \frac{2r^2 + 3r + 2}{r} + 2xr'\frac{r^2 - 1}{r^2}, \qquad \textit{i.e.} \qquad (r+1) \cdot [r(r+2) + 2xr'(r-1)] = 0, \tag{4}$$

or, assuming $r \neq 2$:

$$(r+1) \cdot \frac{r^2}{(r+2)^2} \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x(r+2)^3}{r}\right) = 0. \tag{5}$$

Assuming now that $r \neq 0$ and $r \neq 1$, we get $x(r(x)+2)^3 - cr(x) = 0$ for some constant c. On the other hand, Equation (4) implies that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(4q(x) - 2xr(x) + xr(x)^2\right) = 0, \qquad \text{hence} \qquad q(x) = \frac{1}{2}xr(x) + \frac{1}{4}xr(x)^2 + c'$$

for some constant c'; $q(0) = r(0) = 0$ implies that $c' = 0$. Thus the non affine solutions of Equation (3) have the following form:

$$c + x \cdot (\alpha_c(x) - 1)(2 - \alpha_c(x)) \quad \text{where} \quad x\alpha_c(x)^3 = c(\alpha_c(x) - 1).$$

Initial conditions for q imply that $c = 1$. □

3.2. **Getting a functional equation for simple triangulations.** The case of simple triangulations is very similar to the one described above for quadrangulations; only the decomposition is more complicated.

Proposition 1.5 and Theorem 2.5 describe bijections between symmetric simple triangulations and two families that are hence also in one-to-one correspondence:

Corollary 3.4. *Simple triangulations with $2n$ faces and one marked edge are in one-to-one correspondence with quasi-simple pointed 1-dissections with $2n$ faces.*

Let $\mathcal{T}$ be the family of rooted simple triangulations and let $t(x) = \sum_{n\geqslant 1} t_n x^n$ be the series of $\mathcal{T}$ according to half the number of faces, *i.e.* t_n is the number of rooted simple triangulations with $2n$ faces. We want to use Corollary 3.4 to derive a functional equation for $t(x)$, which requires to first express the family $\mathcal{S}$ of simple triangulations with a marked edge and the family $\mathcal{D}_3$ of quasi-simple pointed 1-dissections in terms of $\mathcal{T}$.

According to the Schnyder tree decomposition of simple triangulations, each edge is canonically associated to one of the three outer edges, hence simple triangulations with a marked edge are exactly rooted simple triangulations with a marked edge chosen among the n edges in the tree rooted at the root edge. Hence:

$$s(x) = \sum_{n\geqslant 1} n t_n x^n = xt'(x). \tag{6}$$

The enumeration of quasi-simple pointed triangulations can be carried out similarly to quasi-simple pointed quadrangulations, but extra care has to be taken in order to deal with

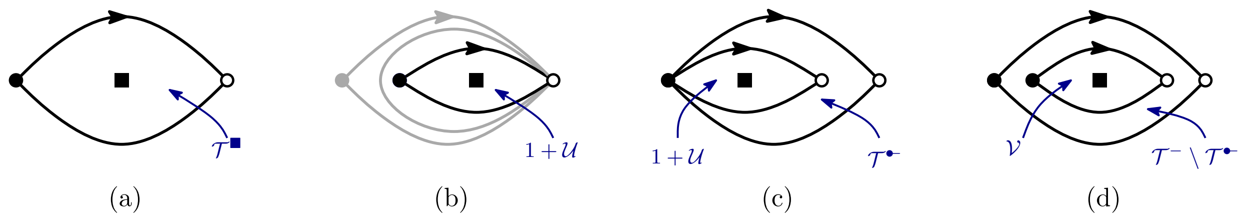


FIGURE 5. The four cases in the decomposition of an element of $\mathcal{U}$.

special conditions for both loops and 2-cycles. Observe that any 1-dissection D in $\mathcal{D}_3$ is implicitly rooted, and that removing the outer loop yields either a unique edge or a quasi-simple pointed triangular 2-dissection that can be canonically rooted with the same root vertex as D, with the additional constraint that there is no loop incident to the root vertex; let us call $\mathcal{U}$ this family and u its generating series according to half the number of triangular faces. Then the generating series d_3 of $\mathcal{D}_3$ is given by:

$$d_3(x) = x \cdot [1 + u(x)].$$

Note that removing the non-root edge incident to the outer face of a triangular 2-dissection produces a rooted triangulation with one face less. Hence families of rooted triangular 2-dissections (such as $\mathcal{U}$) and families of rooted triangulations can be naturally identified. To decompose elements of $\mathcal{U}$, let $\mathcal{V}$ be the family of quasi-simple triangular 2-dissections and $\mathcal{T}^{\blacksquare}$, $\mathcal{T}^{-}$ and $\mathcal{T}^{\bullet\!-}$ be respectively the families of rooted simple triangulations (or equivalently, triangular 2-dissections with neither loop nor 2-cycle except the outer face) with respectively a marked inner vertex, a marked inner edge, and a marked inner edge incident to the root vertex. Let us also denote v, $t^{\blacksquare}$, t^{-} and $t^{\bullet\!-}$ the corresponding generating series according to half the number of triangular faces.

Let us now proceed to the decomposition of an element of $\mathcal{U}$, four different configurations can appear (see Figure 5):

(a) either the outer face is the only 2-cycle;
(b) or the outer face is incident to a single face, the third side of which is a loop;
(c) or the outermost 2-cycle (other than the outer face) is incident to the root vertex;
(d) or the outermost 2-cycle (other than the outer face) is not incident to the root vertex.

Note that each inner edge of a simple triangulation, and in particular of any map T in $\mathcal{T}^{\bullet\!-}$ is naturally oriented according to the minimal 3-orientation of T. Hence there is a canonical way of plugging a rooted 2-dissection in the (rooted) 2-cycle obtained by *opening* the marked edge of a map in $\mathcal{T}^{-}$. Hence:

$$\mathcal{U} = \mathcal{T}^{\blacksquare} + \mathcal{X}\cdot(1+\mathcal{U}) + \mathcal{T}^{\bullet\!-}\cdot\mathcal{U} + (\mathcal{T}^{-}\setminus\mathcal{T}^{\bullet\!-})\cdot\mathcal{V},$$

where $\mathcal{X}$ is an atom representing a triangle. A similar decomposition can be written for $\mathcal{V}$, and putting things together and translating them into the language of generating series, we obtain:

$$\begin{cases} u = t^{\blacksquare} + x(1+u) + t^{\bullet\!-}u + (t^{-} - t^{\bullet\!-})v, \\ v = t^{\blacksquare} + 2x(1+u) + t^{-}v. \end{cases}$$

The series $t^{\blacksquare}$, t^{-} and $t^{\bullet\!-}$ can now be expressed in terms of t as follows. A triangulation with $2n$ faces has $3n$ edges and $n+2$ vertices, so removing the non-root outer edge and taking into account the marked inner vertex or edge, we get:

$$\begin{cases} t^{\blacksquare}(x) = \sum_{n\geqslant 1} n t_n x^n = x t'(x), \\ t^{-}(x) = \sum_{n\geqslant 1} (3n-1) t_n x^n = 3x t'(x) - t(x). \end{cases}$$

The computation of $t^{\bullet\!-}(x)$ is a little more complicated. Let T be a simple rooted triangulation with at least four faces. By merging the two ends of the root edge of T and collapsing the inner triangle incident to it into an edge, we obtain a 2-dissection that is canonically rooted with same root vertex as T, and with a marked edge incident to that vertex. This triangulation is not far from being simple: it has no loop and the only 2-cycles separate the two marked edges. Hence, decomposing this map along the sequence of nested cycles, we get a sequence of elements of $\mathcal{T}^{\bullet\!-}$, hence:

$$\frac{t(x)-x}{x} = \sum_{n\geqslant 2} t_n x^n = \frac{t^{\bullet\!-}(x)}{1-t^{\bullet\!-}(x)}, \quad \text{from which we obtain} \quad t^{\bullet\!-}(x) = \frac{t(x)-x}{t(x)}.$$

Combining all these equations and with the help of computer algebra, we obtain the following expression for the generating series of rooted quasi-simple triangular 1-dissections:

$$d_3(x) = \frac{x\cdot[1+t(x)-2xt'(x)]}{1-2x+2t(x)-3xt'(x)+t(x)^2-3xt'(x)t(x)}. \tag{7}$$

The following result is then obtained by gathering Corollary 3.4 and Equations (6) and (7), after some easy simplifications:

Proposition 3.5. *The generating series $t(x)$ of rooted simple triangulations according to half the number of faces is the unique solution of the following equation:*

$$3xt'(x)^2 + 1 = [t(x)+1]\cdot t'(x), \tag{8}$$

which takes value 0 at 0.

From this equation, written as $t' = (3xt'^2+1)/(t+1)$, one readily extracts the development of $t(x)$ incrementally:

$$t(x) = x + x^2 + 3x^3 + 13x^4 + 68x^5 + 399x^6 + 2530x^7 + 16965x^8 + \dots$$

As in the quadrangular case, the exact expression of the coefficients (first obtained by Tutte from the recursive method [15] and subsequently recovered by Poulalhon and Schaeffer [12] using a bijection with some decorated trees that are in bijection with quaternary trees) can also be recovered from (8):

Corollary 3.6. *For $n \geqslant 1$, the number of rooted simple triangulations with $2n$ faces is equal to:*

$$\frac{2(4n-3)!}{n!(3n-1)!}.$$

Equivalently, the series $t(x)$ is expressed as $t(x) = \dfrac{(\alpha(x)-2)(1-\alpha(x))}{\alpha(x)^2}$, where $\alpha \equiv \alpha(x)$ is the series of rooted quaternary trees, specified by $\alpha = 1 + x\alpha^4$.

Proof. Equation (8) above admits a unique power series solution that is equal to 0 at 0, so it suffices to check that $f \equiv f(x) := (\alpha(x)-2)(1-\alpha(x))/\alpha(x)^2$ is solution of (8). Note that x and $f(x)$ have rational expressions in terms of α, and so does $f'(x) = \frac{\mathrm{d}f}{\mathrm{d}\alpha}/\frac{\mathrm{d}x}{\mathrm{d}\alpha}$:

$$x = \frac{\alpha - 1}{\alpha^4}, \qquad f(x) = \frac{(\alpha(x)-2)(1-\alpha(x))}{\alpha(x)^2}, \quad \text{and} \quad f'(x) = \alpha^2,$$

and these expressions satisfy Equation (8), which concludes the proof.

Now as for Equation (3), a direct proof without guessing the solution is possible; first let $r(x) = t'(x)$, Equation (8) rewrites $1 + t = 3xr + 1/r$, hence $r = 3r' + 3r - r'/r^2$, *i.e.* $u'(3xu^2 - 1) + 2u^3 = 0$. Then we seek for A, B such that $\frac{\mathrm{d}}{\mathrm{d}x}(A(r)x + B(r)) = 0$. Easy computations show that $xr^{3/2} + r^{-1/2}$ is suitable. Using initial conditions, we get $xr^2 + 1 = r^{1/2}$, *i.e.*, with $\alpha = xr^2 + 1$, $1 + x\alpha^4 = \alpha$. □

4. Radial distance of symmetric quadrangular dissections

For $k \geqslant 2$, $i > 0$, and $\mathcal{D}^{(k)}$ a family of k-symmetric dissections, let $\mathcal{D}_i^{(k)}$ be the family of dissections in $\mathcal{D}^{(k)}$ where the central vertex is at distance i from the outer face boundary; define the size of a k-symmetric quadrangular (*resp.* triangular) dissection D as the integer n such that D has kn inner faces (*resp.* $(2n+1)k$ inner faces). Let $D_i^{(k)}(x)$ be the generating series of $\mathcal{D}_i^{(k)}$ with respect to the size.

We compute here the expression of $D_i^{(k)}(x)$ for general, simple, and irreducible k-symmetric quadrangular $2k$-dissections and triangular k-dissections. So from now on $\mathcal{D}^{(k)}$ is either a family of k-symmetric quadrangular $2k$-dissections or a family of k-symmetric triangular k-dissections. Quadrangular and triangular dissections are treated respectively in this section and in the next one (Section 5). Note that, when $k = 2$, $D_i^{(k)}(x) = 0$ for quadrangular irreducible dissections and for triangular simple and irreducible dissections, and when $k = 3$, $D_i^{(k)}(x) = 0$ for triangular irreducible dissections.

We use the letter $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ (instead of $\mathcal{D}$) for the general, simple, and irreducible case respectively. To obtain the generating function expressions, we combine results of Bouttier *et al.* [1, 2] on the 2-point functions of general quadrangulations with quotient and substitution operations, and Lemma 1.2.

Remark 1. Lemma 1.2 implies that a k-symmetric quadrangular dissection has no cycle of length less than $2k$ that strictly encloses the central vertex (indeed the k-quotient has only faces of even degree, hence is bipartite, hence has no loop). And a k-symmetric triangular dissection has no cycle of length less than k that strictly encloses the central vertex.

4.1. Symmetric quadrangular dissections.

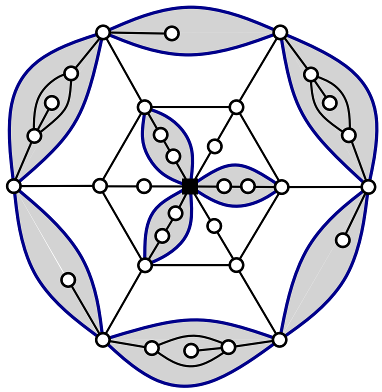
Define the algebraic generating function $P \equiv P(x)$ by

$$P = 1 + 3xP^2, \tag{9}$$

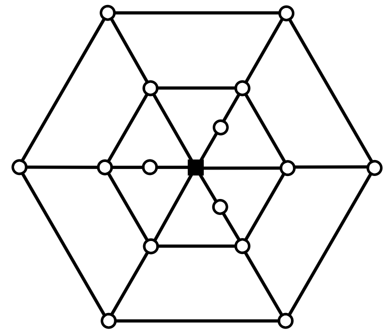
and define the algebraic generating function $X \equiv X(x)$ by

$$X + \frac{1}{X} + 1 = \frac{3}{P-1}. \tag{10}$$

Define also $X_\infty := P$. The following result is very closely related to a result by Bouttier, Di Francesco, and Guitter [1].



(a) The maximal 2-cycles,



(b) and the simple core.

FIGURE 6. A 3-symmetric quadrangulation and its simple 3-symmetric core, obtained by collapsing all maximal 2-cycles into edges.

Proposition 4.1. *For each $i \geqslant 1$ and $k \geqslant 2$, the generating function $F_i^{(k)}(x)$ has the following expression (which does not depend on k):*

$$F_i^{(k)}(x) = X_{i+1} - X_i, \quad \text{where } X_i = X_\infty \frac{(1-X^i)(1-X^{i+3})}{(1-X^{i+1})(1-X^{i+2})}.$$

Proof. If we take the k-quotient of dissection in $\mathcal{F}_i^{(k)}$ we obtain a quadrangular 2-dissection with a pointed inner vertex at distance i from the outer 2-gon, according to Lemma 1.2. Notice that the outer 2-cycle can be contracted (on the sphere) into a single edge e. This yields a quadrangulation of the sphere with a marked edge e and pointed vertex v at distance i from e (i.e., the extremity of e closest from v is at distance i from v). Hence $F_i^{(k)}(x)$ is equal to the generating function $F_i(x)$ of quadrangulations with a marked edge e and a marked vertex v at distance i from e. The algebraic expression of $F_i^{(k)}(x) = F_i(x)$, as $X_{i+1} - X_i$, has been obtained by Bouttier *et al.* [1]. □

4.2. **Symmetric simple quadrangular dissections.** We now compute $y \mapsto G_i^{(k)}(y)$. We classically proceed by substitution (a substitution approach is also discussed in [2] for rooted quadrangulations). The *core* of $\gamma \in \mathcal{F}_i^{(k)}$ is obtained by collapsing each maximal 2-cycle (for the enclosed area) of γ into a single edge (note that no 2-cycle can strictly enclose the central vertex, according to Remark 1), see Figure 6. This yields a simple dissection in $\mathcal{G}_i^{(k)}$, and the distance of the central vertex to the outer boundary is still i (because there is no way of shortening this distance by travelling inside a 2-cycle). Conversely each $\gamma \in \mathcal{F}_i^{(k)}$ is uniquely obtained from $\kappa \in \mathcal{G}_i^{(k)}$ —with nk inner faces— where each of the $(2n+1)k$ edges is either left unchanged or blown into a double edge in the interior of which a rooted quadrangulation is patched, in a way that respects the symmetry of order k (that is, the k edges in an orbit of edges of κ undergo the same substitution operation). Denoting by $f \equiv f(x)$ the series of rooted quadrangulations according to the number of faces, we obtain $F_i^{(k)}(x) = \sum_{n\geqslant 1}[y^n]G_i^{(k)}(y) \cdot x^n \cdot (1+f)^{2n+1}$. In other words,

$$F_i^{(k)}(x) = (1+f) \cdot G_i^{(k)}(x \cdot (1+f)^2). \tag{11}$$

The series $f = f(x)$ is well known to be algebraic, having a rational expression in terms of P: $f = P(4-P)/3 - 1$ (we will need this expression a few times).

Define $Q \equiv Q(y)$ as the algebraic series in y defined by

$$Q = 1 + yQ^3. \tag{12}$$

Lemma 4.2. *Let $K(x)$ and let $L(y)$ be related by $K(x) = L(x(1+f)^2)$. Let $\widehat{K}(P)$ and $\widehat{L}(Q)$ be the expressions of $K(x)$ and $L(y)$ in terms of $P \equiv P(x)$ and $Q \equiv Q(y)$, i.e., $K(x) = \widehat{K}(P(x))$ and $L(y) = \widehat{L}(Q(y))$. Then*

$$\widehat{L}(Q) = \widehat{K}\Big(4 - 3/Q\Big).$$

Proof. The change of variable relation is $y = x(1+f)^2$. We have

$$y = x(1+f)^2 = \frac{P-1}{3P^2}\Big(P\frac{4-P}{3}\Big)^2 = \frac{P-1}{3}\Big(\frac{4-P}{3}\Big)^2$$

Hence, if we write $Q \equiv Q(y) = 3/(4-P(x))$, we have $y = (1-1/Q)Q^{-2}$, so that $Q = 1+yQ^3$. In addition $P(x) = 4 - 3/Q(y)$. □

For the rest of this subsection, when we have y and x together in an equation we assume y and x to be related by $y = x(1+f)^2$. Define $Y_\infty(y) := X_\infty(x)/(1+f)$, $Y_i(y) := X_i(x)/(1+f)$, and $Y(y) := X(x)$. The expression of $X_i(x)$ in terms of $X_\infty(x)$ and $X(x)$ ensures that (we do not need Lemma 4.2 at this step):

$$Y_i = Y_\infty \frac{(1-Y^i)(1-Y^{i+3})}{(1-Y^{i+1})(1-Y^{i+2})}.$$

In addition, since $G_i^{(k)}(y) = F_i^{(k)}(x)/(1+f)$, we have

$$G_i^{(k)}(y) = Y_{i+1} - Y_i.$$

We now apply Lemma 4.2 to get an algebraic expression of $G_i^{(k)}(y)$, written in terms of $Q(y)$. We have

$$Y_\infty(y) = \frac{X_\infty(x)}{1+f(x)} = \frac{3}{4-P}.$$

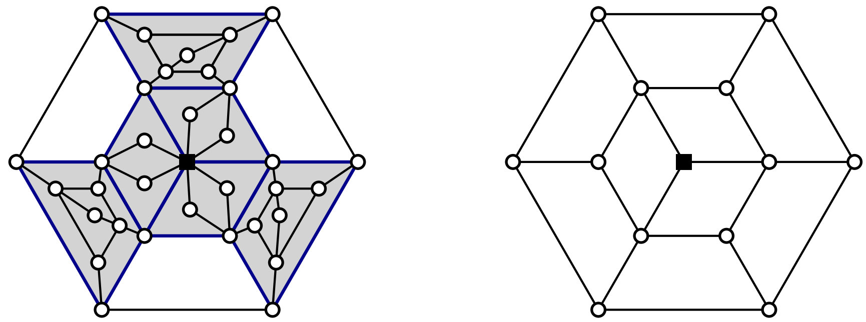
Then Lemma 4.2 ensures that $Y_\infty(y) = Q(y)$. Lemma 4.2 and the relation $X + 1/X + 1 = 3/(P-1)$ ensure that $Y \equiv Y(y)$ is the algebraic generating function specified by

$$Y + \frac{1}{Y} = \frac{1}{Q-1}. \tag{13}$$

To summarize we obtain:

Proposition 4.3. *For each $i \geqslant 1$ and $k \geqslant 2$, the generating function $G_i^{(k)}(y)$ has the expression (with $Y_\infty = Q$):*

$$G_i^{(k)}(y) = Y_{i+1} - Y_i, \quad \text{where } Y_i = Y_\infty \frac{(1-Y^i)(1-Y^{i+3})}{(1-Y^{i+1})(1-Y^{i+2})}.$$



(a) The maximal non empty 4-cycles,   (b) and the irreductible core.

Figure 7. A 3-symmetric simple quadrangulation and its irreductible 3-symmetric core, obtained by emptying all maximal 4-cycles into faces.

4.3. **Symmetric irreducible quadrangular dissections.** We now use a substitution approach at faces (instead of edges) to get an expression for $H_i^{(k)}(z)$ for $k \geqslant 3$ and $i > 0$. The *core* of $\gamma \in \mathcal{G}_i^{(k)}$ is obtained by emptying each maximal (for the enclosed area) 4-cycle of γ (note that no 4-cycle strictly encloses the central vertex, according to Remark 1), see Figure 7. This yields a symmetric irreducible dissection $\kappa \in \mathcal{H}_i^{(k)}$, and the distance of the pointed vertex to the outer boundary is still i (because there is no way of shortening this distance by travelling inside a 4-cycle). Conversely each $\gamma \in \mathcal{G}_i^{(k)}$ is uniquely obtained from $\kappa \in \mathcal{H}_i^{(k)}$ where at each face a rooted simple quadrangulation (with at least one inner face) is patched, in a way that respects the symmetry of order k (i.e., the k faces of an orbit undergo the same patching operation). Denoting by $g \equiv g(y)$ the series of rooted simple non-degenerated quadrangulations according to the number of inner faces, we obtain, for $k \geqslant 3$ and $i > 0$:

$$G_i^{(k)}(y) = H_i^{(k)}(g(y)).$$

Again the series $g \equiv g(y)$ is well known to be algebraic, having a rational expression in terms of Q: $g = 3Q - Q^2 - 2$.

Define the algebraic series $R \equiv R(z)$ by

$$R = z + R^2. \tag{14}$$

Lemma 4.4. *Let $L(y)$ and let $M(z)$ be related by $L(y) = M(g(y))$. Let $\widehat{L}(Q)$ and $\widehat{M}(R)$ be the expressions of $L(y)$ and $M(z)$ in terms of $Q \equiv Q(y)$ and $R \equiv R(z)$, i.e., $L(y) = \widehat{L}(Q(y))$ and $M(z) = \widehat{M}(R(z))$. Then*

$$\widehat{M}(R) = \widehat{L}(R+1).$$

Proof. The change of variable relation is $z = g(y)$. We have

$$z = g(y) = 3Q - Q^2 - 2 = -(Q-1)^2 + (Q-1)$$

Hence, if we write $R \equiv R(z) = Q(y) - 1$, we have $z = -R^2 + R$, so that $R = z + R^2$. In addition $Q(y) = R(z) + 1$. □

For the rest of this subsection, when we have z and y together in an equation we assume z and y to be related by $z = g(y)$. Define $Z_\infty(z) := Y_\infty(y)$, $Z_i(z) := Y_i(y)$, and $Z(z) := Y(y)$.

The expression of $Y_i(y)$ in terms of $Y_\infty(y)$ and $Y(y)$ ensures that (we do not need Lemma 4.4 at this step):

$$Z_i = Z_\infty \frac{(1-Z^i)(1-Z^{i+3})}{(1-Z^{i+1})(1-Z^{i+2})}.$$

In addition, since $H_i^{(k)}(z) = G_i^{(k)}(y)$, we have

$$H_i^{(k)}(z) = Z_{i+1} - Z_i.$$

We now apply Lemma 4.4 to get an algebraic expression of $H_i^{(k)}(z)$, written in terms of $R(z)$. We have

$$Z_\infty(z) = Y_\infty(y) = Q(y) = R(z) + 1.$$

Lemma 4.4 and the relation $Y + 1/Y = 1/(Q-1)$ also ensure that $Z \equiv Z(z)$ is the algebraic generating function specified by

$$Z + \frac{1}{Z} = \frac{1}{R}. \tag{15}$$

To summarize we obtain:

Proposition 4.5. *For each $i \geqslant 1$ and $k \geqslant 3$, the generating function $H_i^{(k)}(z)$ has the expression (with $Z_\infty = R + 1$):*

$$H_i^{(k)}(z) = Z_{i+1} - Z_i, \quad \text{where } Z_i = Z_\infty \frac{(1-Z^i)(1-Z^{i+3})}{(1-Z^{i+1})(1-Z^{i+2})}.$$

5. Radial distance of symmetric triangular dissections

5.1. **Symmetric triangular dissections.** Define the algebraic generating function $P \equiv P(x)$ by

$$P^2 = 1 + 8xP^3, \tag{16}$$

and define the algebraic generating function $X \equiv X(x)$ by

$$X + \frac{1}{X} + 2 = \frac{8}{P^2 - 1}. \tag{17}$$

Define also $X_\infty \equiv X_\infty(x)$ and $A_\infty \equiv A_\infty(x)$ by

$$X_\infty = P, \quad {A_\infty}^2 = \frac{2P(P-1)}{(1+P)}. \tag{18}$$

As for quadrangulations the following result is very closely related to a result by Bouttier and Guitter [3].

Proposition 5.1. *For each $i \geqslant 1$ and $k \geqslant 2$, the generating function $F_i^{(k)}(x)$ has the following expression (which does not depend on k):*

$$F_i^{(k)}(x) = X_{i+1} - X_{i-1} + {A_i}^2 - {A_{i-1}}^2,$$

where

$$X_i = X_\infty \cdot \frac{(1-X^i)(1-X^{i+2})}{(1-X^{i+1})^2}, \quad A_i = A_\infty \cdot \Big(1 - \frac{P+1}{4} X^i \frac{(1-X)(1-X^2)}{(1-X^{i+1})(1-X^{i+2})}\Big).$$

Proof. We recall the result in [3] about triangulations:

- the series U_i of triangulations of the sphere with a pointed vertex u and a marked edge $e = (v, v')$ with v at distance i and v' at distance $i-1$ from u is given by $U_i = X_i - X_{i-1}$,
- the series V_i of triangulations of the sphere with a pointed vertex u and a marked oriented edge $e = (v, v')$ with v and v' at distance i from u is given by $V_i = {A_i}^2 - {A_{i-1}}^2$. (Note that the series V_i decomposes as $V_{i,\mathrm{loop}} + V_{i,\mathrm{distinct}}$, whether the extremities of the marked edge are equal or distinct.)

The k-quotient of a dissection in $\mathcal{F}_i^{(k)}$ is a triangular 1-dissection D with a pointed inner vertex u at distance i from the outer loop, according to Lemma 1.2. The vertex incident to the outer loop is called the *root-vertex* and denoted by v. If we delete the outer loop we obtain a map $\widetilde{D}$ with an outer face of degree 2 and all inner faces of degree 3. Two cases can occur: either the outer face contour of $\widetilde{D}$ is a 2-gon or is made of two adjacent loops. In the first case let v' be the other vertex of the 2-gon: v' is at distance either $i-1$, $i+1$, or i from u, giving respective contributions U_i, U_{i+1}, $V_{i,\mathrm{distinct}}$. In the second case we have an ordered pair D_1, D_2 of 1-dissections, and one of these two dissections contains a marked vertex at distance i from v. Orient the outer loop of D_1 clockwise and the outer loop of D_2 counterclockwise, and paste D_1 and D_2 together at their outer loops. This yields a triangulation of the sphere with a pointed vertex u and a marked oriented loop whose incident vertex is at distance i from u. The corresponding contribution is $V_{i,\mathrm{loop}}$. Gathering all cases we obtain $F_i^{(k)} = U_i + U_{i+1} + V_{i,\mathrm{distinct}} + V_{i,\mathrm{loop}} = U_i + U_{i+1} + V_i$. $\square$

5.2. **Symmetric simple triangular dissections.** Call a rooted triangulation *simply-rooted* if the root-edge is not a loop. To compute $y \mapsto G_i^{(k)}(y)$, we proceed very similarly as for quadrangulations. Each $\gamma \in \mathcal{F}_i^{(k)}$ (for $k \geqslant 3$) is uniquely obtained from $\kappa \in \mathcal{G}_i^{(k)}$ —with $(2n+1)k$ inner faces— where each of the $(3n+2)k$ edges is either left unchanged or blown into a double edge in the interior of which a simply-rooted triangulation is patched, in a way that respects the symmetry of order k (that is, the k edges in an orbit of edges of κ undergo the same substitution operation). Denoting by $f \equiv f(x)$ the series of simply-rooted triangulations according to half the number of faces, we obtain $F_i^{(k)}(x) = \sum_{n\geqslant 1}[y^n]G_i^{(k)}(y) \cdot x^n \cdot (1+f)^{3n+2}$. In other words,

$$F_i^{(k)}(x) = (1+f)^2 \cdot G_i^{(k)}(x \cdot (1+f)^3). \tag{19}$$

The series $f = f(x)$ is algebraic, with a rational expression in terms of P: $1 + f = -P(P^2 - 9)/8$.

Define $Q \equiv Q(y)$ as the algebraic series in y given by

$$Q = \frac{y}{(1-Q)^3}, \tag{20}$$

and define also $\widetilde{Q} \equiv \widetilde{Q}(y)$ as $\widetilde{Q} := (1+8Q)^{1/2}$.

Lemma 5.2. *Let $K(x)$ and $L(y)$ be related by $K(x) = L(x(1+f)^3)$. Let $\widehat{K}(P)$ and $\widetilde{L}(\widetilde{Q})$ be the expressions of $K(x)$ and $L(y)$ in terms of $P \equiv P(x)$ and $\widetilde{Q} \equiv \widetilde{Q}(y)$, i.e., $K(x) = \widehat{K}(P(x))$ and $L(y) = \widetilde{L}(\widetilde{Q}(y))$. Then the expressions are the same, i.e.,*

$$\widetilde{L} = \widehat{K}.$$

Proof. The change of variable relation is $y = x(1+f)^3$. We have:

$$y = x(1+f)^3 = -\frac{1}{2^{12}}(P^2-1)(P^2-9)^3.$$

Hence, if we write $Q \equiv Q(y) = (P(x)^2-1)/8$, we have $y = Q(1-Q)^3$. In addition $P(x) = \widetilde{Q}(y)$, hence $\widetilde{L} = \widehat{K}$. □

Define (with y and x related by $y = x(1+f)^3$):

$$Y_\infty(y) := \frac{X_\infty(x)}{(1+f)^2},\ Y_i(y) := \frac{X_i(x)}{(1+f)^2},\ Y(y) := X(x),$$

and define

$$B_\infty(y) = \frac{A_\infty(x)}{(1+f)},\ B_i(y) := \frac{A_i(x)}{(1+f)}.$$

The expression of $X_i(x)$ in terms of $X_\infty(x)$ and $X(x)$ ensures that

$$Y_i = Y_\infty \frac{(1-Y^i)(1-Y^{i+2})}{(1-Y^{i+1})^2}.$$

The expression of $A_i(x)$ in terms of $A_\infty(y)$ and $X(x)$ and Lemma 5.2 (to replace P by $\widetilde{Q}$ in the expression) ensure that

$$B_i = B_\infty \cdot \Big(1 - \frac{\widetilde{Q}+1}{4} Y^i \frac{(1-Y)(1-Y^2)}{(1-Y^{i+1})(1-Y^{i+2})}\Big).$$

In addition, since $G_i^{(k)}(y) = F_i^{(k)}(x)/(1+f)^2$, we have

$$G_i^{(k)}(y) = Y_{i+1} - Y_{i-1} + {B_i}^2 - {B_{i-1}}^2.$$

Using Lemma 5.2 we obtain (after simplifications)

$$Y_\infty = \frac{1}{\widetilde{Q}(1-Q)^2},\quad {B_\infty}^2 = \frac{16 \cdot Q}{\widetilde{Q}(1+\widetilde{Q})^2(1-Q)^2}. \tag{21}$$

Lemma 5.2 and the relation $X + 1/X + 1 = 8/(P^2-1)$ also ensure that $Y \equiv Y(y)$ is the algebraic generating function specified by

$$Y + \frac{1}{Y} + 2 = \frac{1}{Q}. \tag{22}$$

To summarize we obtain:

Proposition 5.3. *For each $i \geqslant 1$ and $k \geqslant 3$, the generating function $G_i^{(k)}(y)$ has the following expression (which does not depend on k):*

$$G_i^{(k)}(y) = Y_{i+1} - Y_{i-1} + {B_i}^2 - {B_{i-1}}^2,$$

where

$$Y_i = Y_\infty \frac{(1-Y^i)(1-Y^{i+2})}{(1-Y^{i+1})^2},\quad B_i = B_\infty \cdot \Big(1 - \frac{\widetilde{Q}+1}{4} Y^i \frac{(1-Y)(1-Y^2)}{(1-Y^{i+1})(1-Y^{i+2})}\Big).$$

5.3. **Symmetric irreducible triangular dissections.** We now use a substitution approach at faces to get an expression for $H_i^{(k)}(z)$ for $k \geqslant 4$ and $i > 0$. Similarly as in the quadrangulated case, each $\gamma \in \mathcal{G}_i^{(k)}$ (for $k \geqslant 4$) is uniquely obtained from $\kappa \in \mathcal{H}_i^{(k)}$ —with $(2n+1)k$ inner faces— where at each face a rooted simple triangulation (with at least one inner face) is patched, in a way that respects the symmetry of order k (i.e., the k faces of an orbit undergo the same patching operation). Denoting by $g \equiv g(y)$ the series of rooted simple triangulations according to half the number of faces, we obtain, for $k \geqslant 4$ and $i > 0$:

$$G_i^{(k)}(y) = \frac{g}{y} H_i^{(k)}(g^2/y).$$

Again the series $g \equiv g(y)$ is well known to be algebraic, having a rational expression in terms of Q: $g = Q - 2Q^2$.

Define the algebraic series $R \equiv R(z)$ by

$$R = \frac{z}{(1-R)^2}. \tag{23}$$

Define also $\widetilde{R} \equiv \widetilde{R}(z)$ as $\widetilde{R} := \sqrt{1+9R}/\sqrt{1+R}$.

Lemma 5.4. *Let $L(y)$ and let $M(z)$ be related by $L(y) = M(g(y)^2/y)$. Let $\widetilde{L}(\widetilde{Q})$ and $\widetilde{M}(\widetilde{R})$ be the expressions of $L(y)$ and $M(z)$ in terms of $\widetilde{Q} \equiv \widetilde{Q}(y)$ and $\widetilde{R} \equiv \widetilde{R}(z)$, i.e., $L(y) = \widetilde{L}(\widetilde{Q}(y))$ and $M(z) = \widetilde{M}(\widetilde{R}(z))$. Then the expressions are the same, i.e.,*

$$\widetilde{M} = \widetilde{L}.$$

Proof. The change of variable relation is $z = g(y)^2/y$. We have

$$z = g(y)^2/y = Q\frac{(1-2Q)^2}{(1-Q)^3}.$$

Hence, if we write $R \equiv R(z) = Q(y)/(1-Q(y))$, we have $z = R(1-R)^2$, so that $R = z/(1-R)^2$. In addition we have $Q(y) = R(z)/(1+R(z))$, so that $\widetilde{Q}(y) = (1+8Q(y))^{1/2} = \sqrt{1+9R(z)}/\sqrt{1+R(z)} = \widetilde{R}(z)$. Hence $\widetilde{M} = \widetilde{L}$. ☐

Define (with z and y related by $z = g(y)^2/y$):

$$Z_\infty(z) := \frac{y}{g} Y_\infty(y),\ Z_i(z) := \frac{y}{g} Y_i(y),\ Z(z) := Y(y),$$

and define

$$C_\infty(z)^2 := \frac{y}{g} B_\infty(y)^2,\ C_i(z)^2 := \frac{y}{g} B_i(y)^2.$$

The expression of $Y_i(y)$ in terms of $Y_\infty(y)$ and $Y(y)$ ensures that

$$Z_i = Z_\infty \frac{(1-Z^i)(1-Z^{i+2})}{(1-Z^{i+1})^2}.$$

The expression of $B_i(y)$ in terms of $B_\infty(y)$ and $Y(y)$ and Lemma 5.4 (to replace $\widetilde{Q}$ by $\widetilde{R}$ in the expression) ensure that

$$C_i = C_\infty \cdot \Big(1 - \frac{\widetilde{R}+1}{4} Z^i \frac{(1-Z)(1-Z^2)}{(1-Z^{i+1})(1-Z^{i+2})}\Big).$$

In addition, since $H_i^{(k)}(z) = \frac{y}{g} G_i^{(k)}(y)$, we have

$$H_i^{(k)}(z) = Z_{i+1} - Z_{i-1} + {C_i}^2 - {C_{i-1}}^2.$$

Using Lemma 5.4 we obtain (after simplifications):

$$Z_\infty = \frac{1}{\widetilde{R}(1-R)}, \quad {C_\infty}^2 = \frac{16 \cdot R}{(\widetilde{R}+1)^2 \widetilde{R}(1-R^2)}. \tag{24}$$

Lemma 5.4 and the relation $Y + 1/Y + 2 = 1/Q$ also ensure that $Z \equiv Z(z)$ is the algebraic generating function specified by

$$Z + \frac{1}{Z} + 1 = \frac{1}{R}. \tag{25}$$

To summarize we obtain:

Proposition 5.5. *For each $i \geqslant 1$ and $k \geqslant 4$, the generating function $H_i^{(k)}(z)$ has the following expression (which does not depend on k):*

$$H_i^{(k)}(z) = Z_{i+1} - Z_{i-1} + {C_i}^2 - {C_{i-1}}^2,$$

where

$$Z_i = Z_\infty \frac{(1-Z^i)(1-Z^{i+2})}{(1-Z^{i+1})^2}, \quad C_i = C_\infty \cdot \Big(1 - \frac{\widetilde{R}+1}{4} Z^i \frac{(1-Z)(1-Z^2)}{(1-Z^{i+1})(1-Z^{i+2})}\Big).$$

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