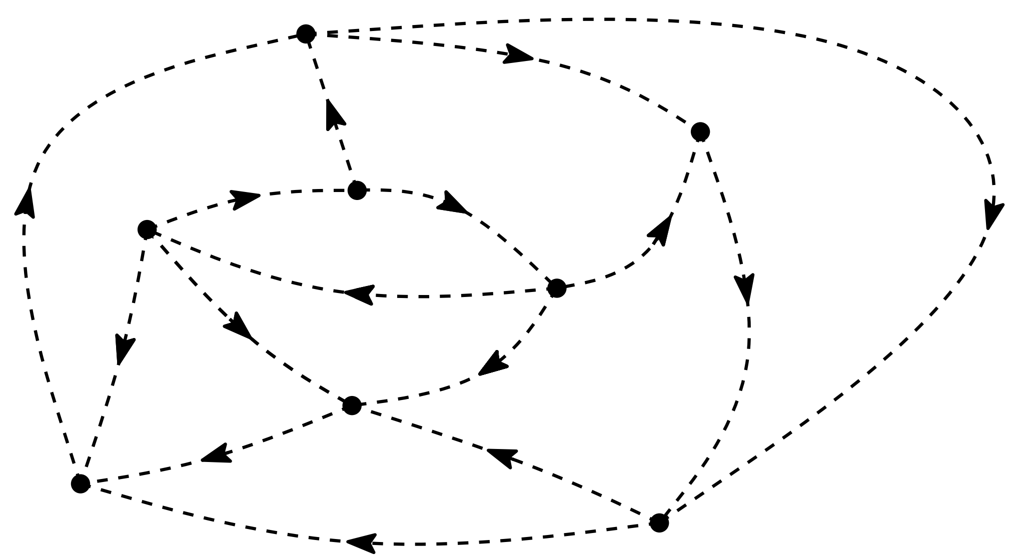
AN INTRODUCTION TO ORIENTATIONS ON MAPS

1st lecture — May, 15th 2017

Marie Albenque (CNRS, LIX, École Polytechnique)



Mini-school on Random Maps and the Gaussian Free Field — ProbabLyon

Overview

Today : Construction of orientations, existence, uniqueness

1 - Some definitions : maps, orientations.
2 - Existence of orientations
3 - Flip and flop : the lattice of orientations

Tuesday : Applications : graph drawings, couplings, bijections

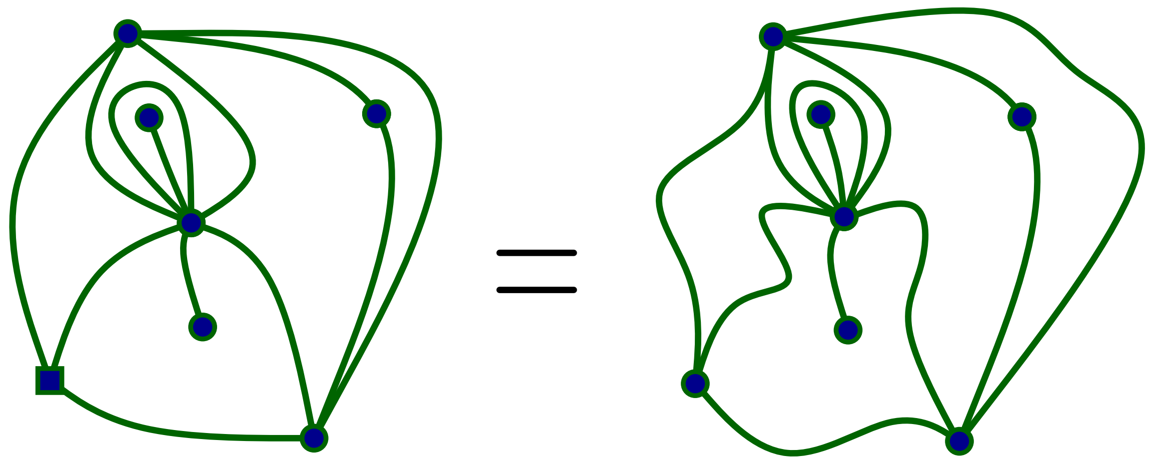
1 - Schnyder woods and graph drawings.
2 - Couplings and spanning trees.
3 - Orientations and blossoming trees

Thursday : Why should you care ?

1 - Higher genus
2 - Scaling limits for simple triangulations.
3 - Scaling limits for maps with an orientation ?

Plane maps — Definition

A **plane map** is the proper embedding of a connected planar graph in the plane (considered up to continuous deformations).



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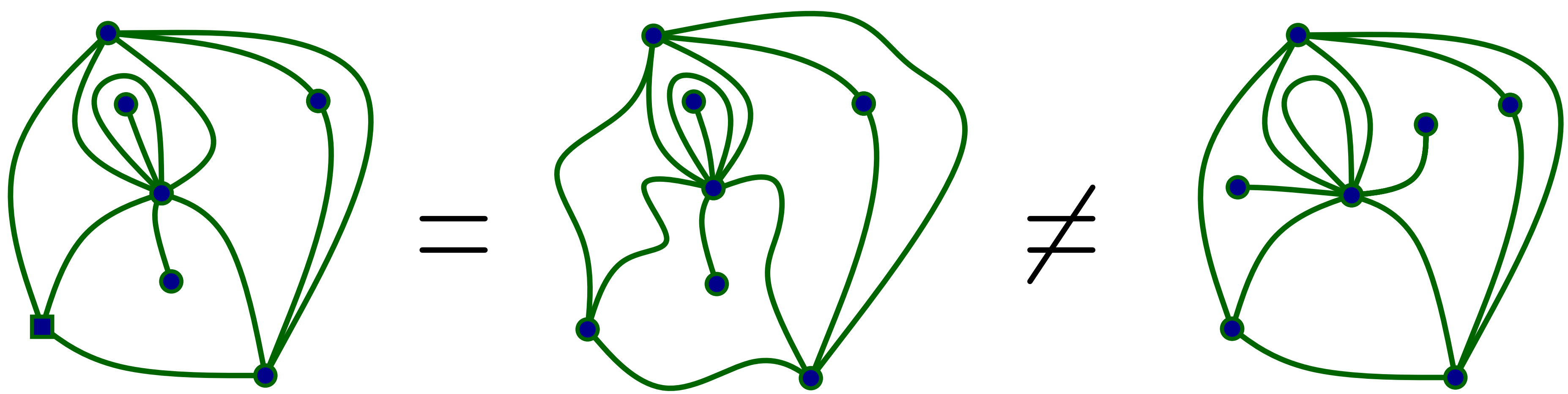

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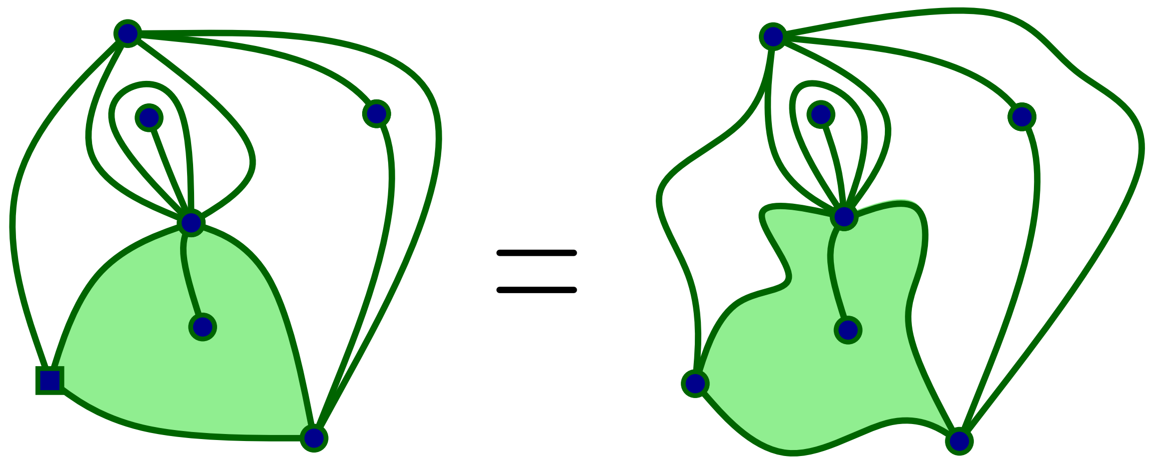

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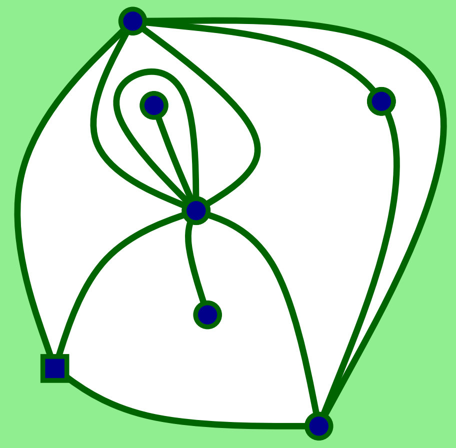

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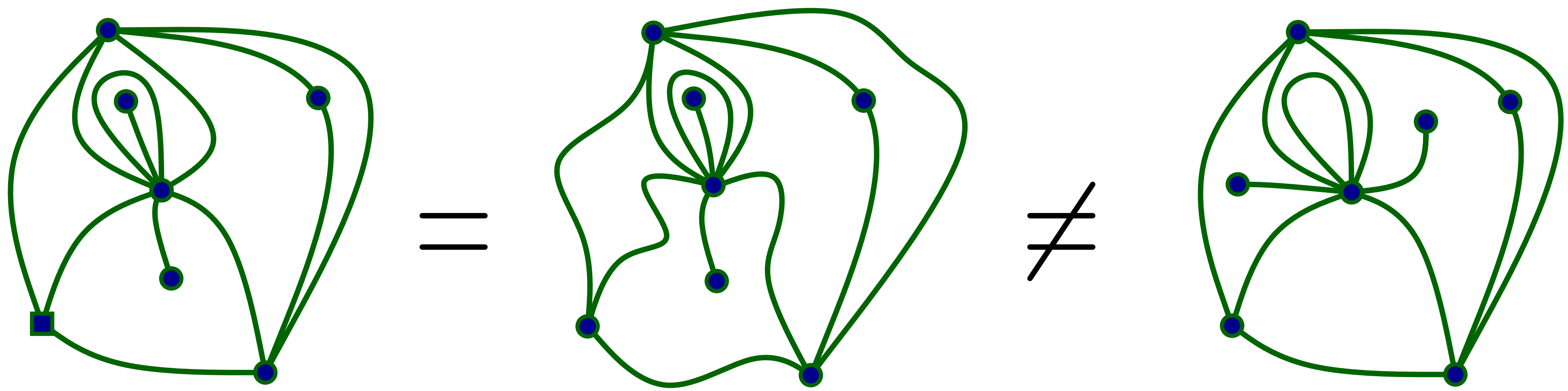

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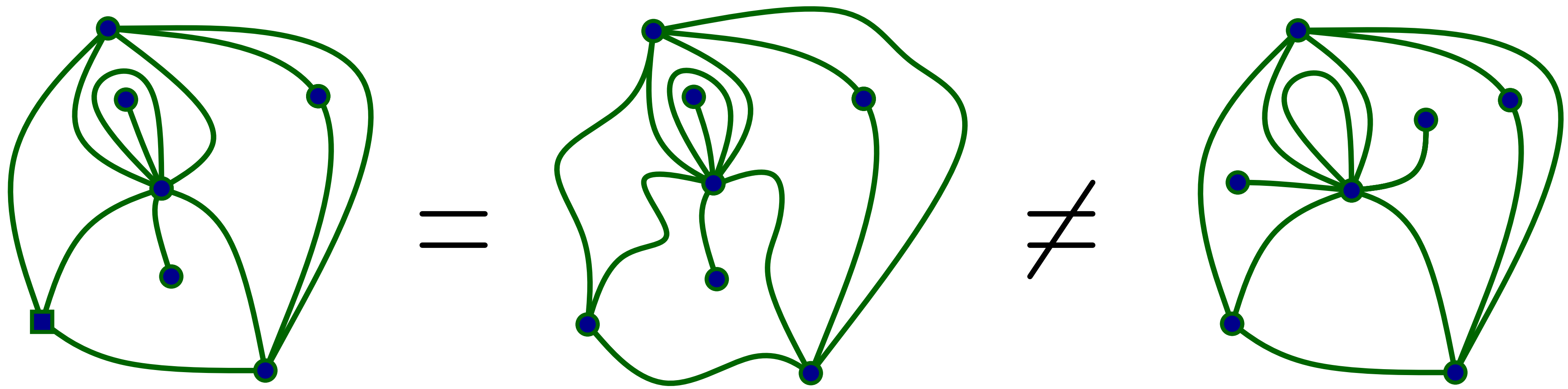
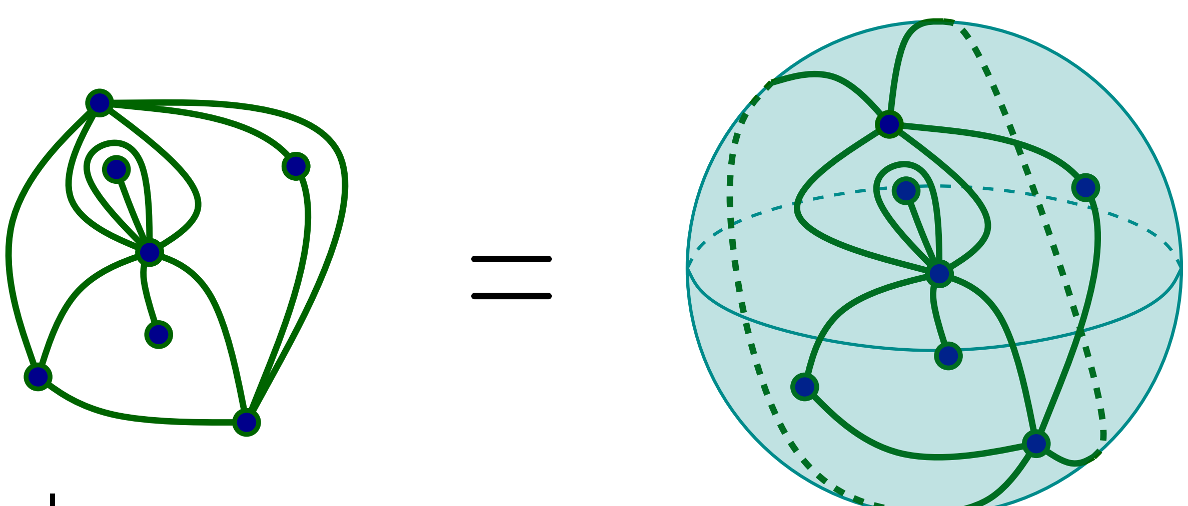


plane map ≈ plane graph + cyclical ordering around the vertices.



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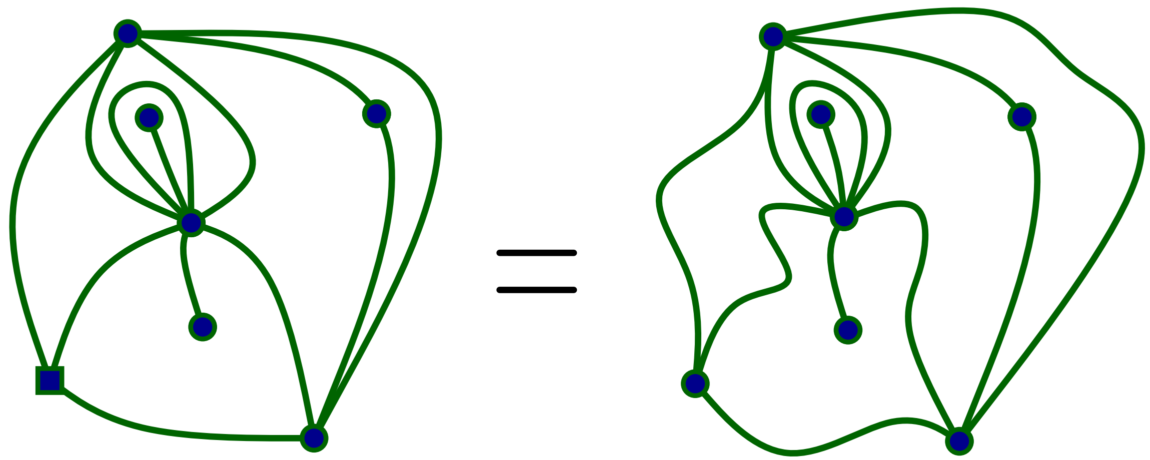

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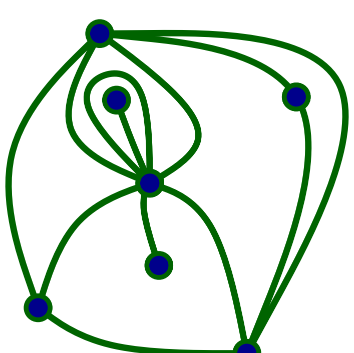
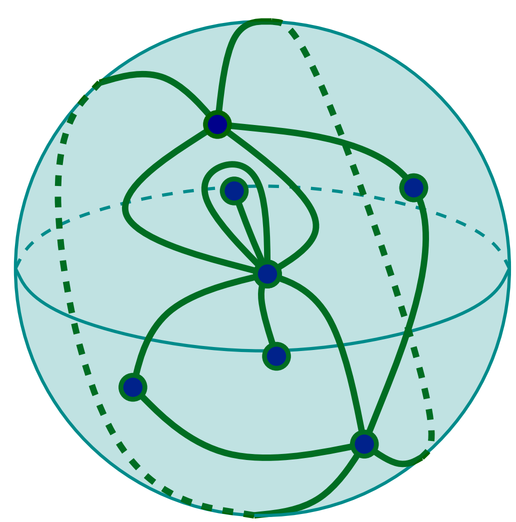


plane map

=



planar map

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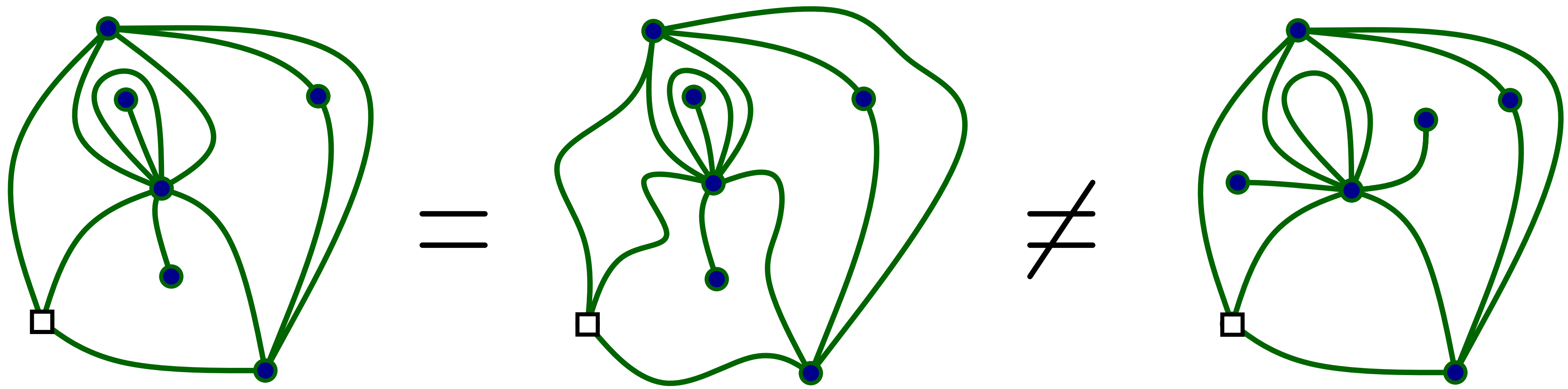

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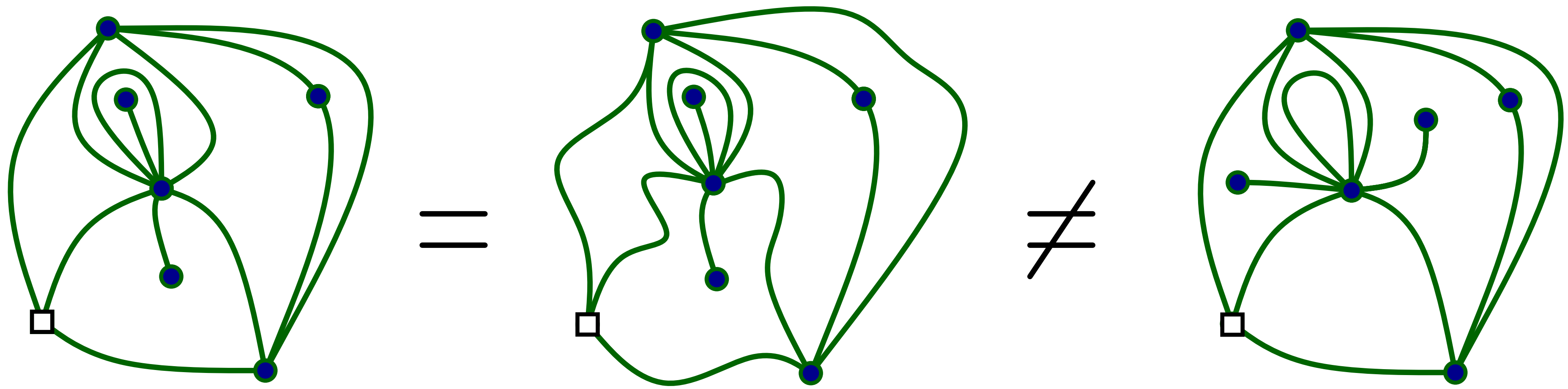

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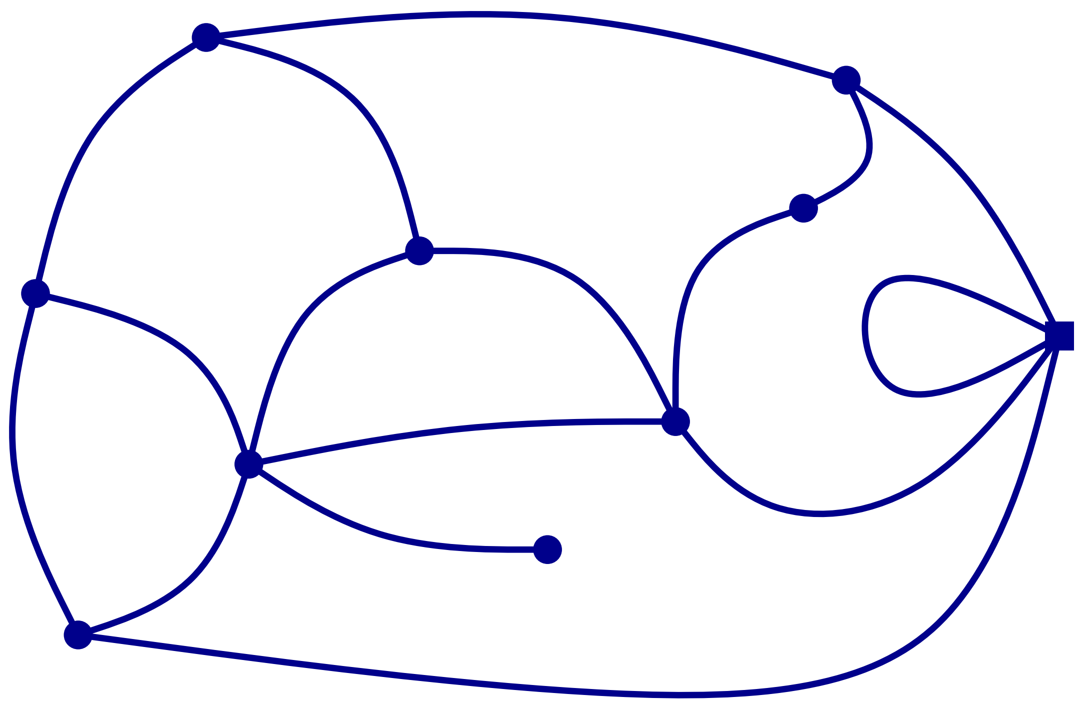
Notation : $V(M) = \{\text{vertices of } M\}$
$E(M) = \{\text{edges of } M\}$
$F(M) = \{\text{faces of } M\}$

Digression : Euler Formula

Euler Formula

$$|V(M)| + |F(M)| = 2 + |E(M)|$$



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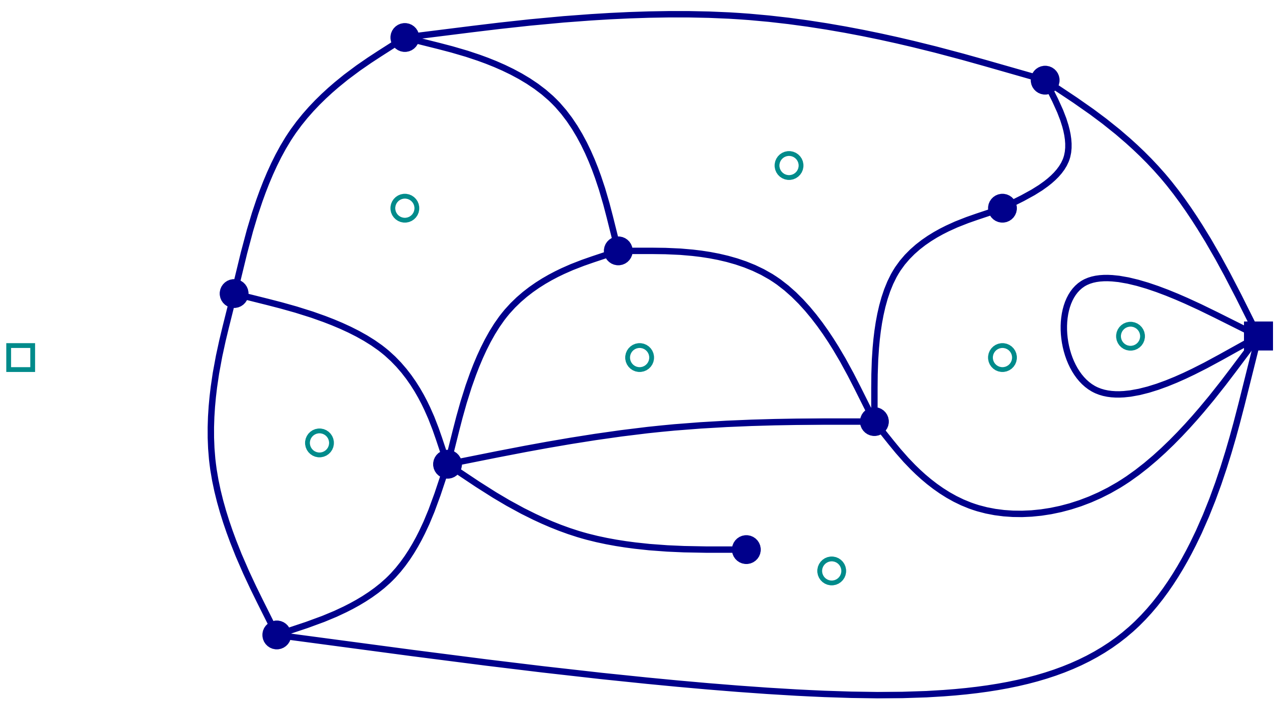

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$M^\star$ its dual map

$V(M^\star) = F(M)$

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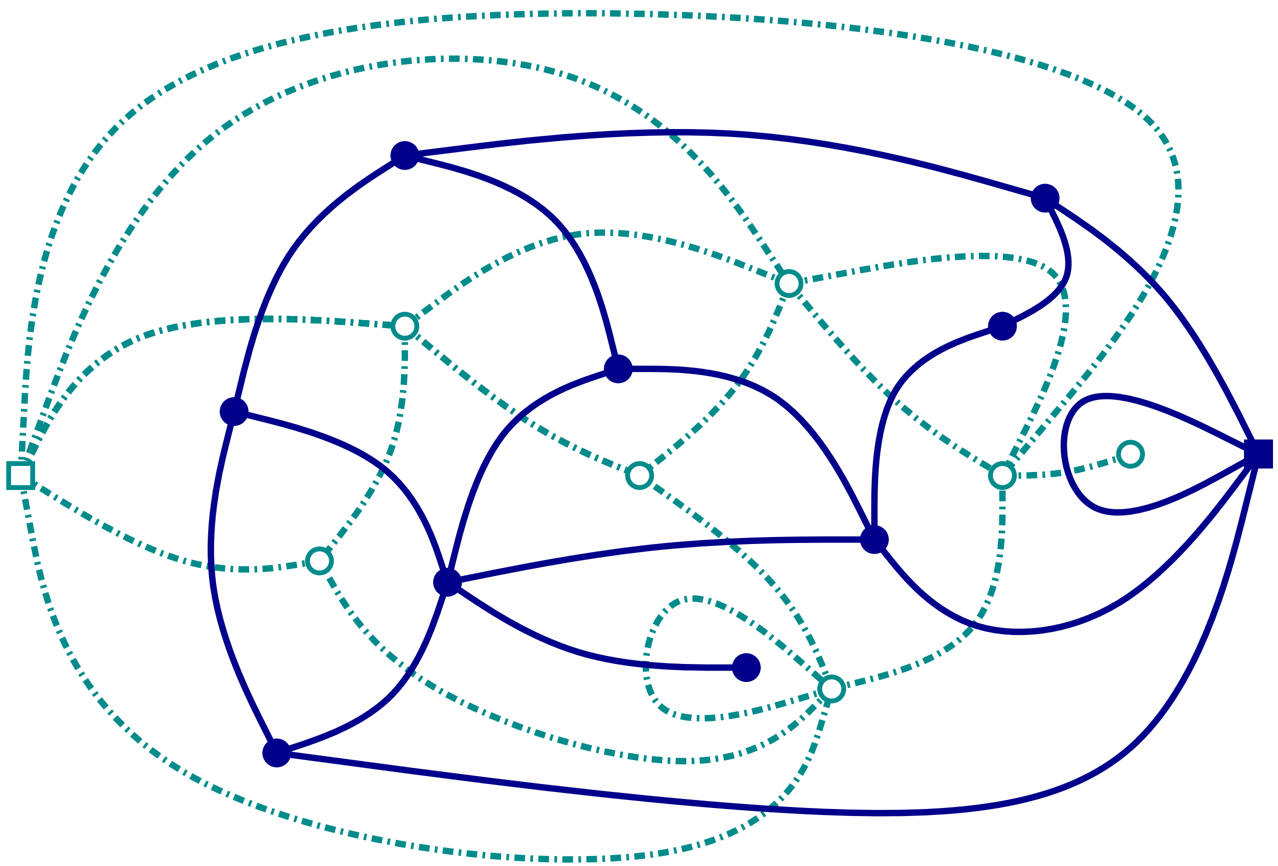

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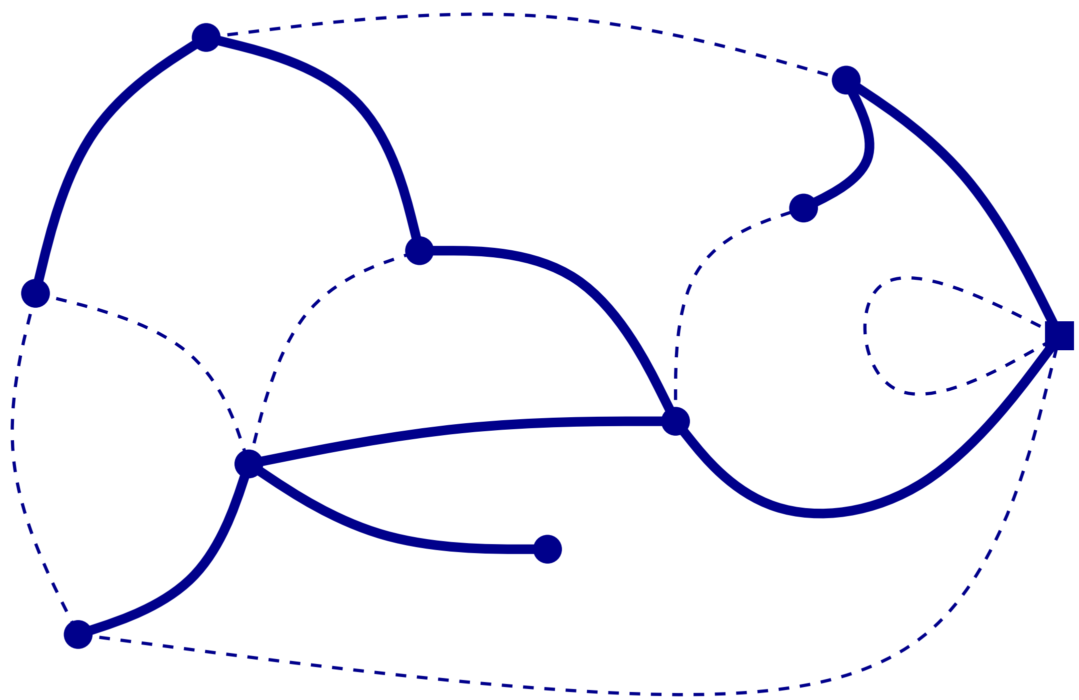

M a rooted plane map

T spanning tree of M

$V(T) = V(M)$,
$|E(T)| = |V(M)| - 1$

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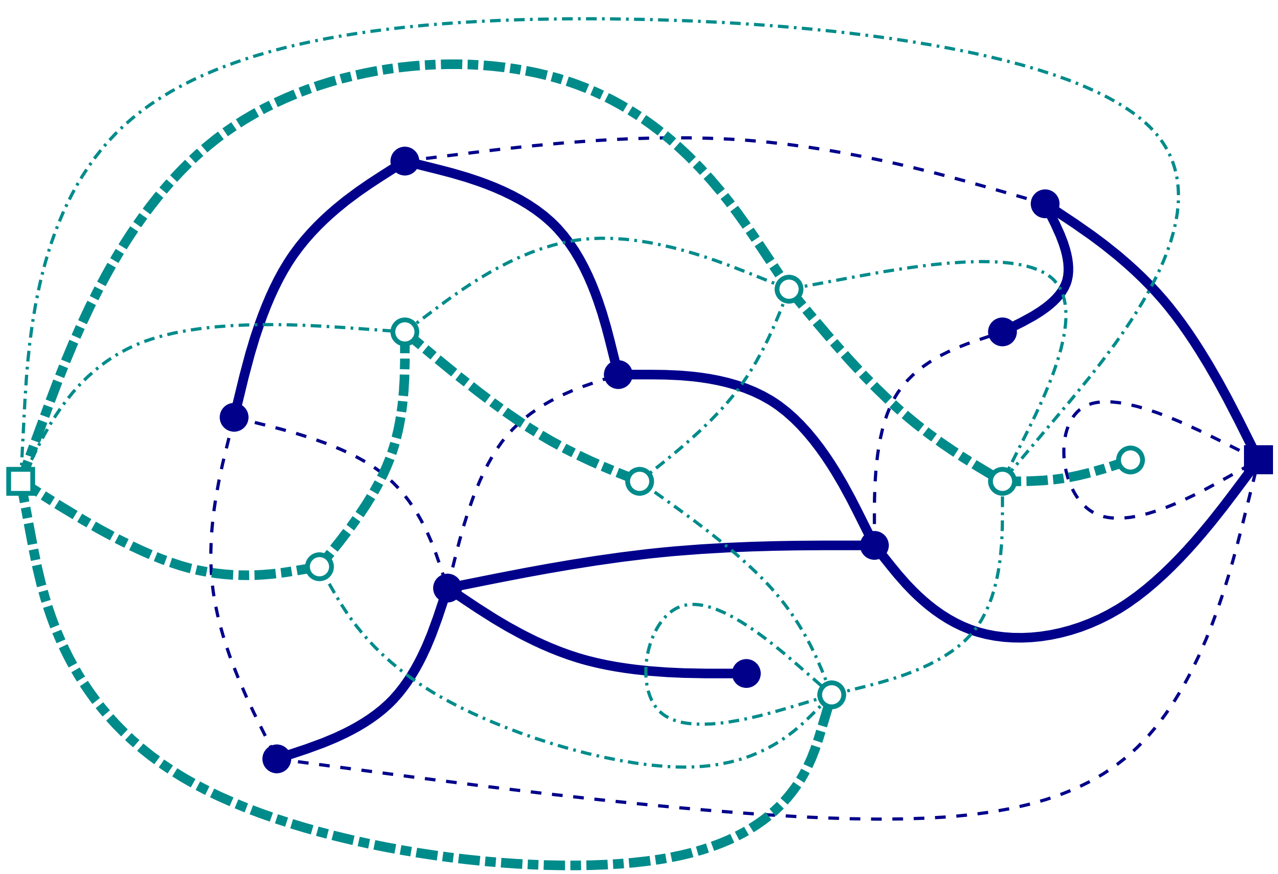

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$M^\star$ its dual map

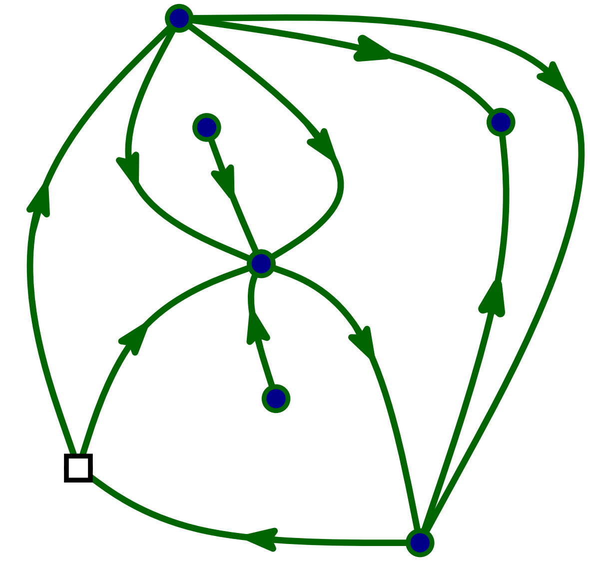
$V(M^\star) = F(M)$

$T^\star$ = complement of T

$\Longrightarrow T^\star$ = spanning tree of $M^\star$

α-Orientations — Definition

An **orientation** of a plane map is the choice of one orientation for each of its edges.



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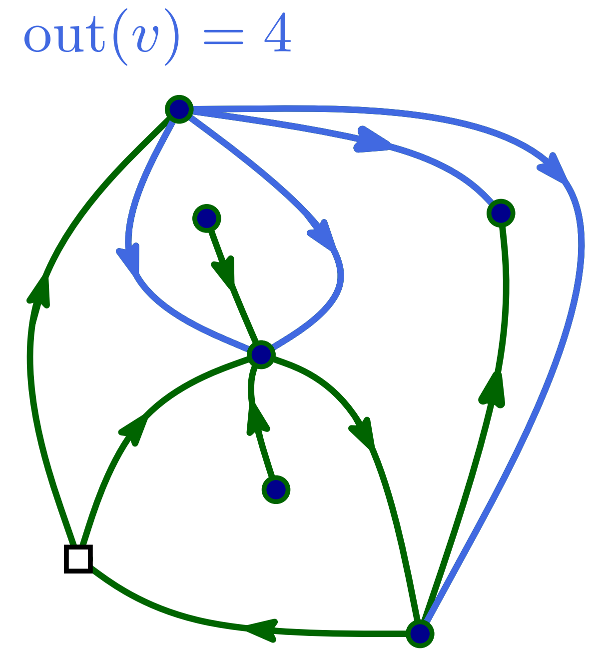
We choose to characterize an orientation by the outdegree of each vertex.

$\mathrm{out}(v)$ = outdegree of v



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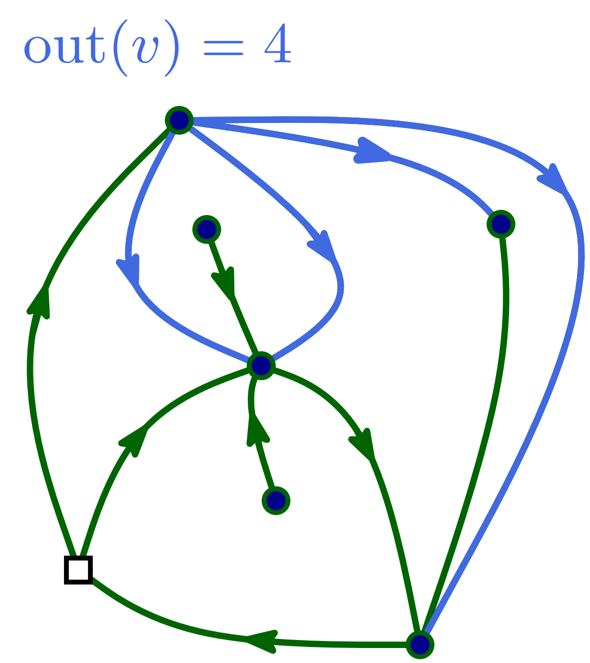

We choose to characterize an orientation by the outdegree of each vertex.

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Let $\alpha : V(M) \to \mathbb{N}$, an α-**orientation** is an orientation such that :

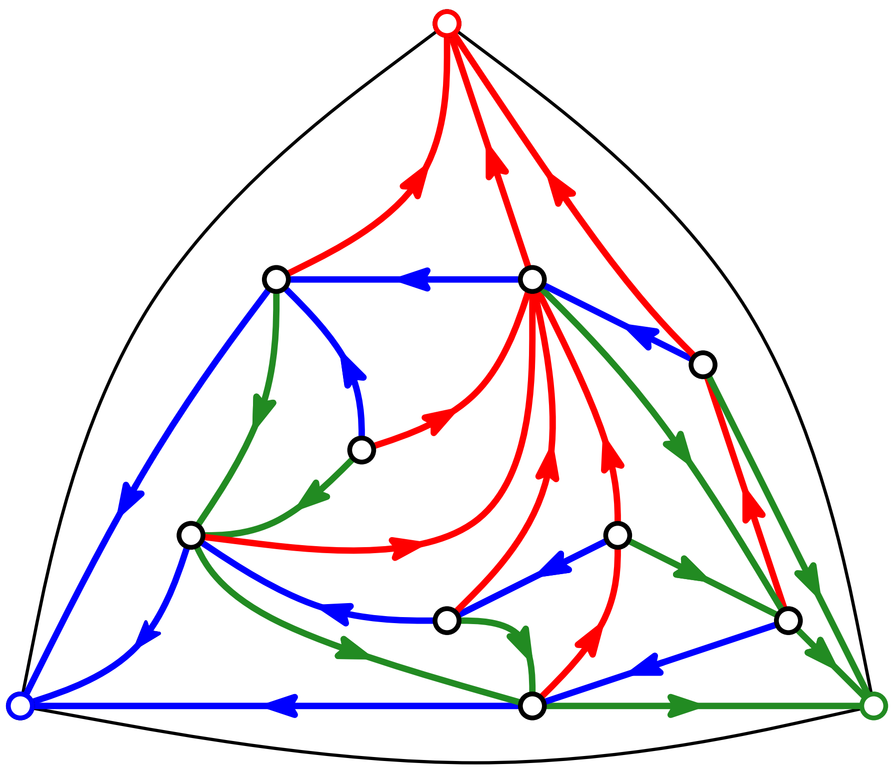
$$\text{out}(v) = \alpha(v), \text{ for all } v$$

[Propp '93], [Ossona de Mendez '94], [Felsner '04]

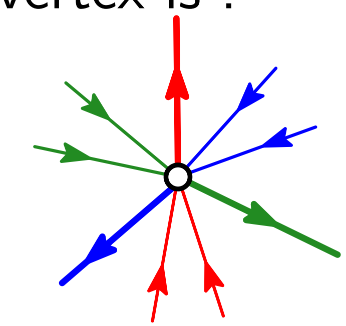
Why α-orientations ? Some motivations.

Schnyder woods [Schnyder '89] : Initial motivation.

More details in the coming lectures.



Orientation and coloring of the edges of a simple triangulation such that the local configuration around an inner vertex is :



The red (resp. blue or green) edges form a spanning tree of the inner vertices rooted at one outer vertex.

In particular $\mathrm{out}(v) = 3$ for any inner vertex v.

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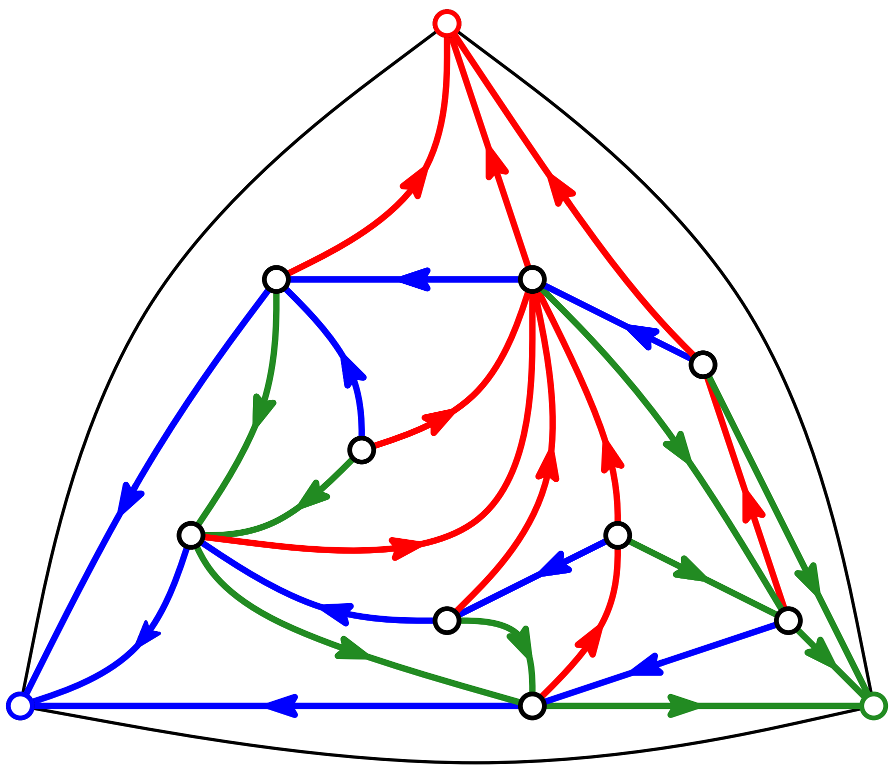

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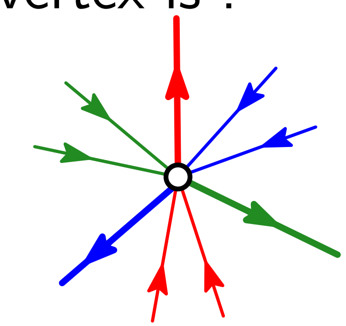

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Theorem :

Schnyder woods are in bijection with 3-orientations on a simple triangulation.

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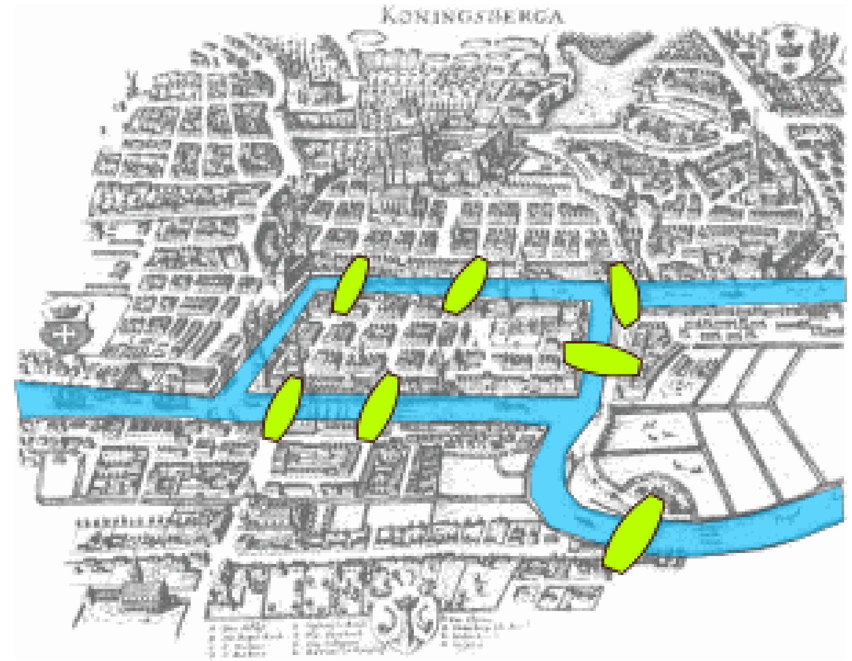
Eulerian orientations :

Eulerian tour of a graph : walk which takes once every edge and stops at its starting point.



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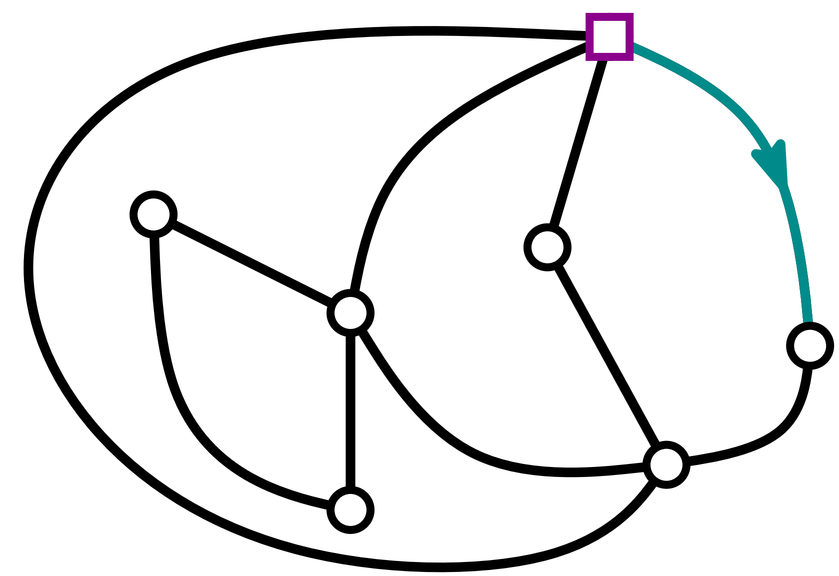

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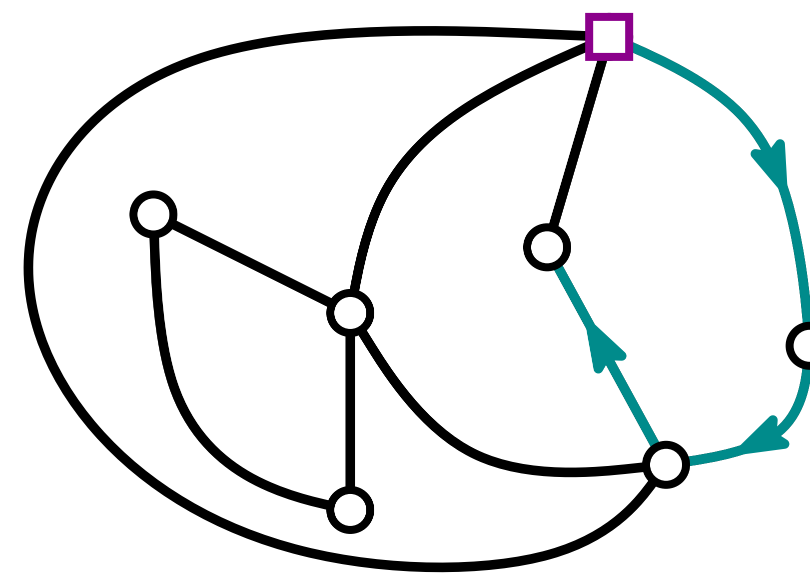

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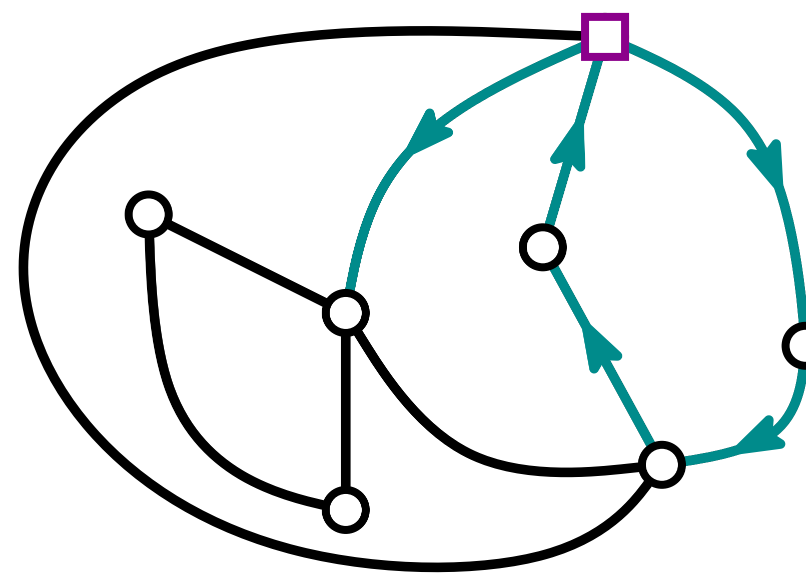

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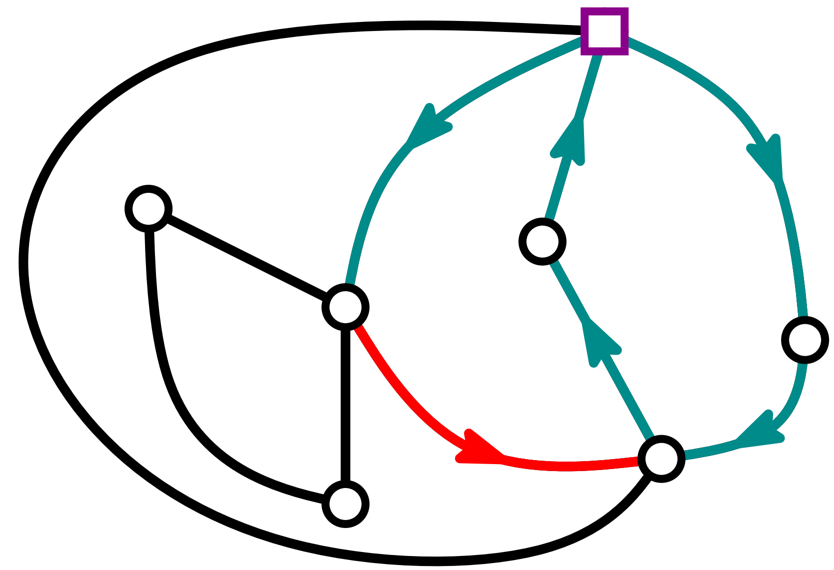

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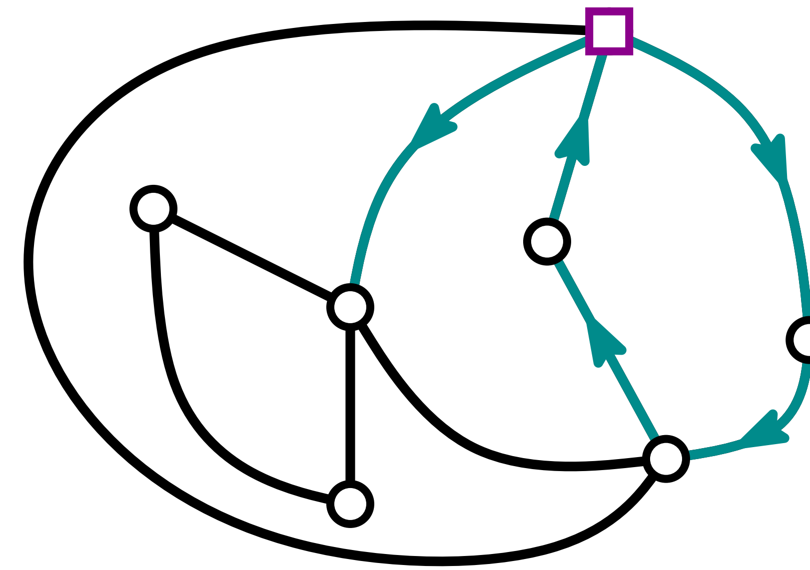

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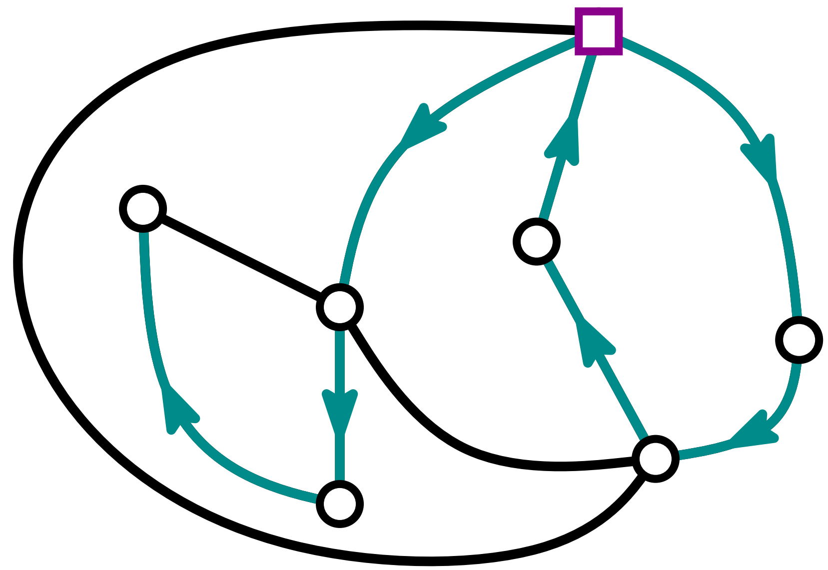

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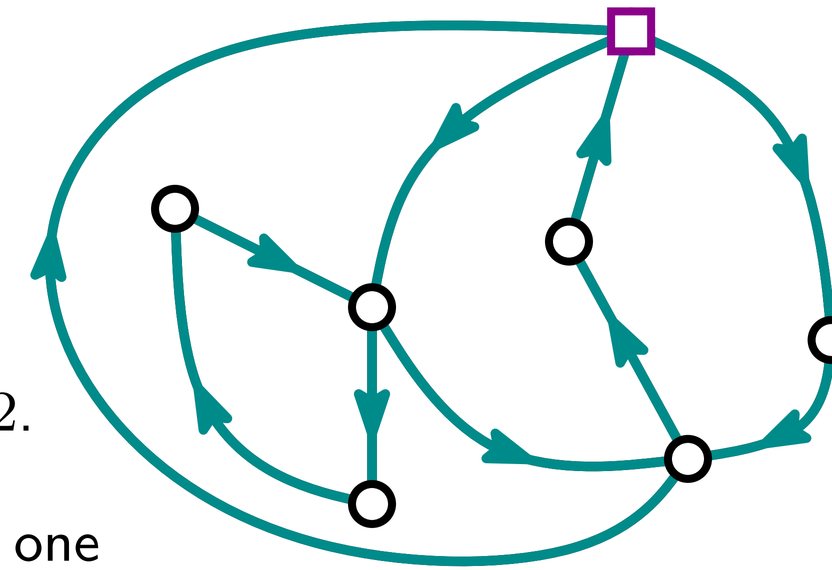

Eulerian tour of a graph : walk which takes once every edge and stops at its starting point.

Eulerian orientation : for any vertex v, $\mathrm{in}(v) = \mathrm{out}(v)$,
i.e. $\mathrm{out}(v) = \deg(v)/2$.

Each tour gives naturally birth to a Eulerian orientation : the one obtained by orienting the edges according to their direction in the tour.

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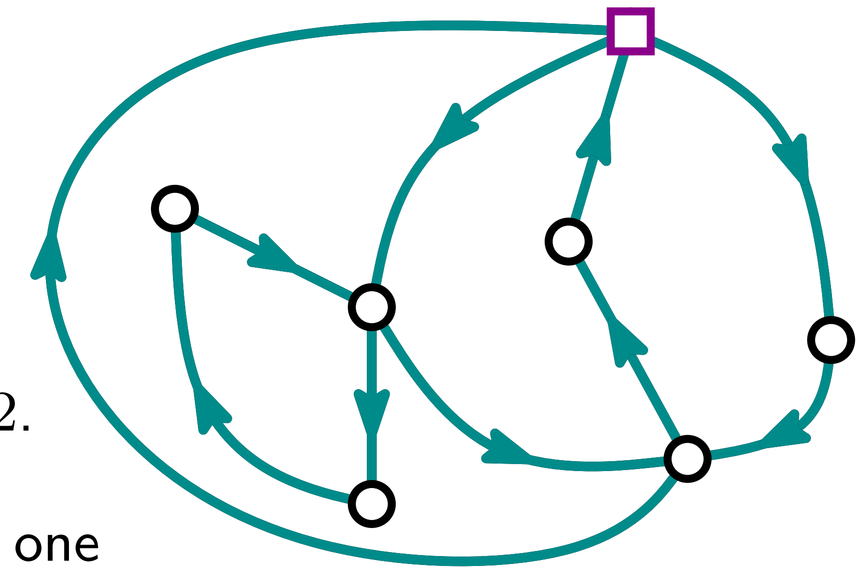

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Theorem : Euler (1759), Hierholzer (1873)

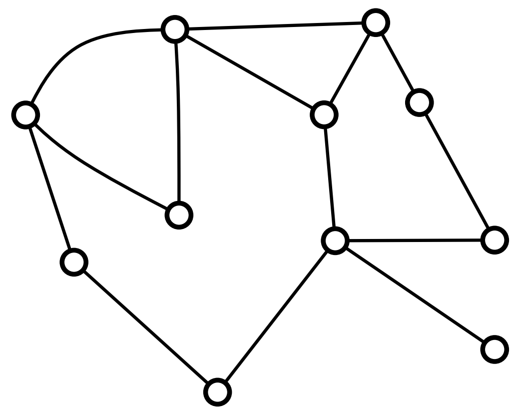
There exists a Eulerian tour for a connected graph iff it is Eulerian ($=$ even degree $\forall v$).

$\Rightarrow$ A graph admits a Eulerian orientation iff it is Eulerian.

Why α-orientations ? Some motivations.

Perfect matching in bipartite graphs :

Matching in a graph : set of edges such that each vertex belongs at most to one edge.



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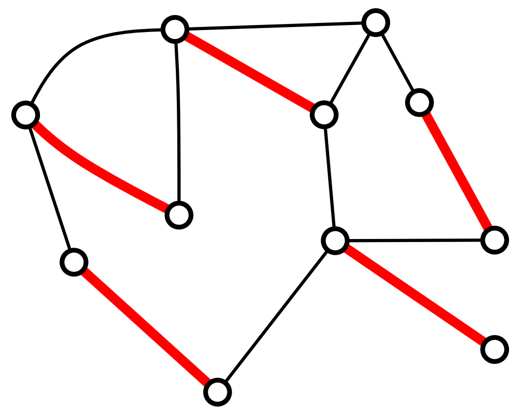

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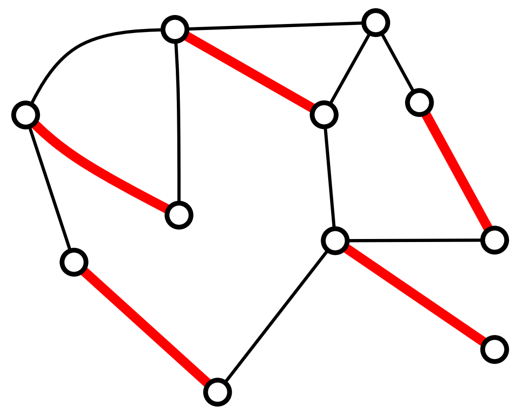

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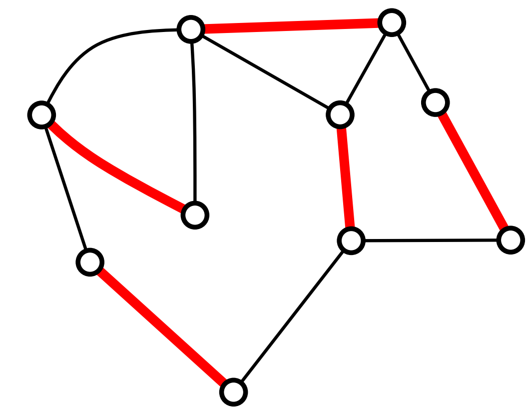

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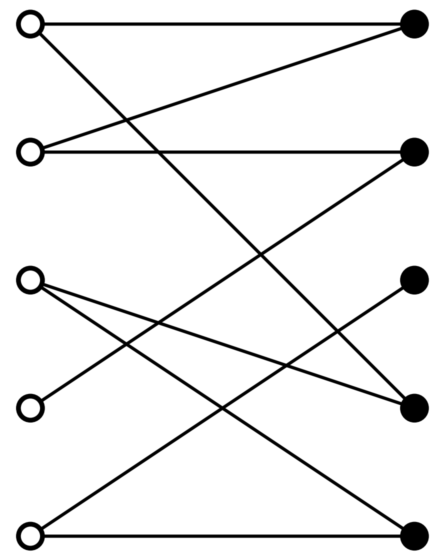

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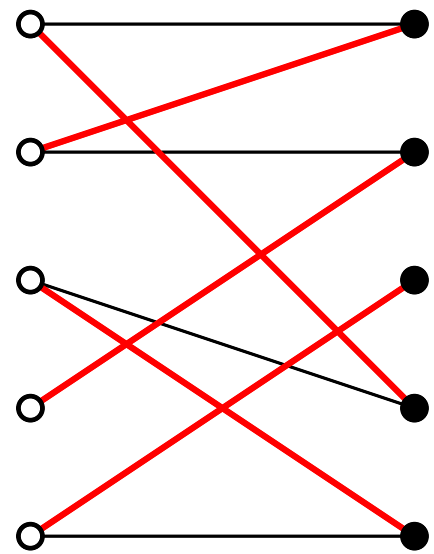

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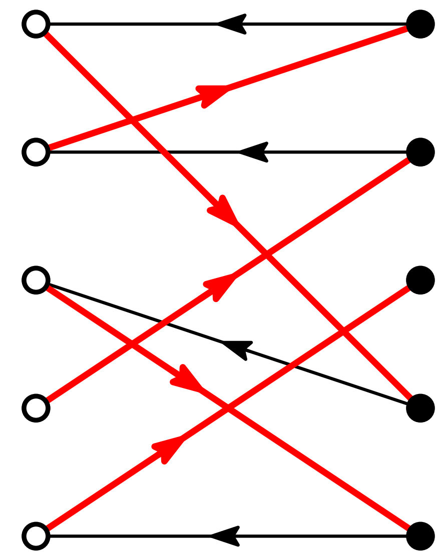

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$$\left.\begin{array}{l}\alpha(\bullet) = \deg - 1 \\ \alpha(\circ) = 1\end{array}\right\}$$ The perfect matchings of a bipartite graph and its α-orientations are in bijection.

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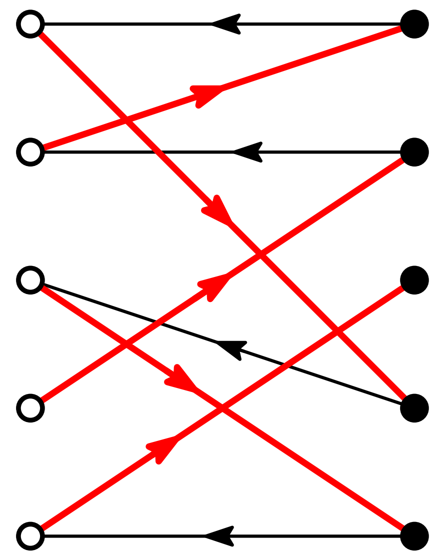

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Does a perfect matching always exist ?

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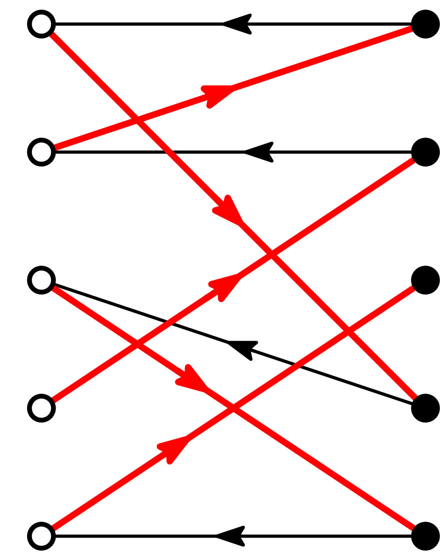

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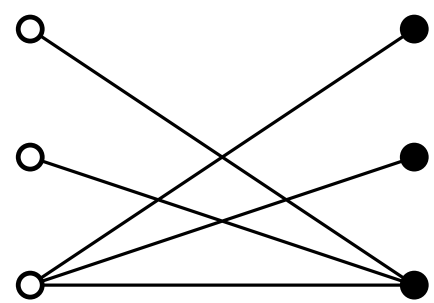

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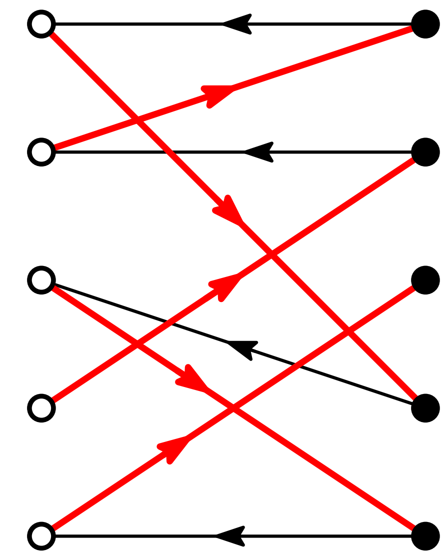

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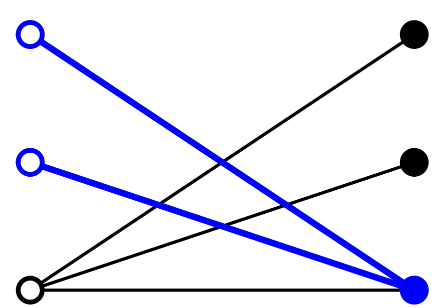

Perfect matching ?
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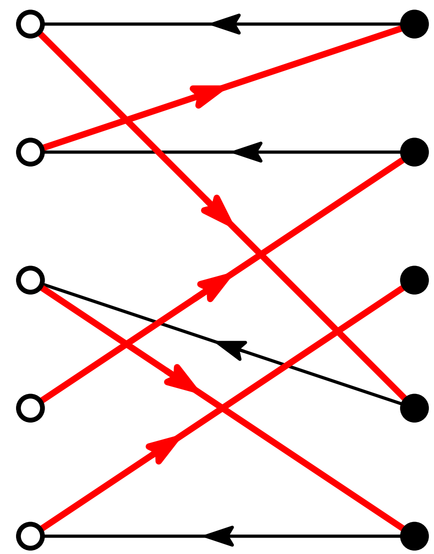

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Theorem : Hall (1935)

A bipartite graph admits a perfect matching iff $\forall$ subset W of white vertices,

$$|W| \leq \left| \cup_{w \in W} \{ \text{ neighbours of } w \} \right|$$

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Existence of orientations : necessary conditions

plane map M
$\alpha : V(M) \to \mathbb{N}$ ⟶ α is **feasible** iff $\exists$ an α-orientation on M

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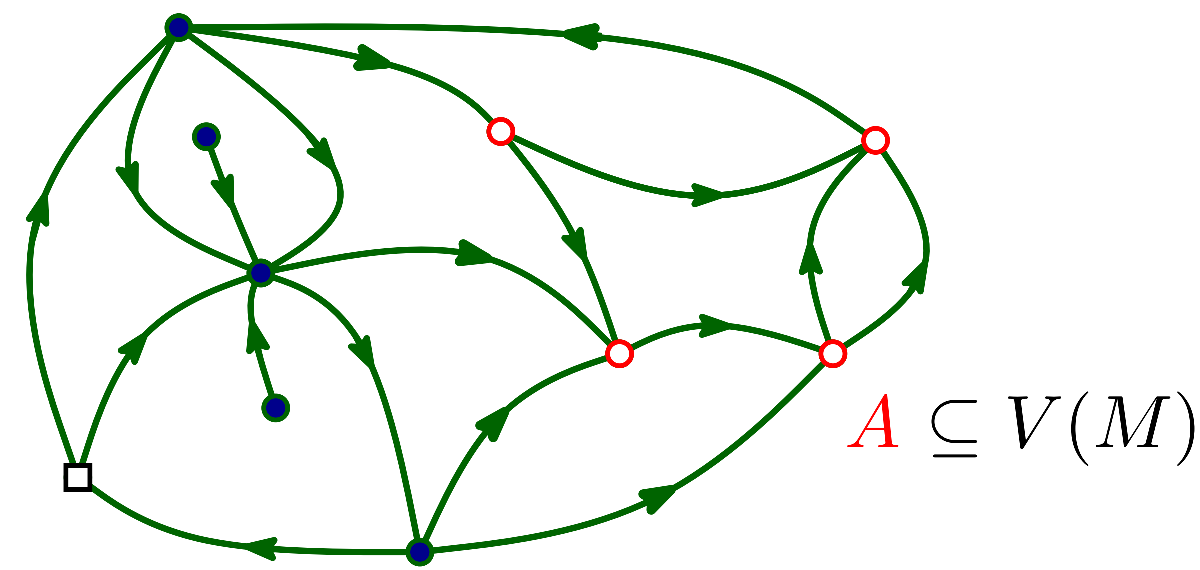

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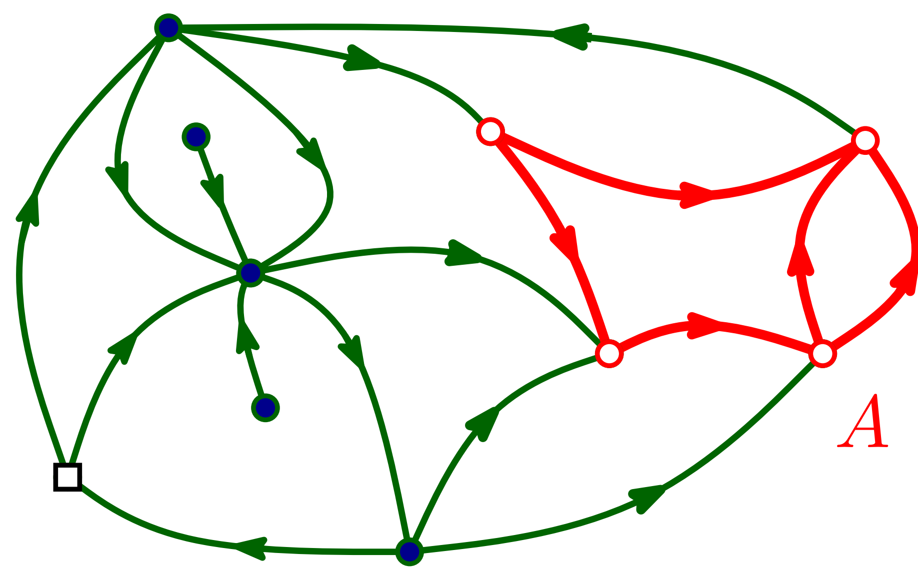

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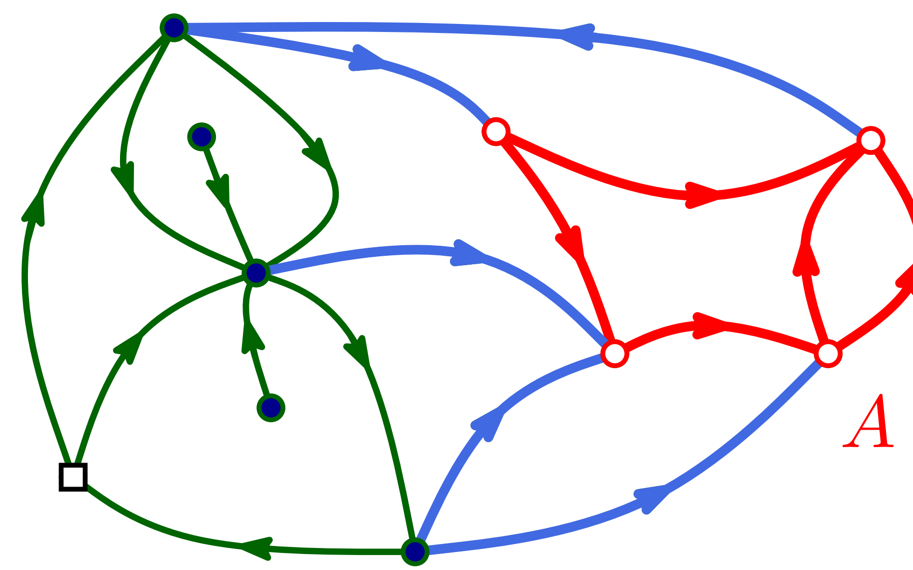

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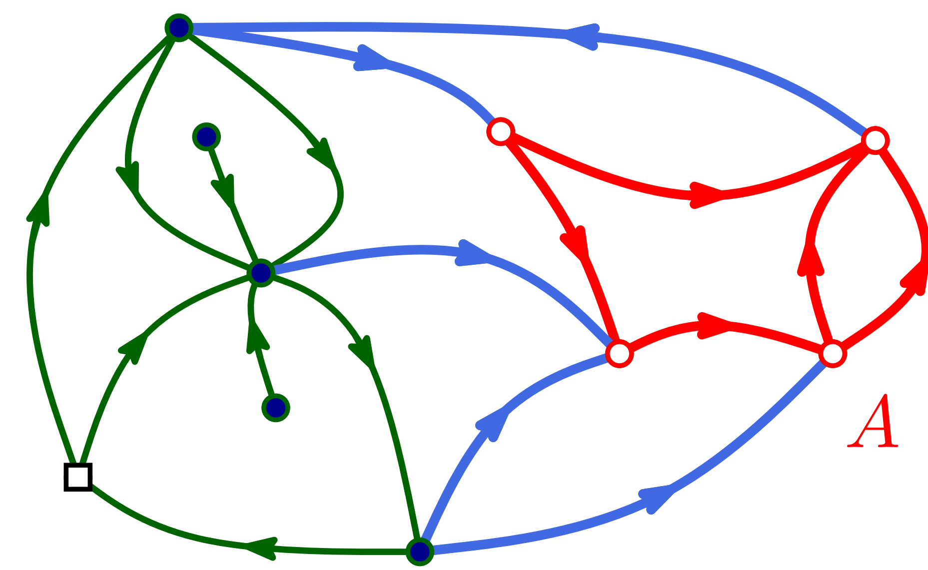

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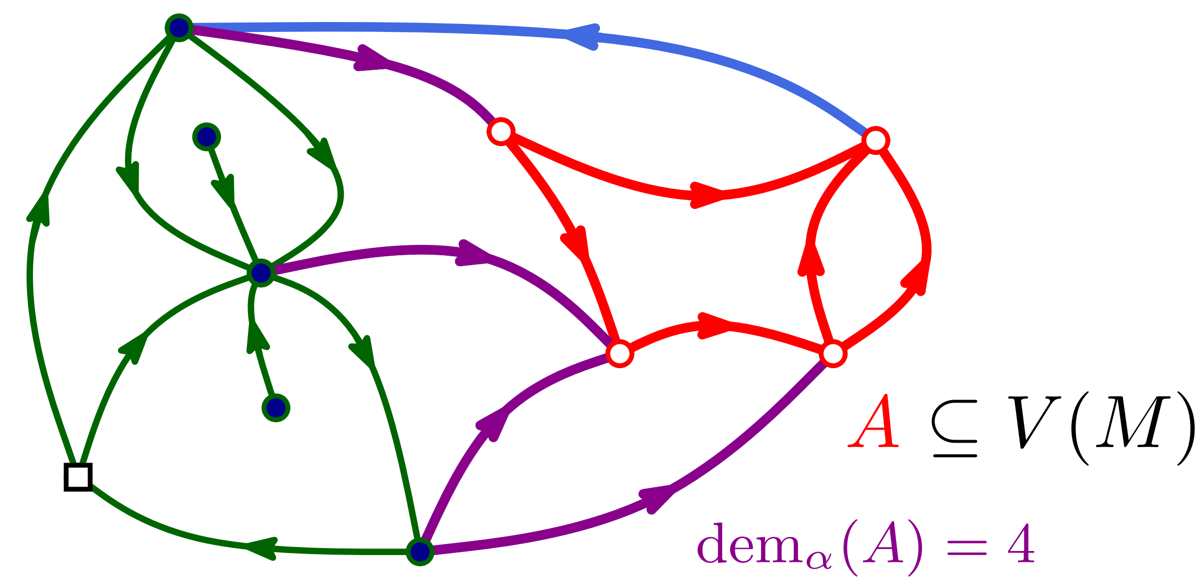

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$\Leftrightarrow$

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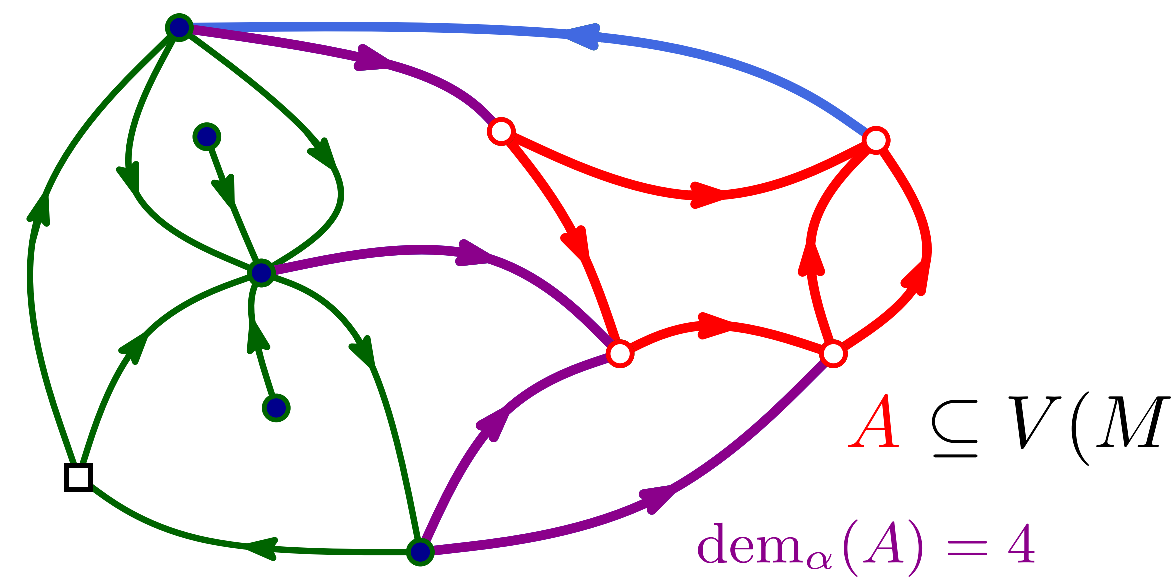

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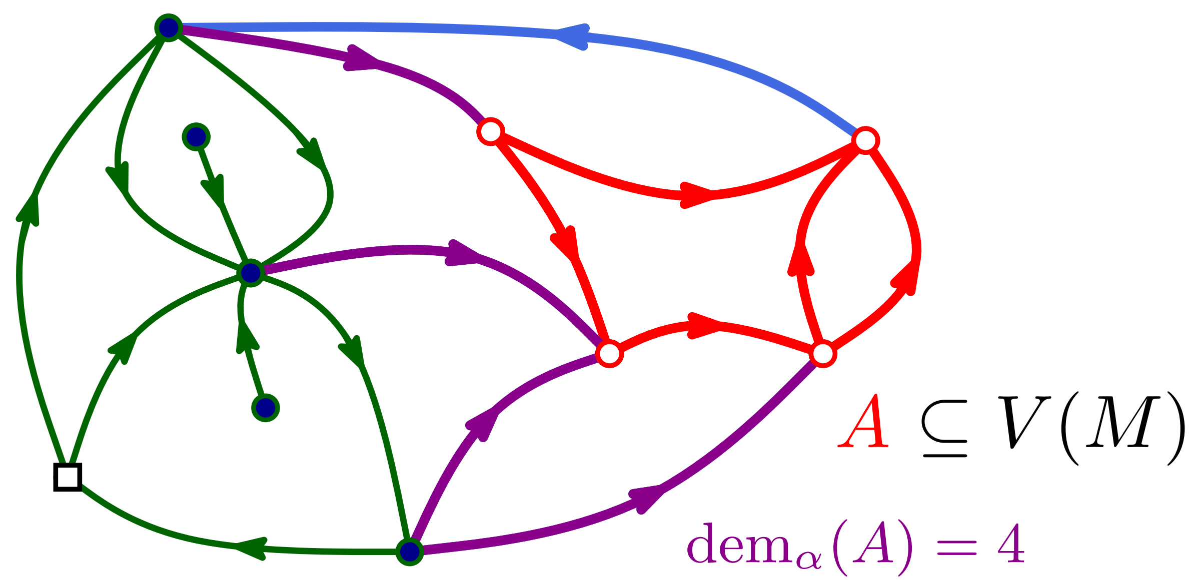

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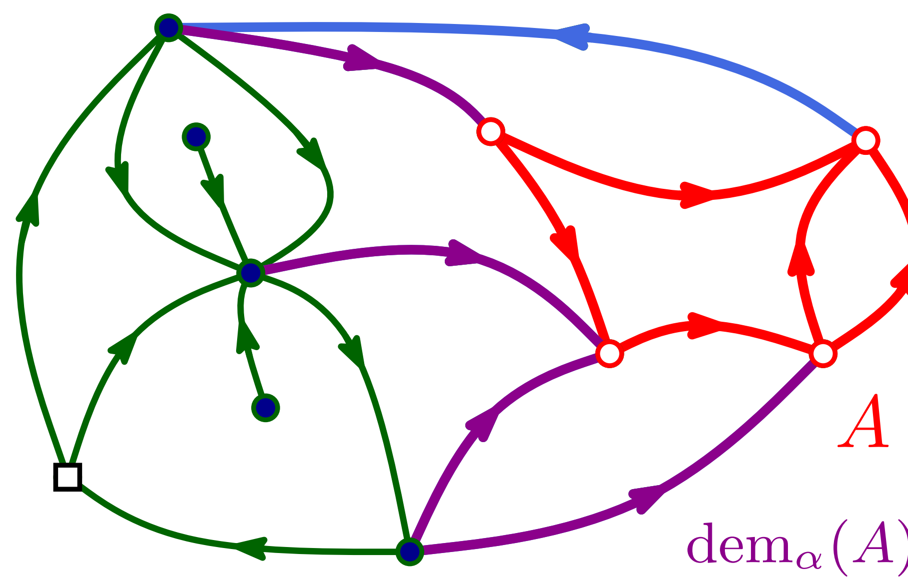
Theorem :

Those conditions are sufficient.



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Existence of orientations : sufficient conditions

Theorem :

Let M and α be such that :

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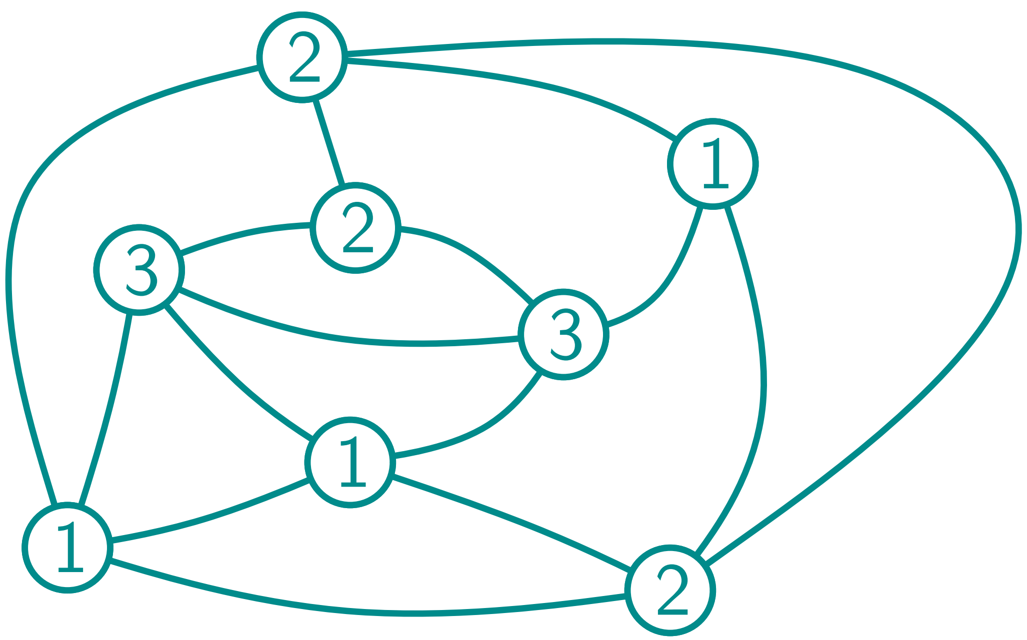
2 - For all $A \subsetneq V(M)$, $0 \leq \mathrm{dem}_\alpha(A) \leq |E_{\mathrm{cut}}[A]|$.

then α is **feasible**.

Proof (by example) :



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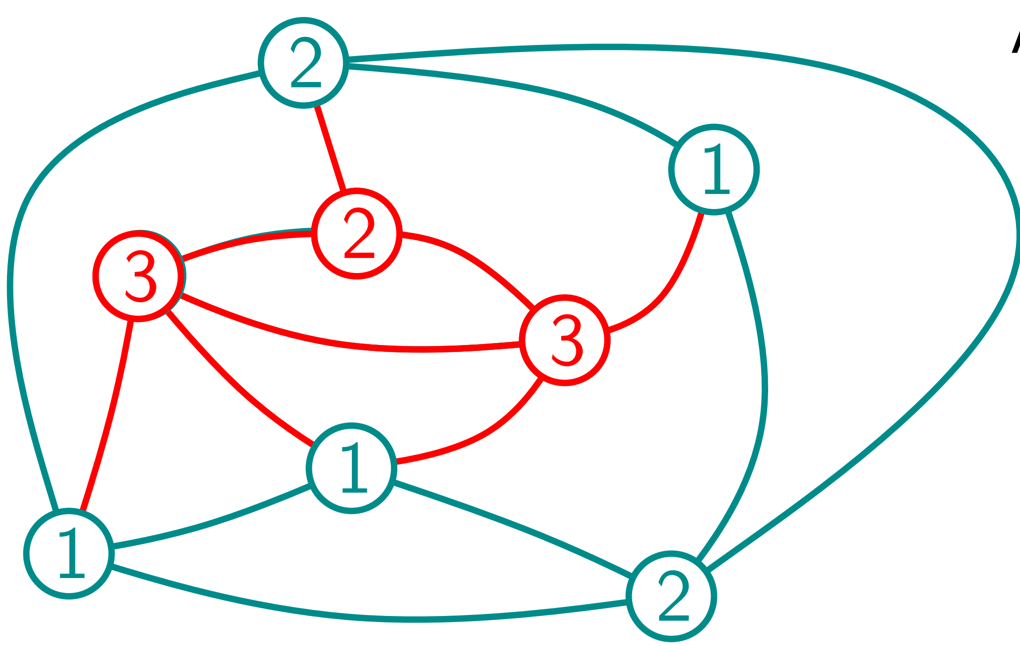

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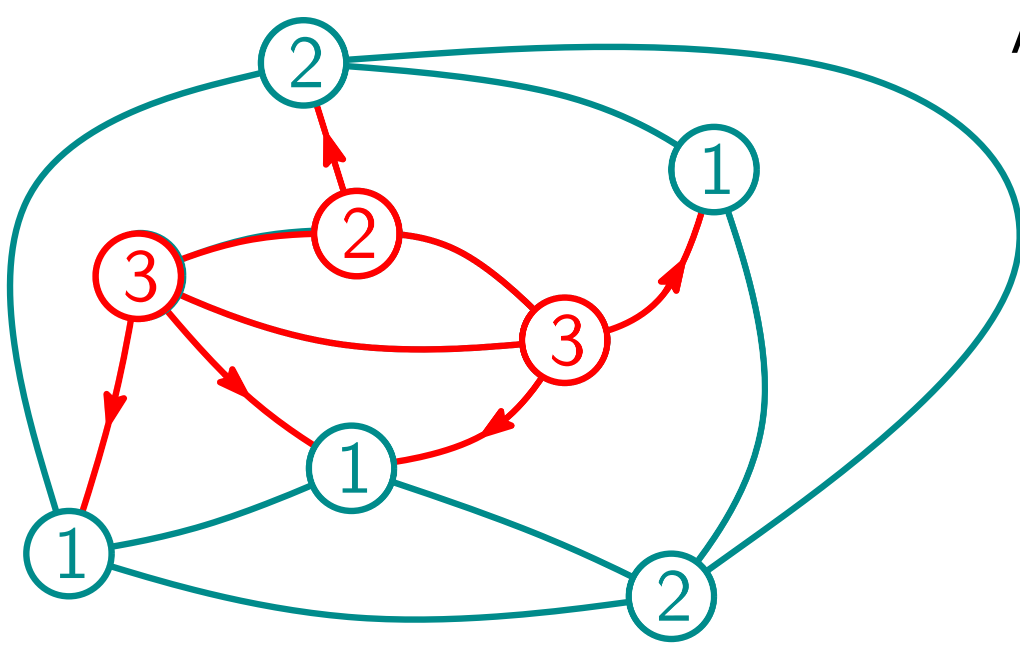

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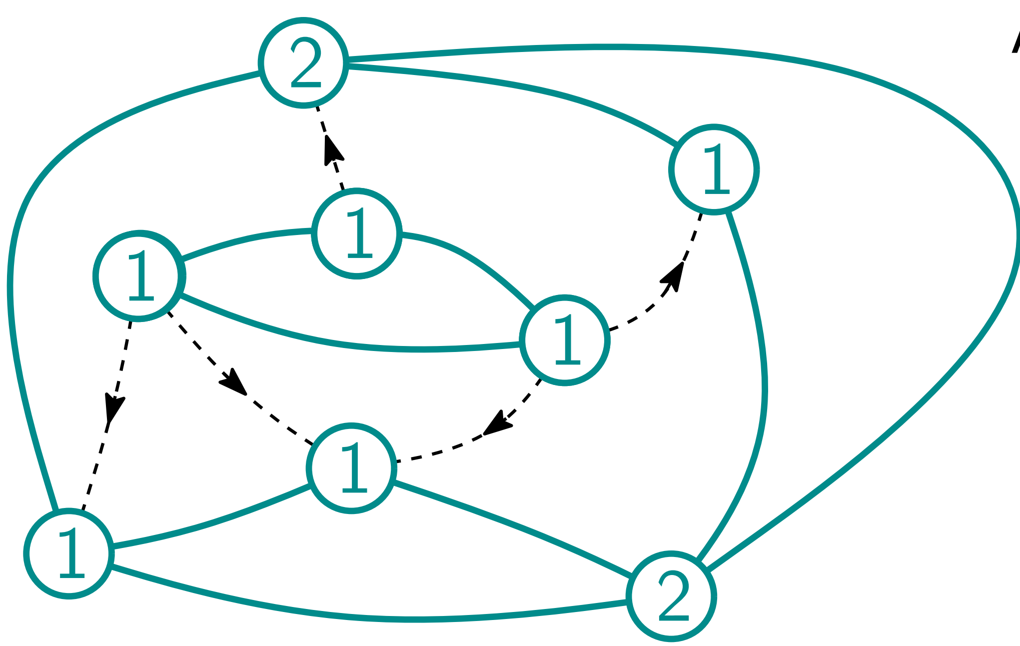

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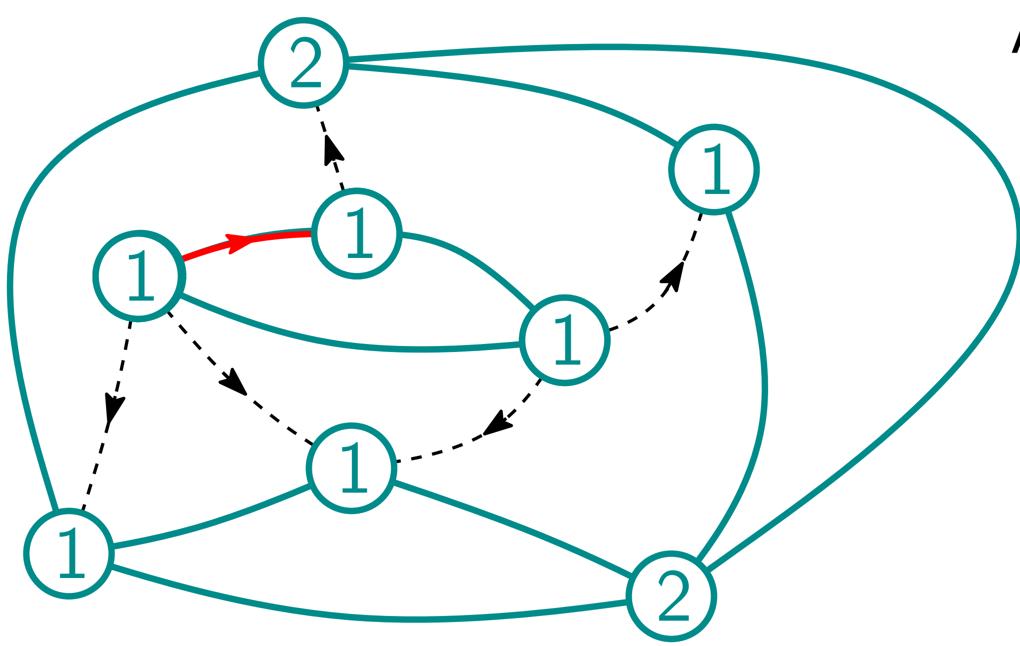

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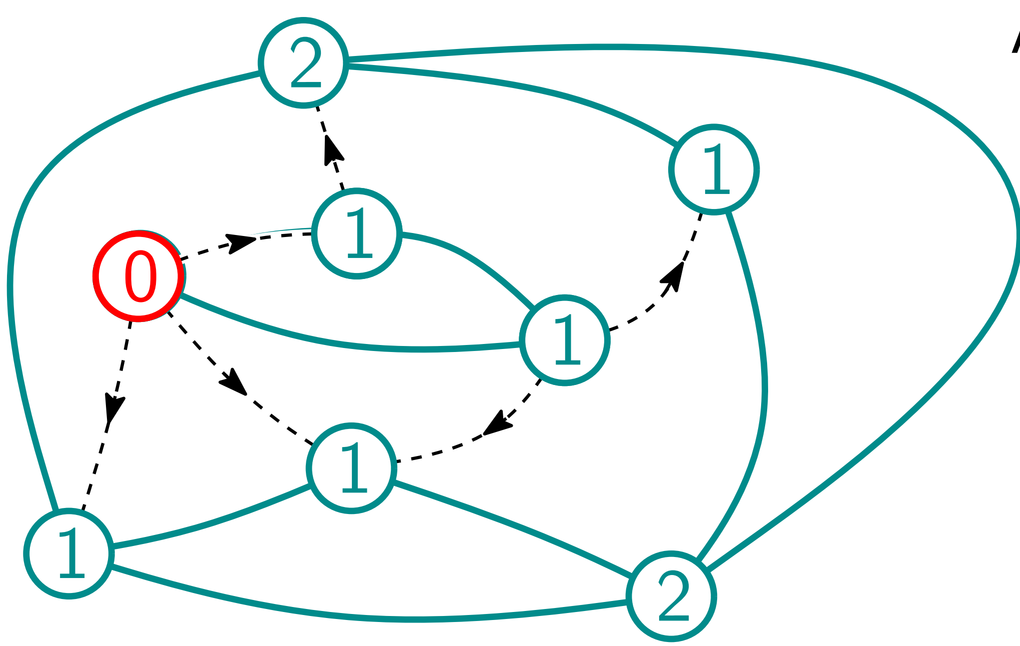

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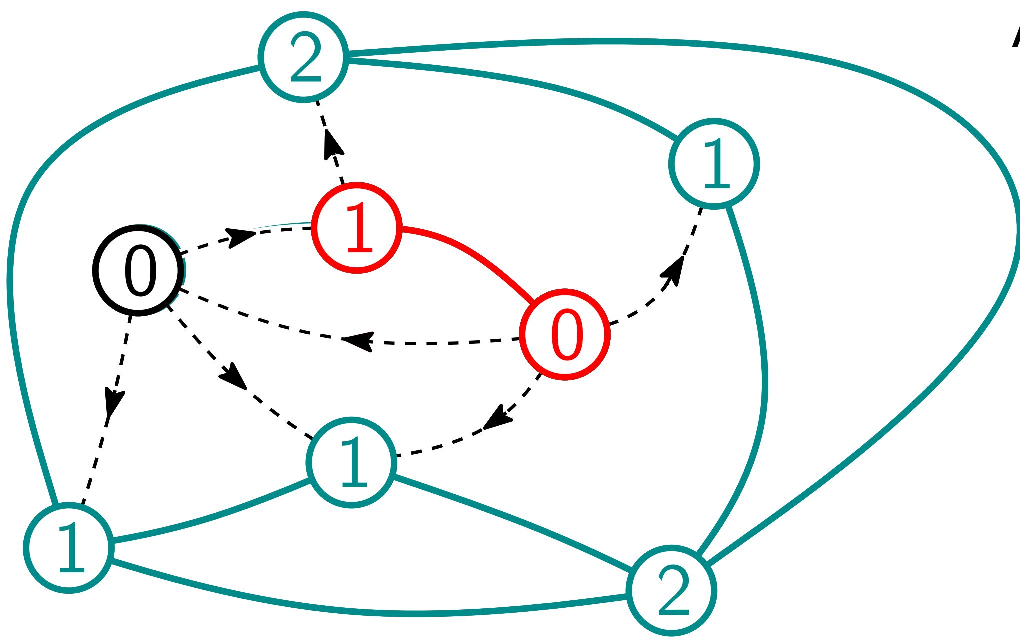

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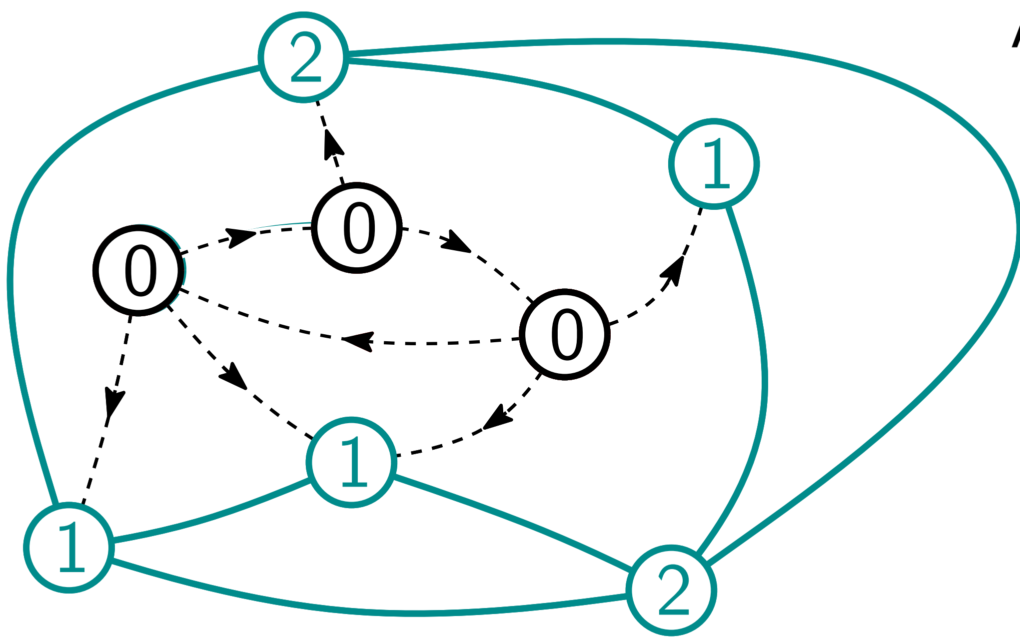

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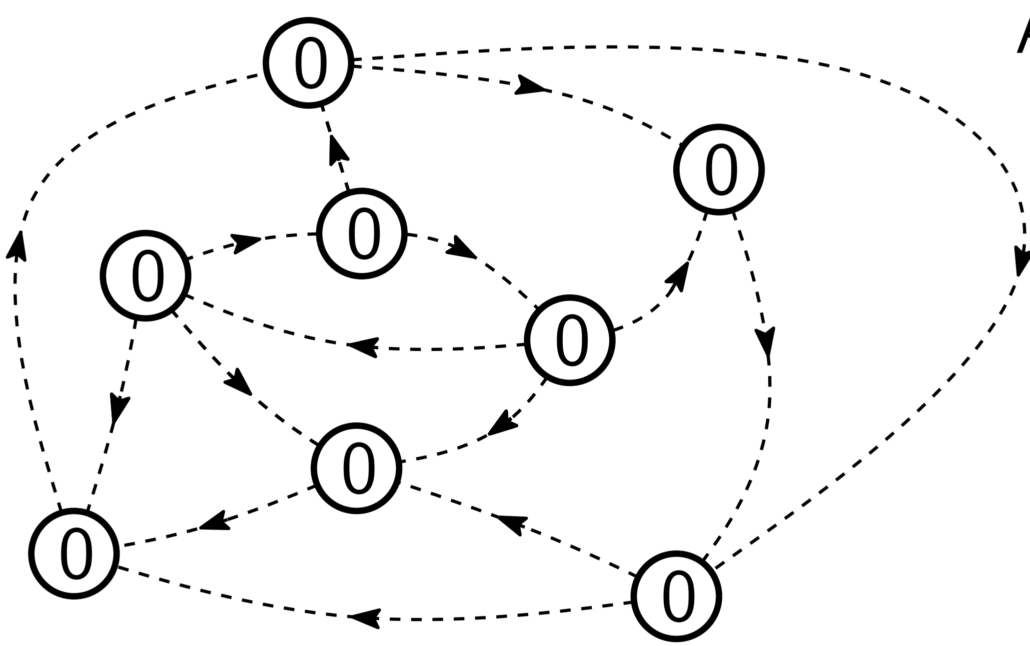

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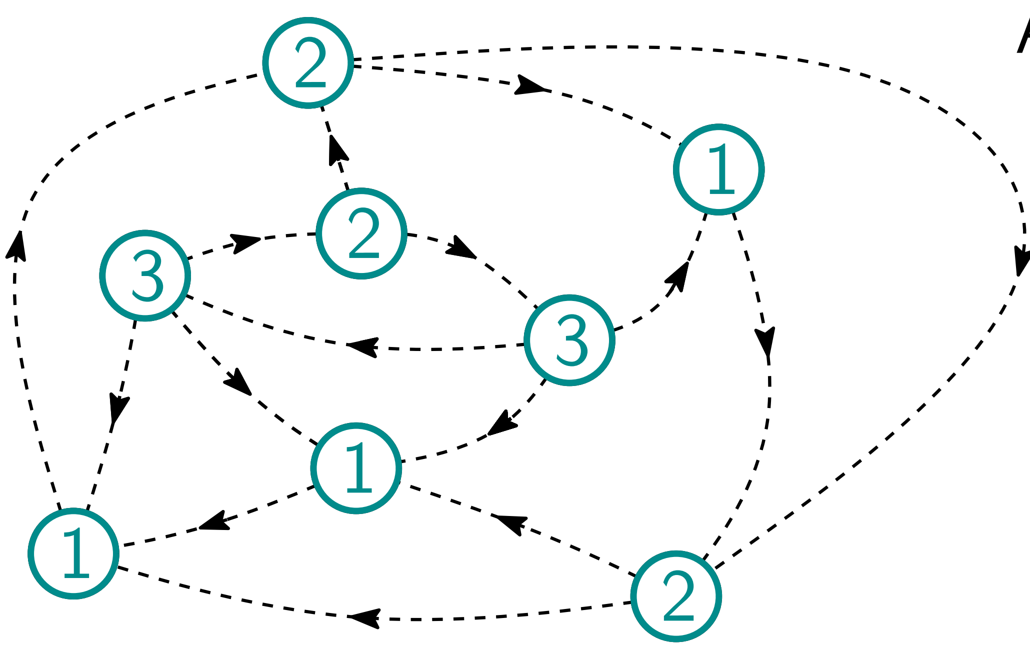

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$$\left.\begin{array}{l}\text{a plane map } M \\ \text{a feasible } \alpha\end{array}\right\}$$ What can we say about the α-orientations ?

Set of α-orientations

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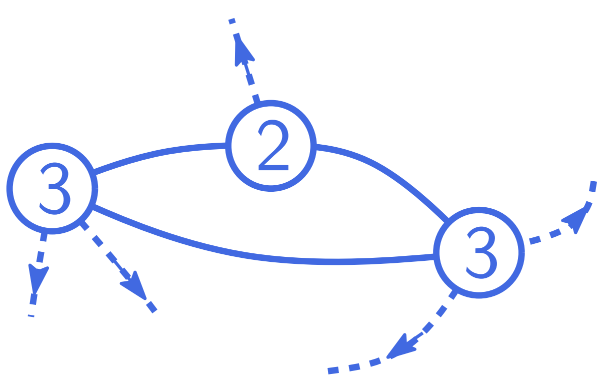
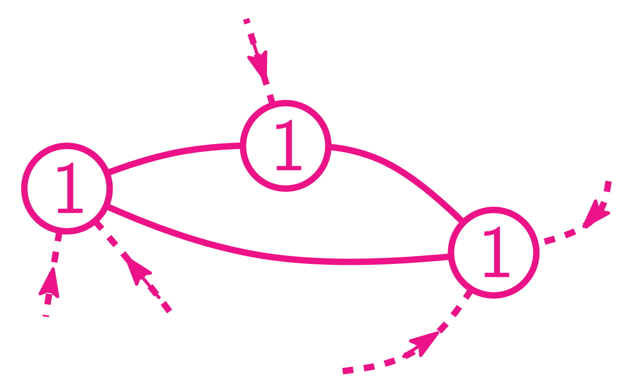
1 - Rigid edges

if there exists A such that or $\begin{cases} \mathrm{dem}_\alpha(A) = 0 \\ \mathrm{dem}_\alpha(A) = |E_{\mathrm{cut}}(A)| \end{cases}$

the edges of $E_{\mathrm{cut}}(A)$ are **rigid** (= no choice for their orientation)





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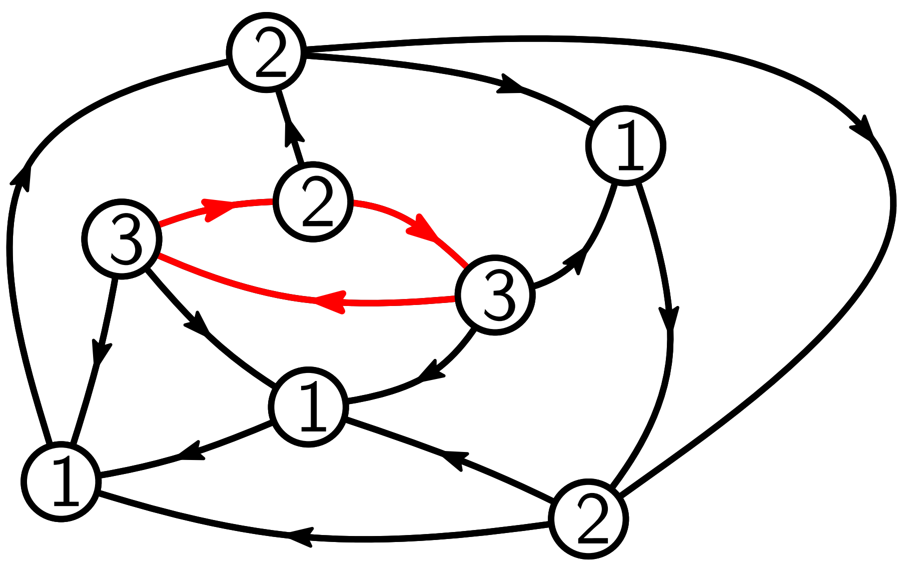

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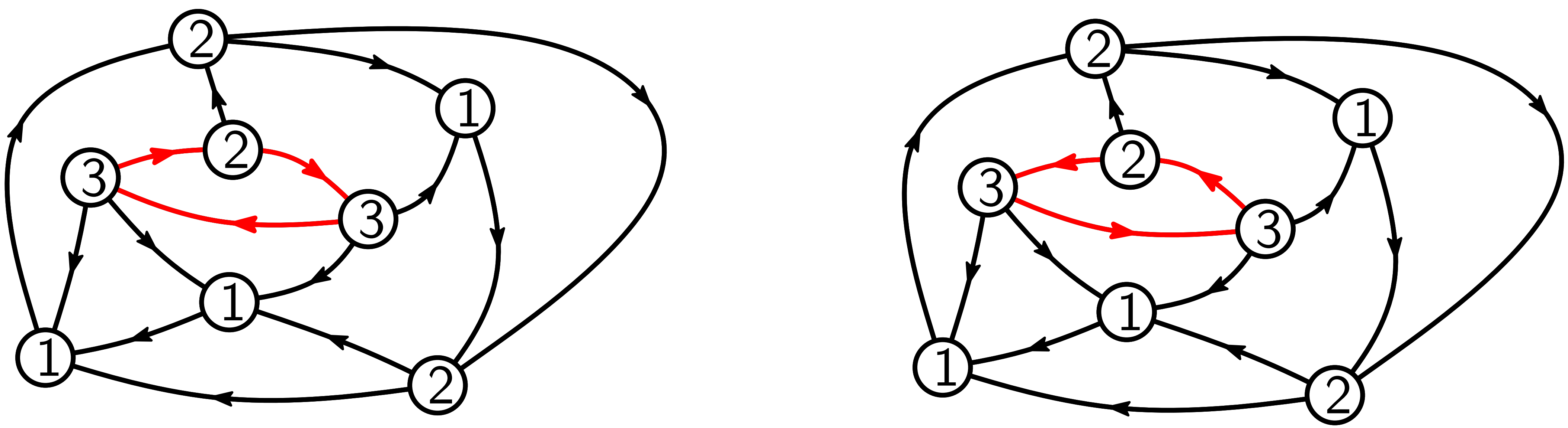

The flip on an oriented cycle gives a new α-orientation.

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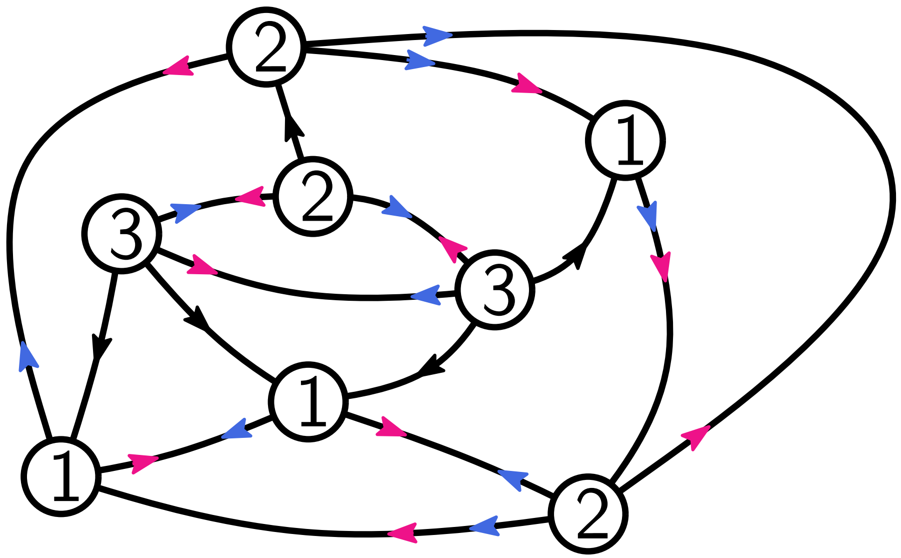

rigid edges in black
α-orientations : O_1 and O_2

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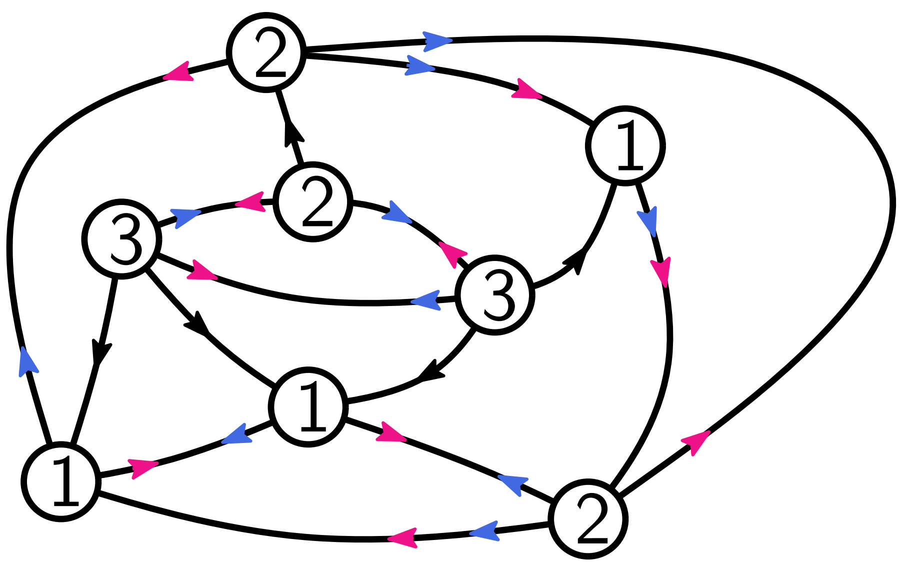

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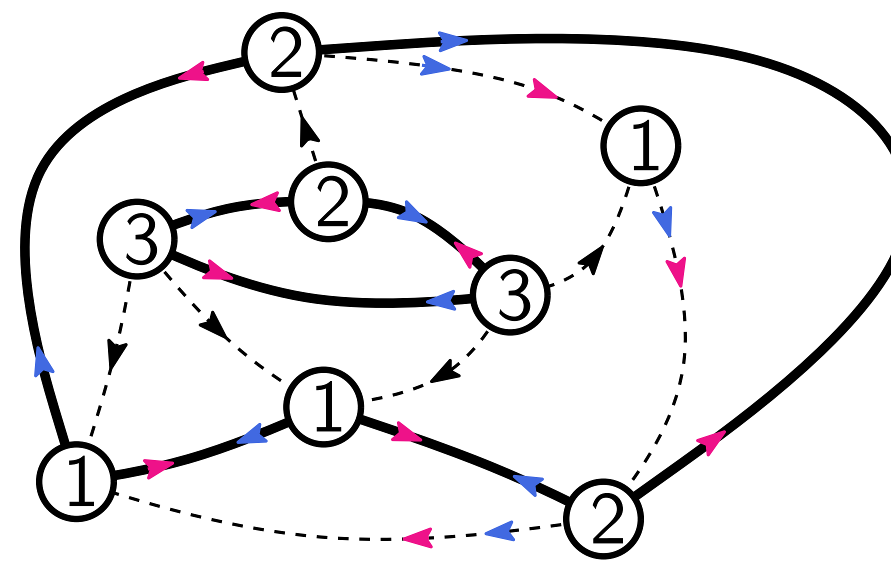

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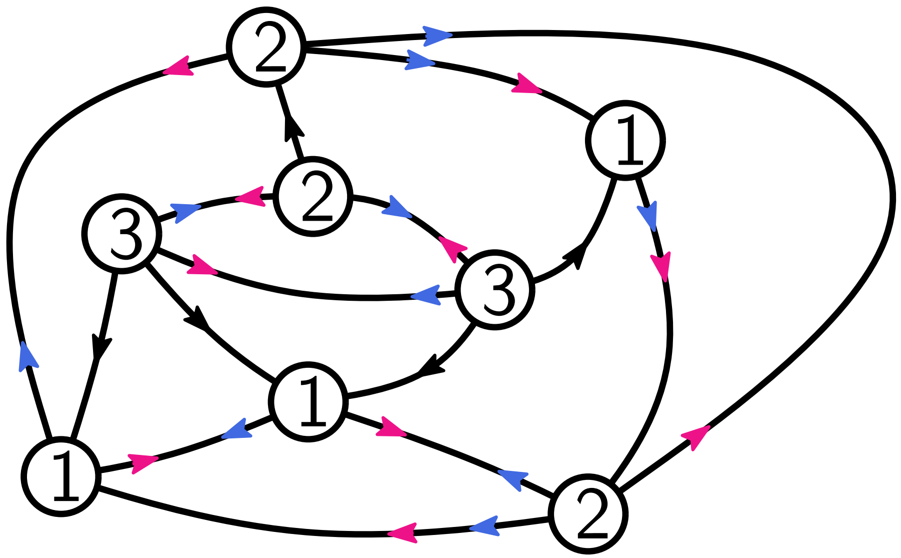

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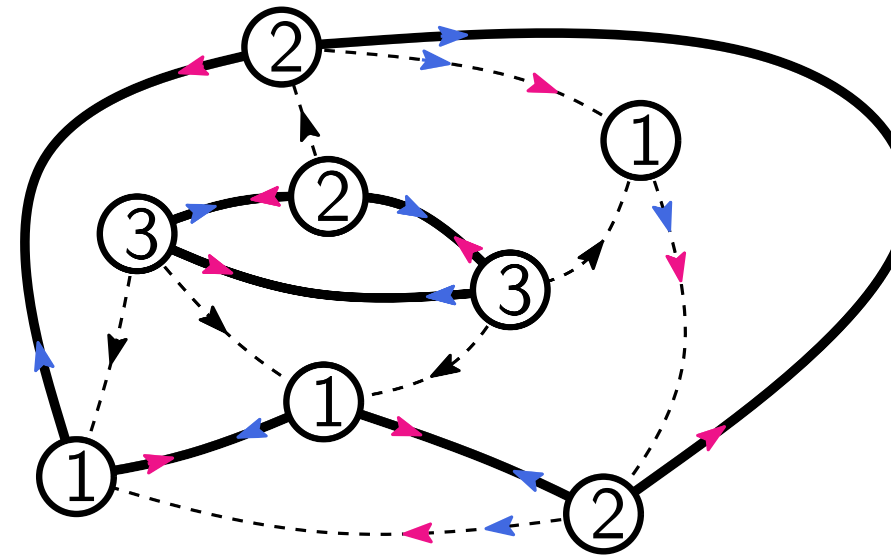

Property :

All vertices have even degree.
(i.e. it is a union of cycles)

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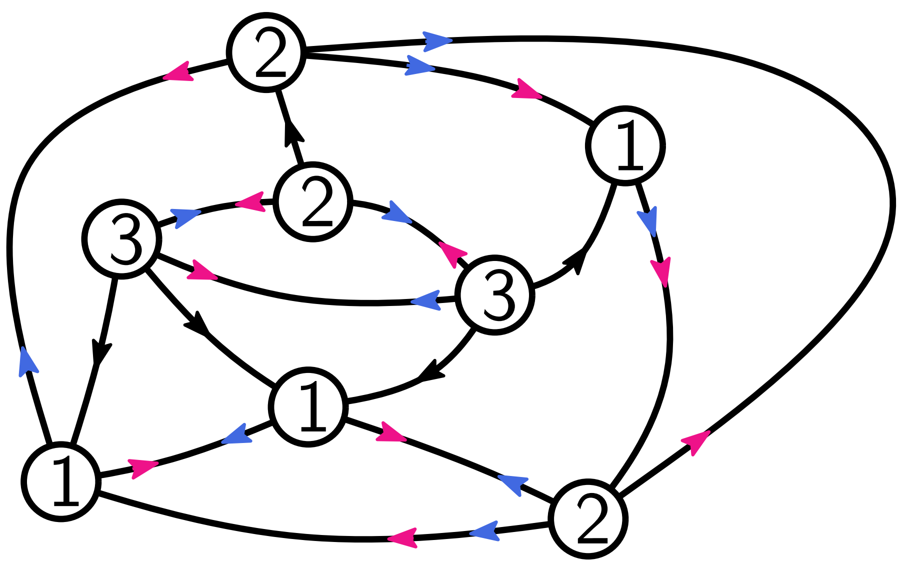

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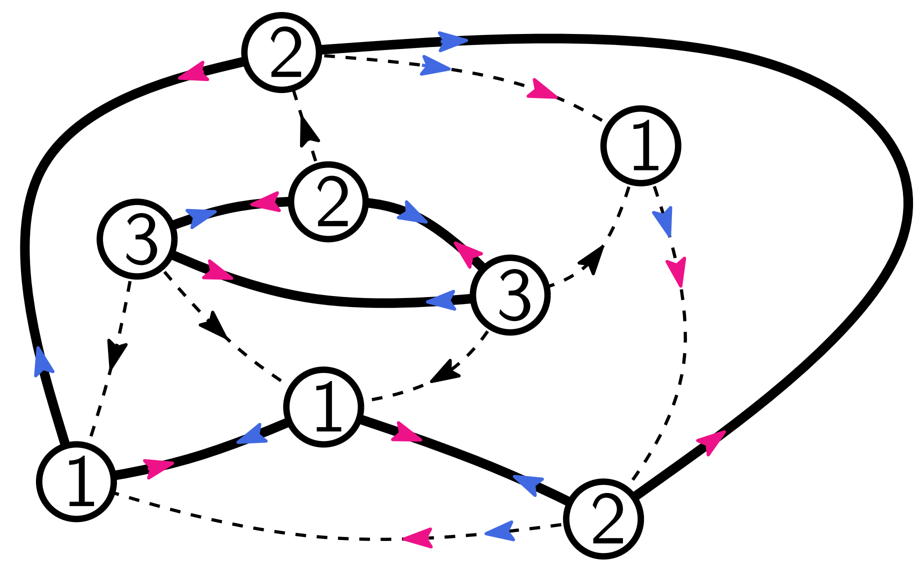

Theorem :

We can go from one α-orientation to another by a sequence of flips of directed cycles.

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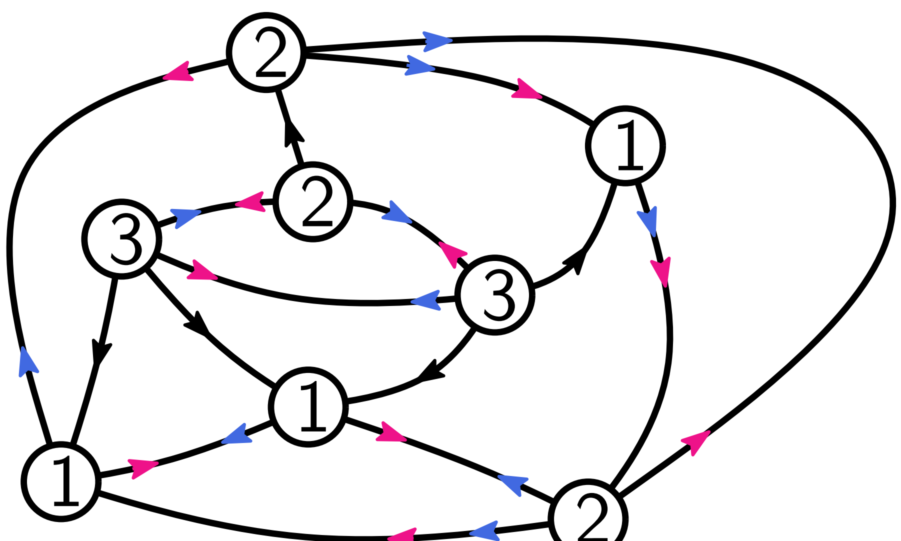

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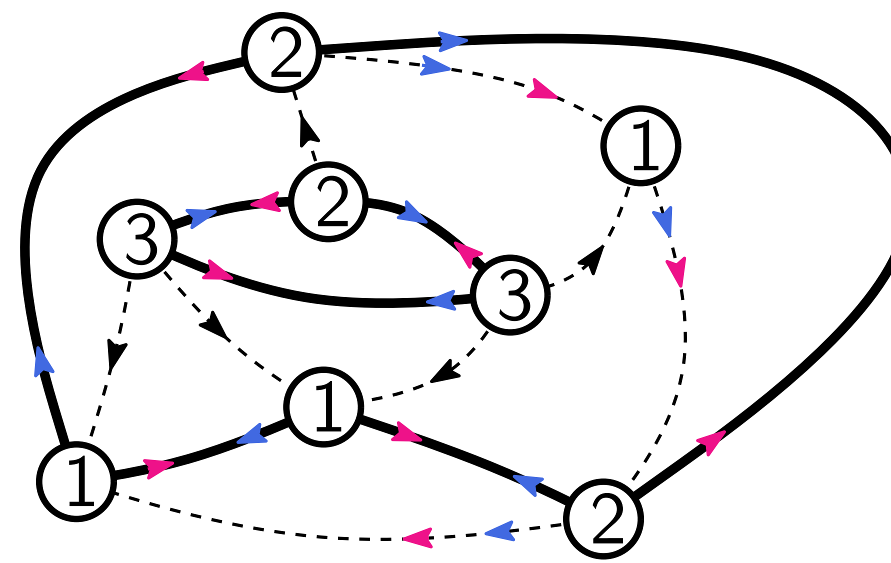

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We can go from one α-orientation to another by a sequence of flips of directed cycles.

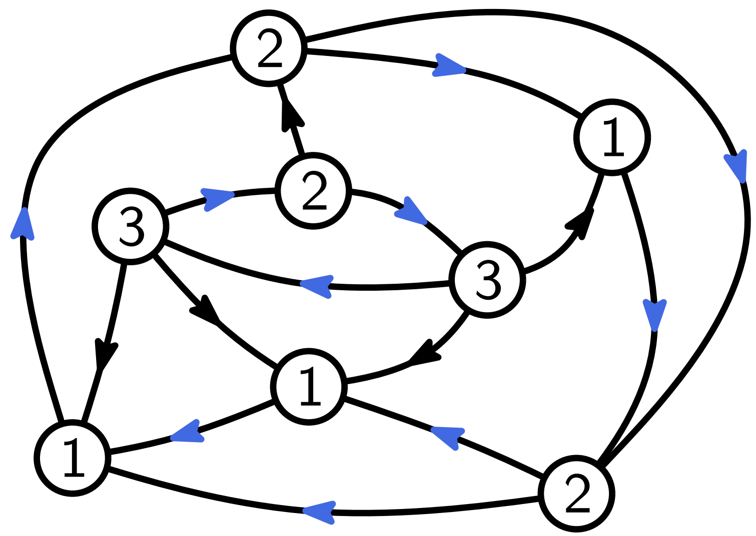
Essential cycles

Theorem :

We can go from one α-orientation to another by a sequence of successive flips of directed cycles.

A cycle C is **essential** iff :

- C is simple and chordless
- if $E_{\mathrm{cut}}[I_C]$ is rigid ($I_C =$ intérieur de C)
- $\exists$ an α-orientation in which C is a directed cycle.



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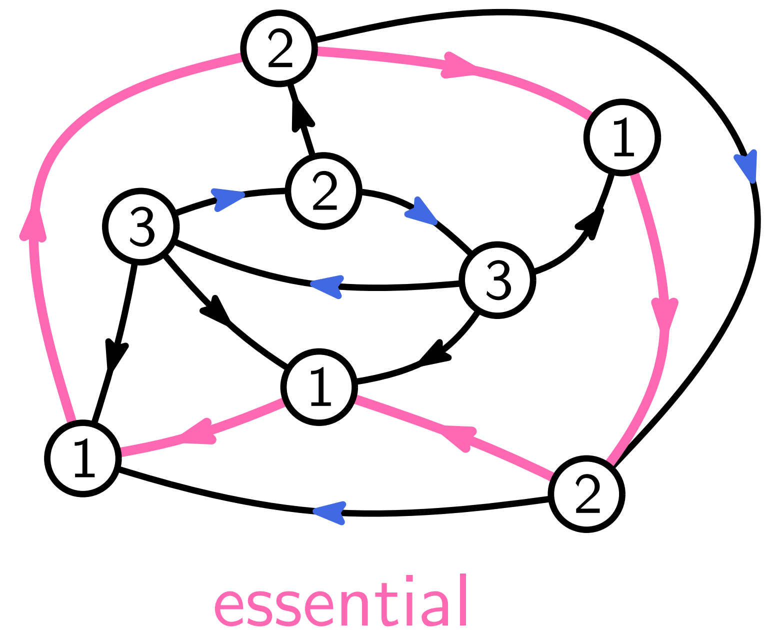

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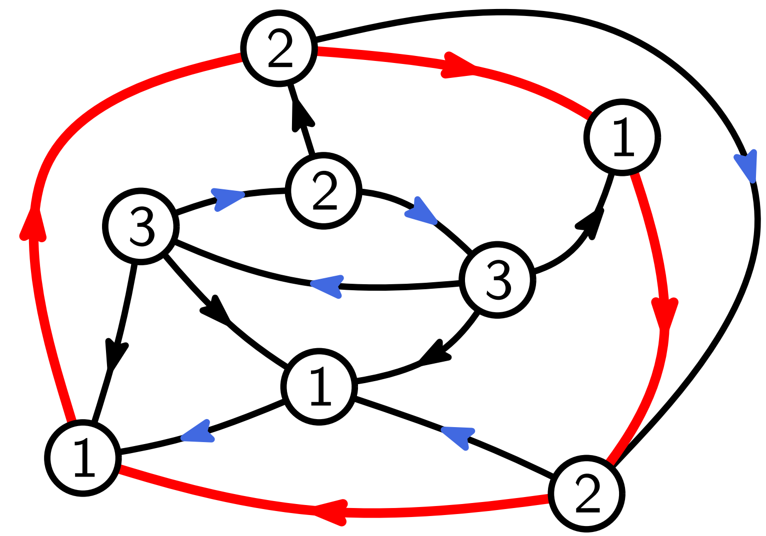

non-essential

Essential cycles

Theorem (Felsner '04) :

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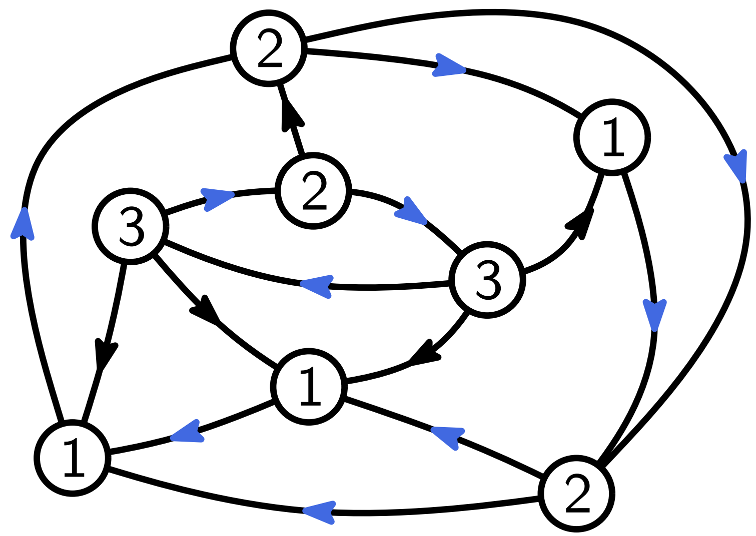

Essential cycles

Theorem (Felsner '04) :

We can go from one α-orientation to another by a sequence of successive flips of directed **essential** cycles (= **flips/flops**).

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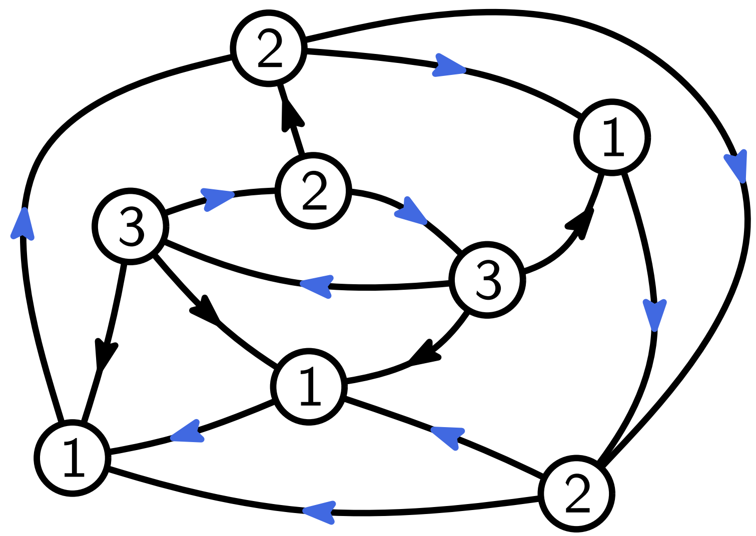

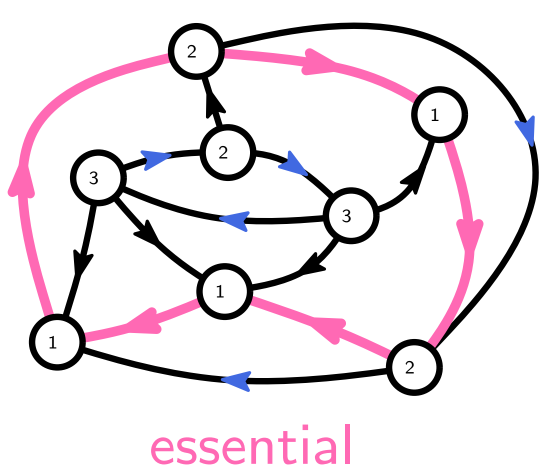

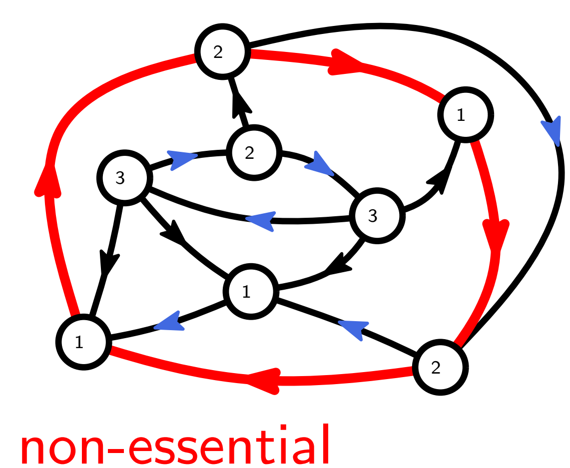

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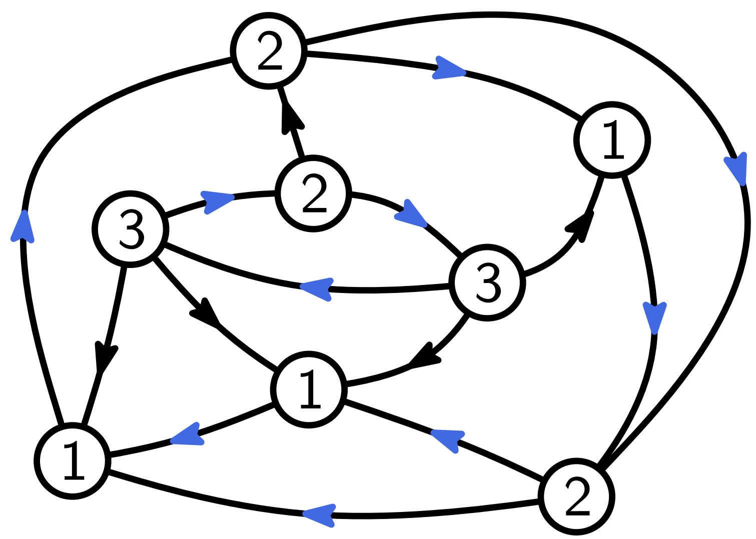

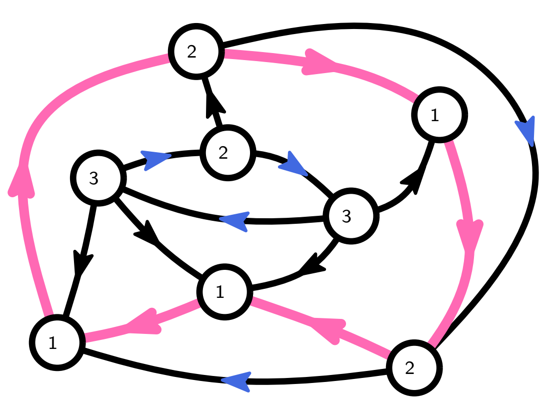

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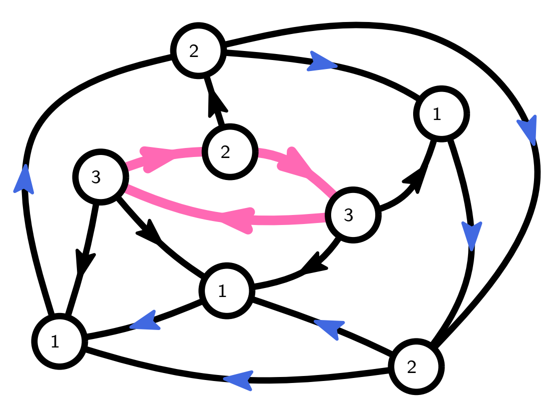

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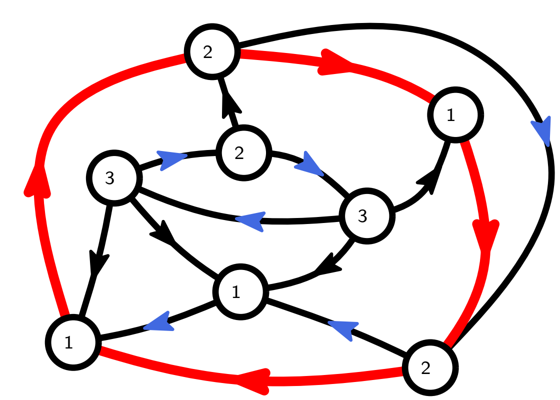

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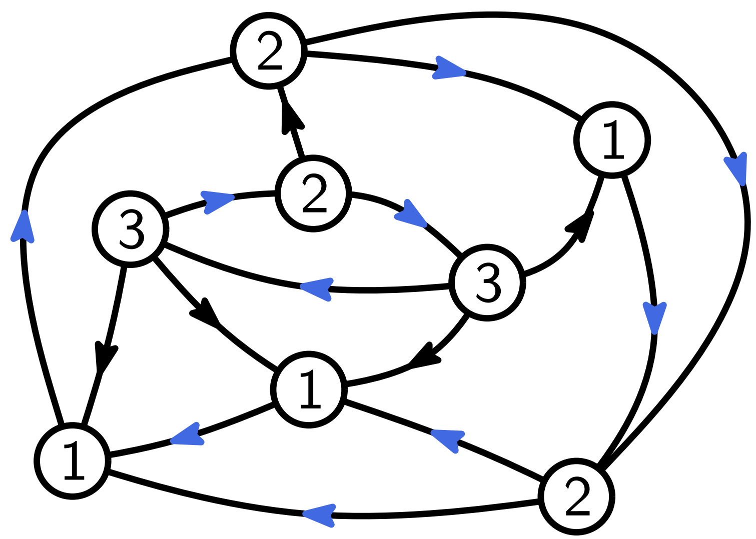

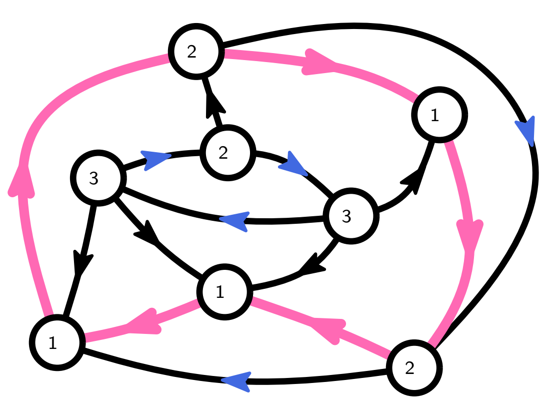

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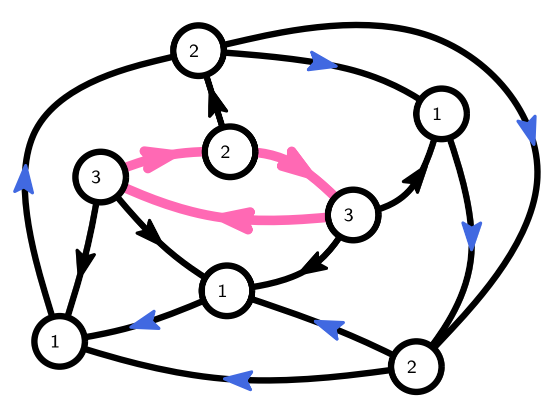

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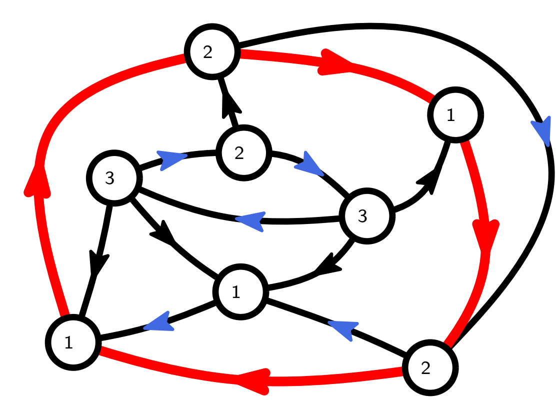

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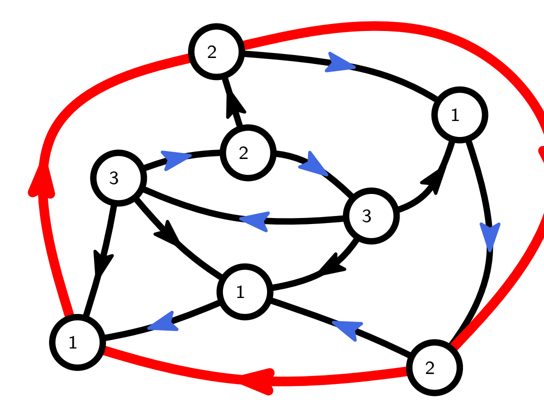

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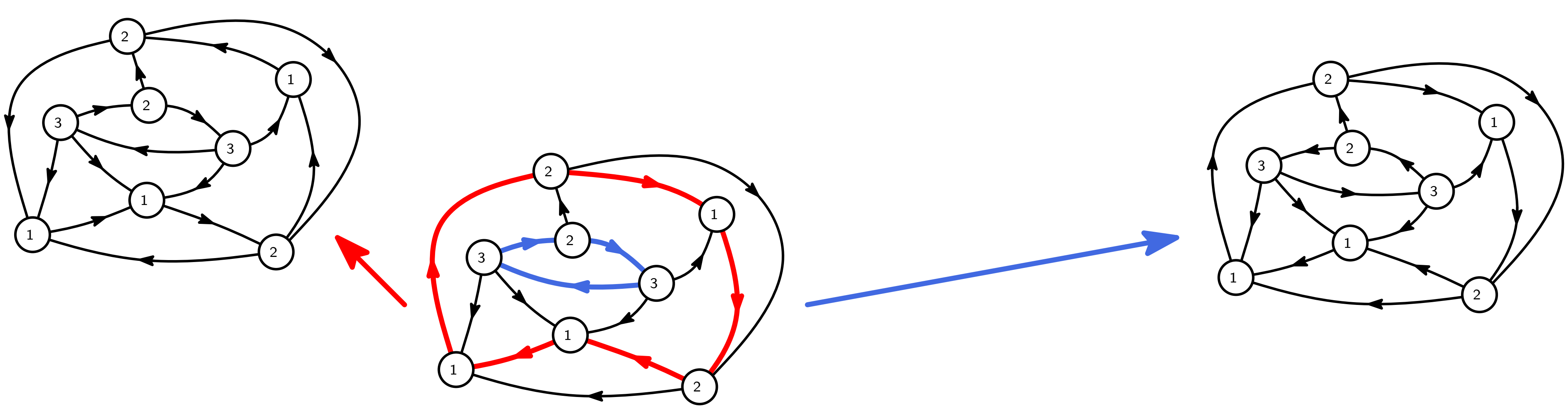
Set of α-orientations



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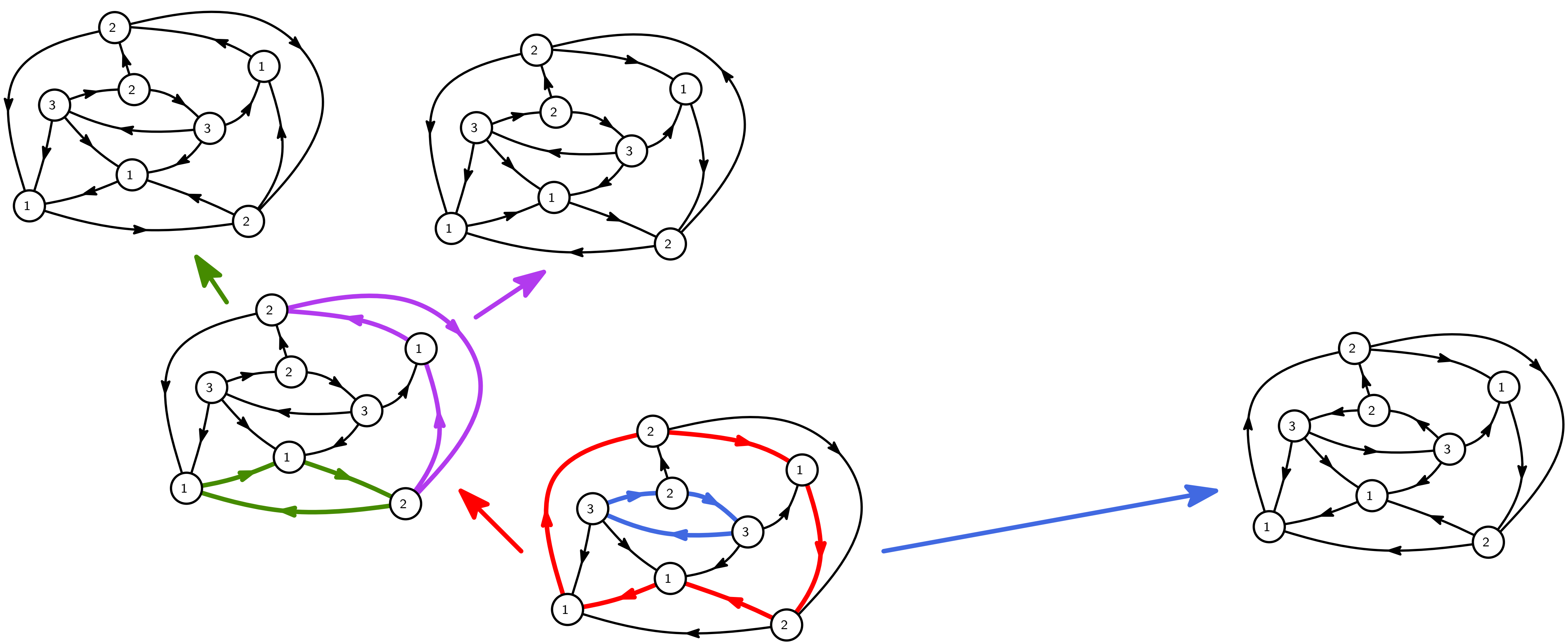

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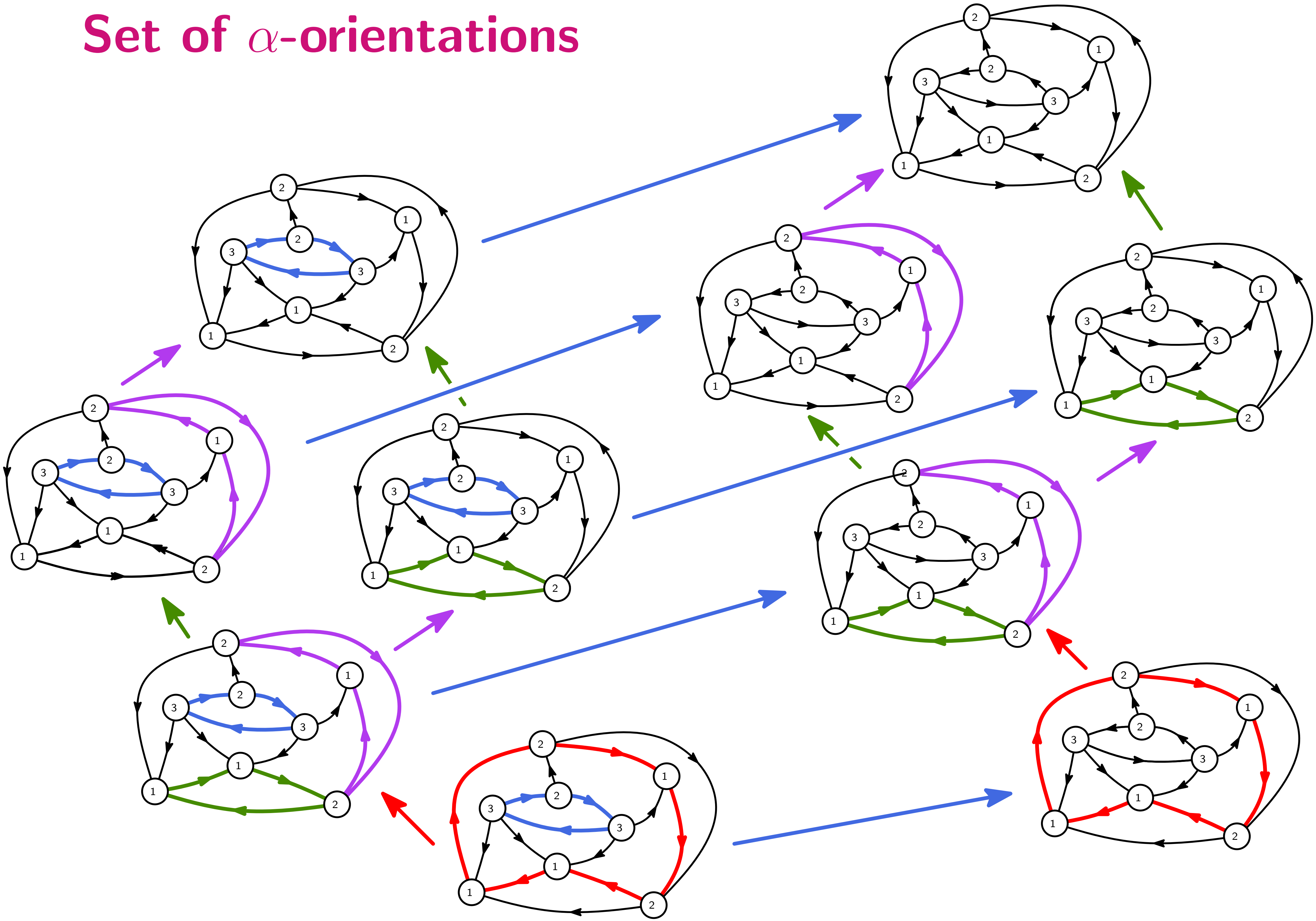

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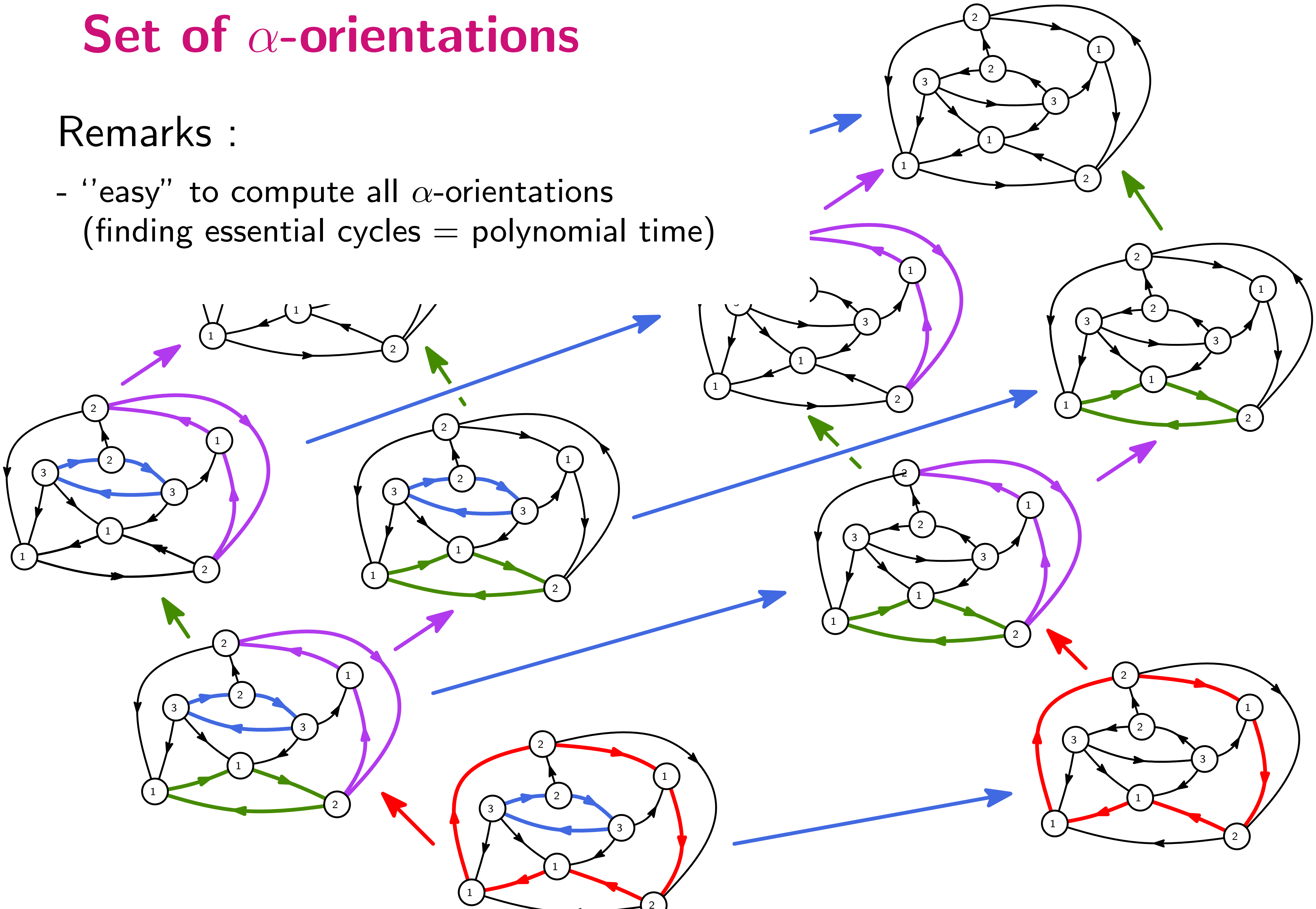
Remarks :

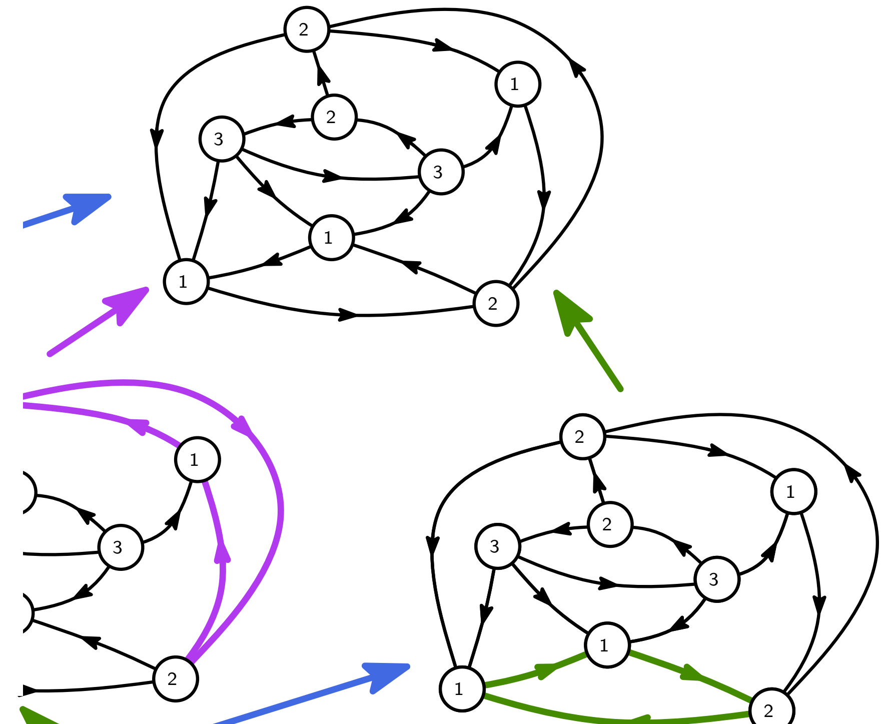
- ''easy'' to compute all α-orientations
  (finding essential cycles = polynomial time)

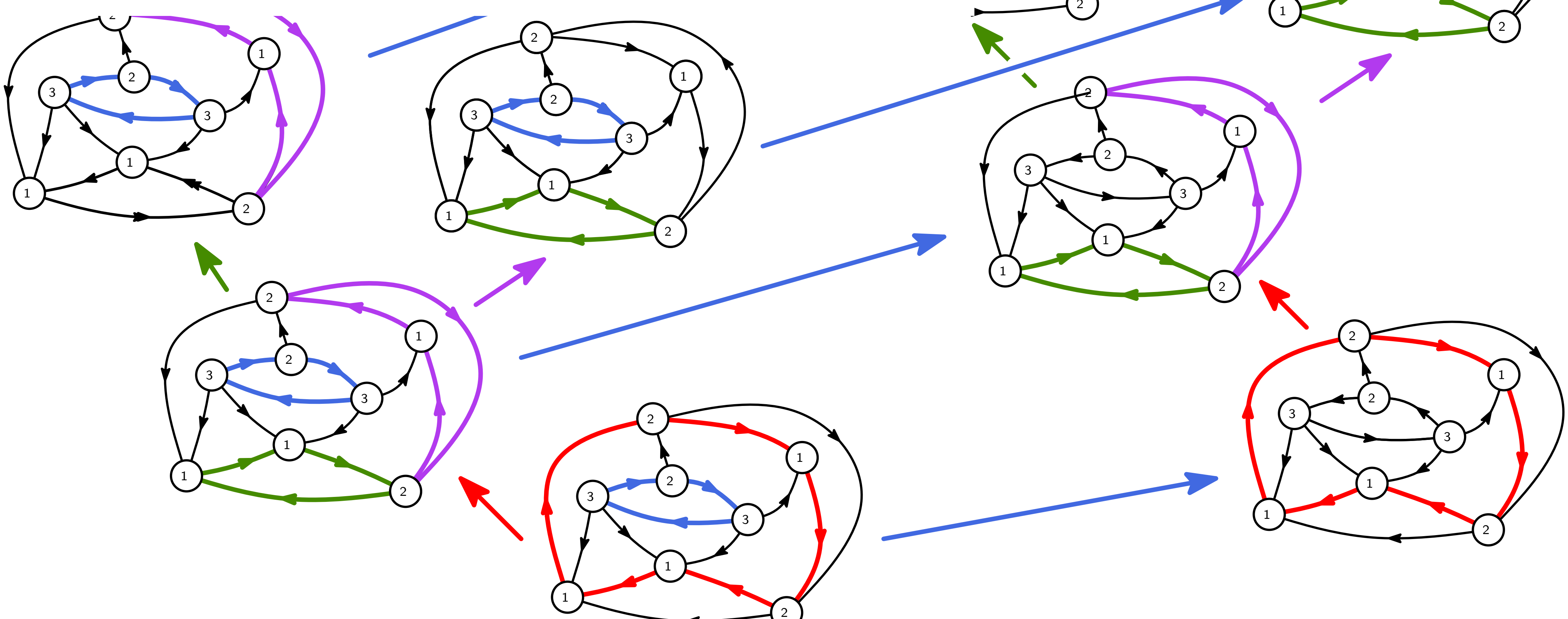


Set of α-orientations

Remarks :

- ''easy'' to compute all α-orientations (finding essential cycles = polynomial time)
- Flipping a cw essential cycle into a ccw cycle defines an **ordering** on the set of orientations.





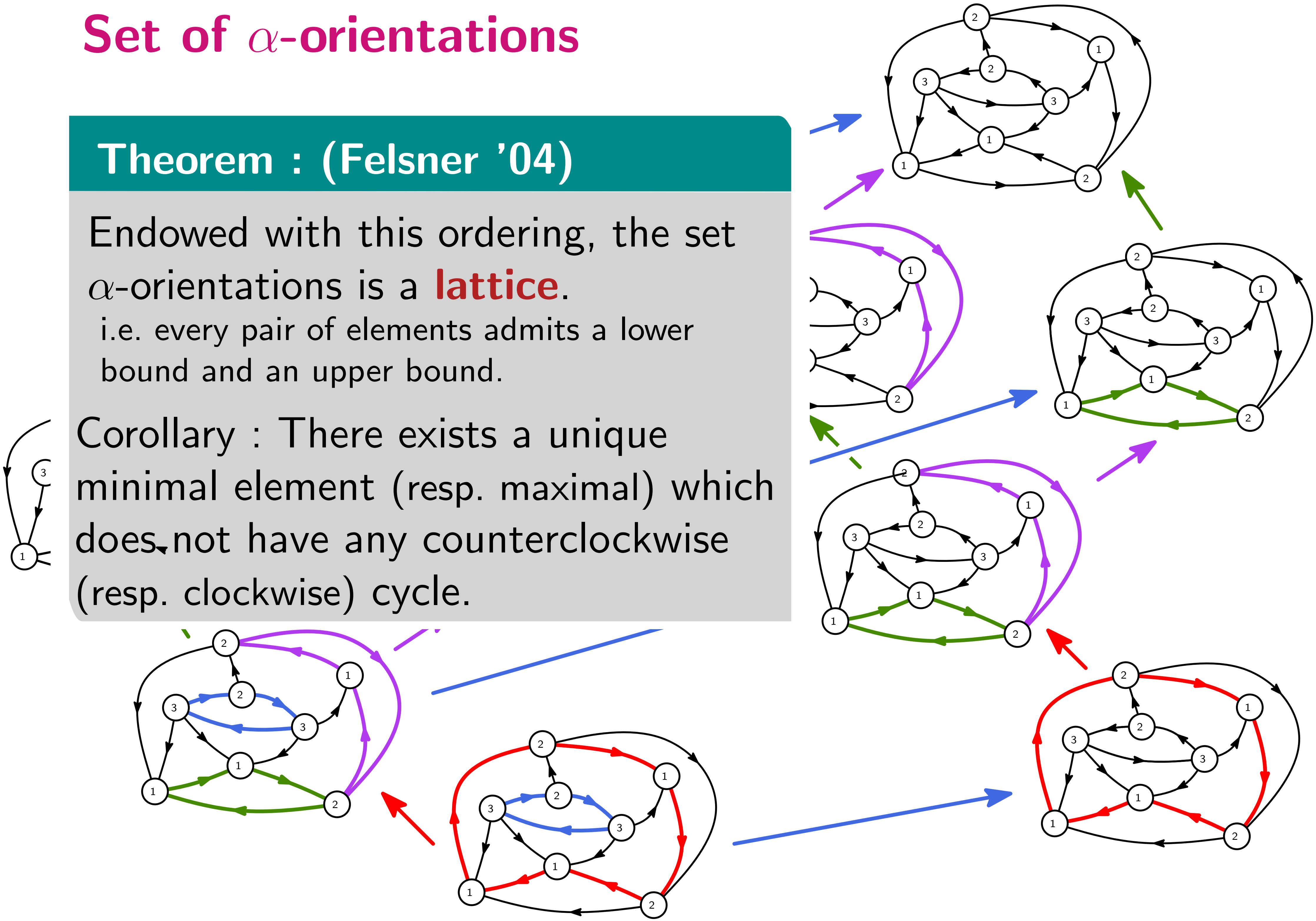
Set of α-orientations

Theorem : (Felsner '04)

Endowed with this ordering, the set α-orientations is a **lattice**.

i.e. every pair of elements admits a lower bound and an upper bound.

Corollary : There exists a unique minimal element (resp. maximal) which does not have any counterclockwise (resp. clockwise) cycle.



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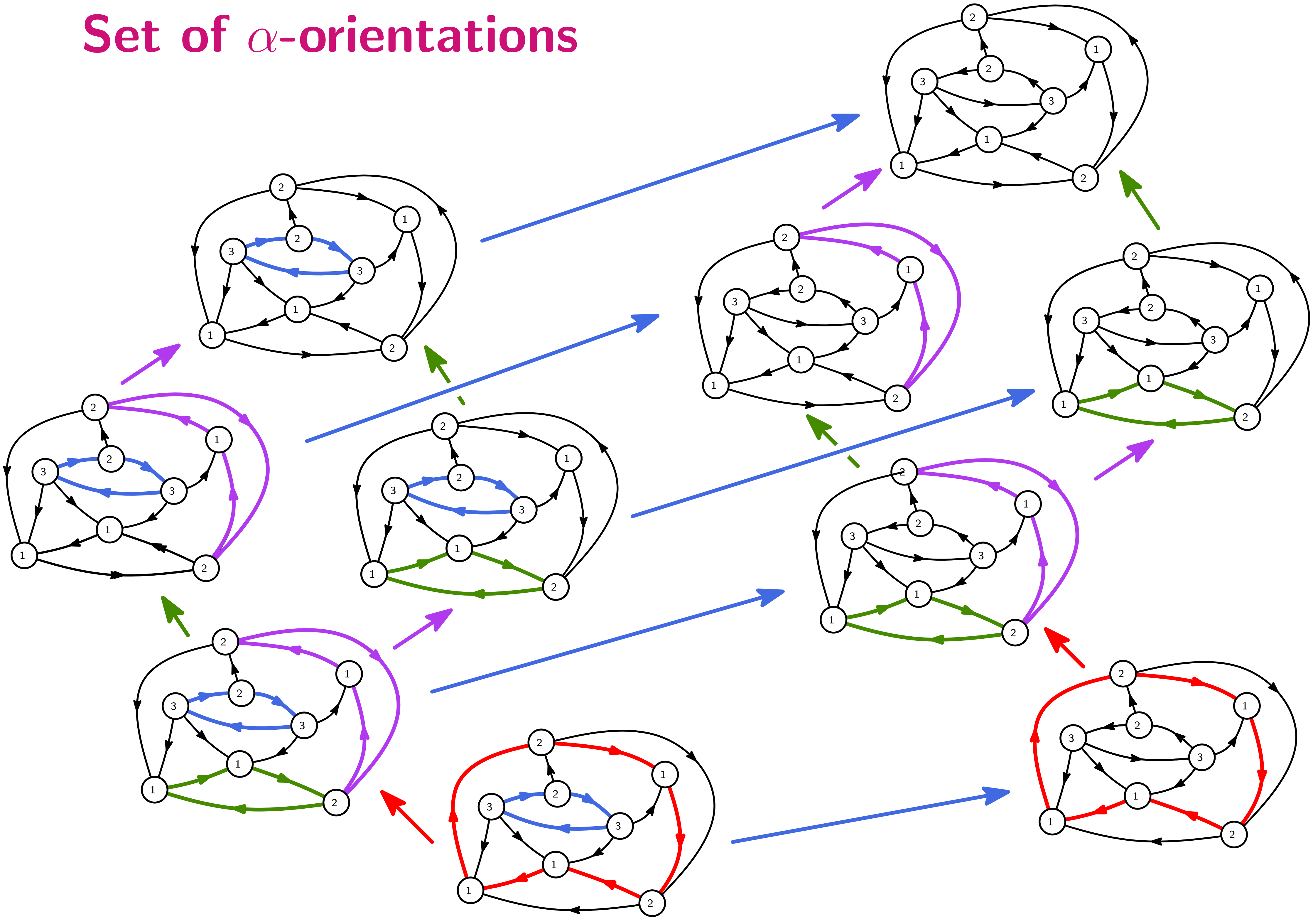

Existence of a maximal element

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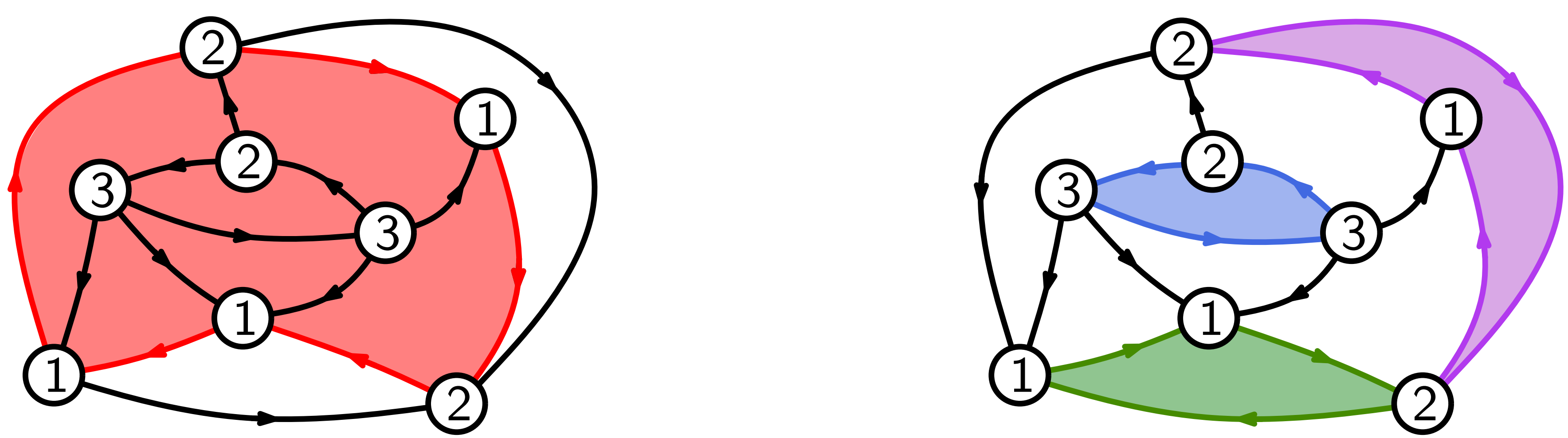
Property :

An edge belongs at most to 2 essential cycles.
The interior of those cycles are disjoint.



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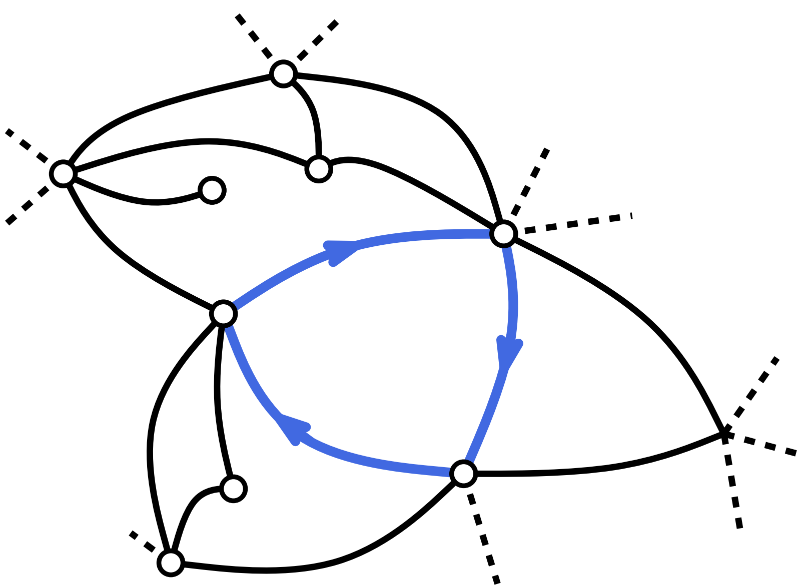

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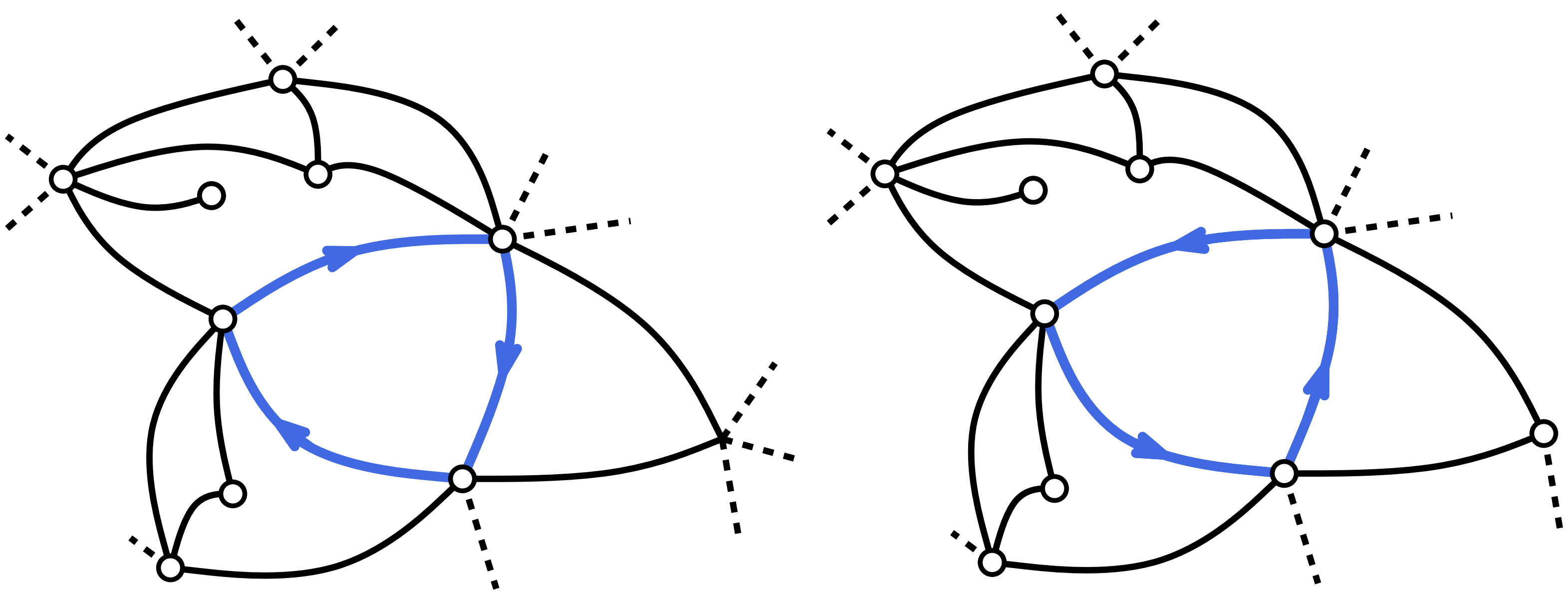

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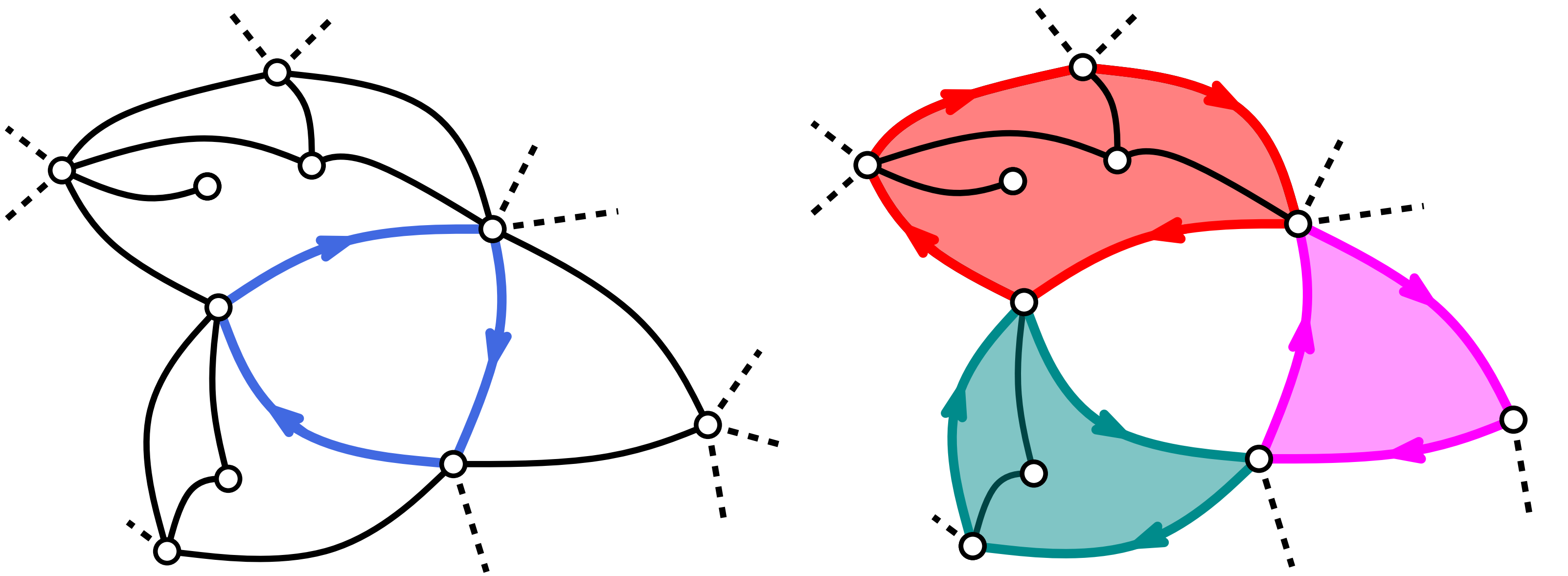

To flip again these edges, one must flip each one of them first.

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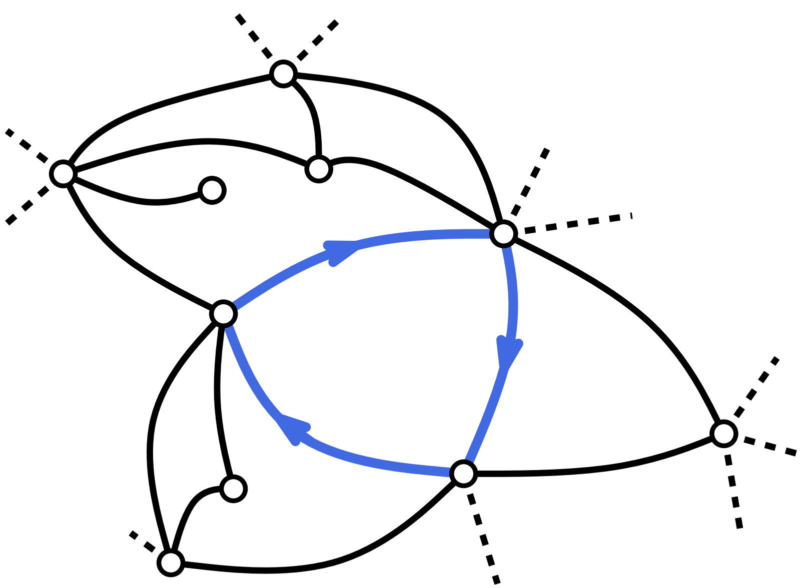

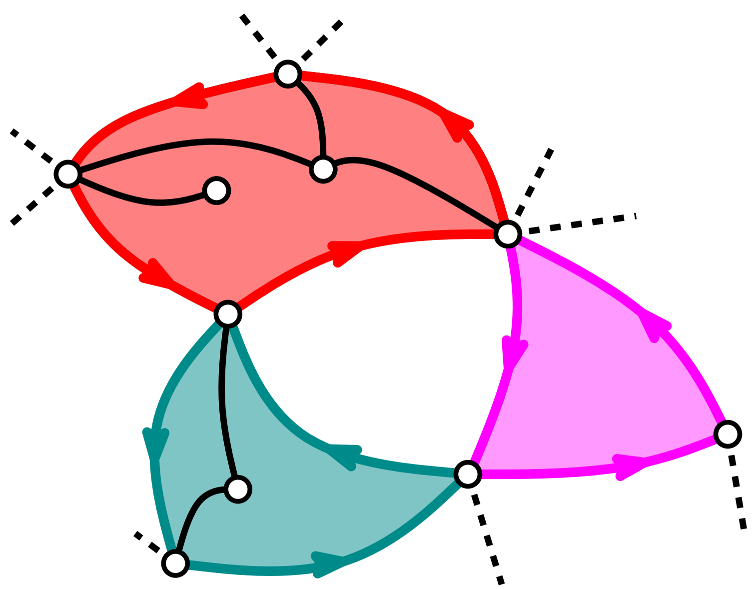

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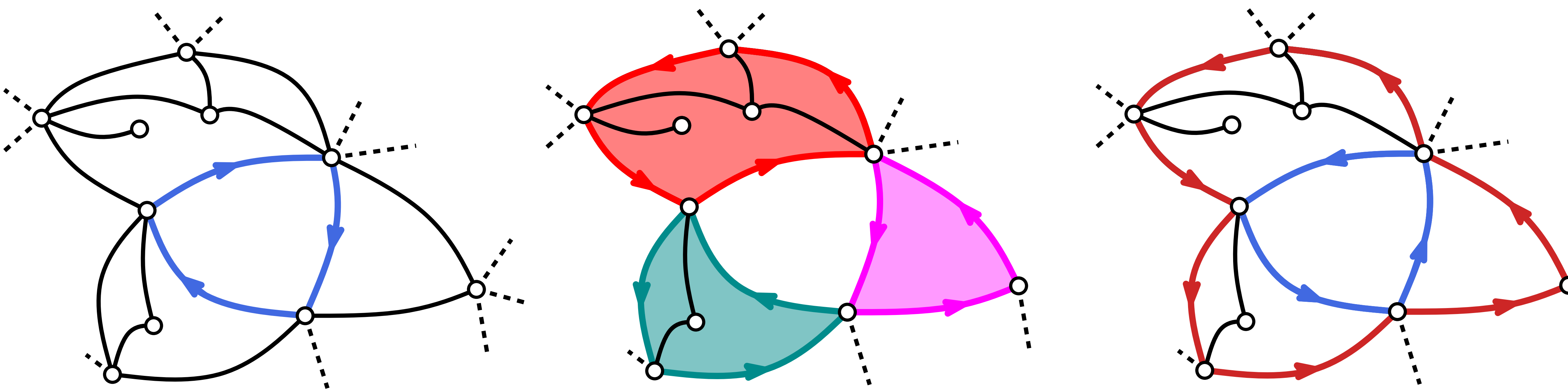

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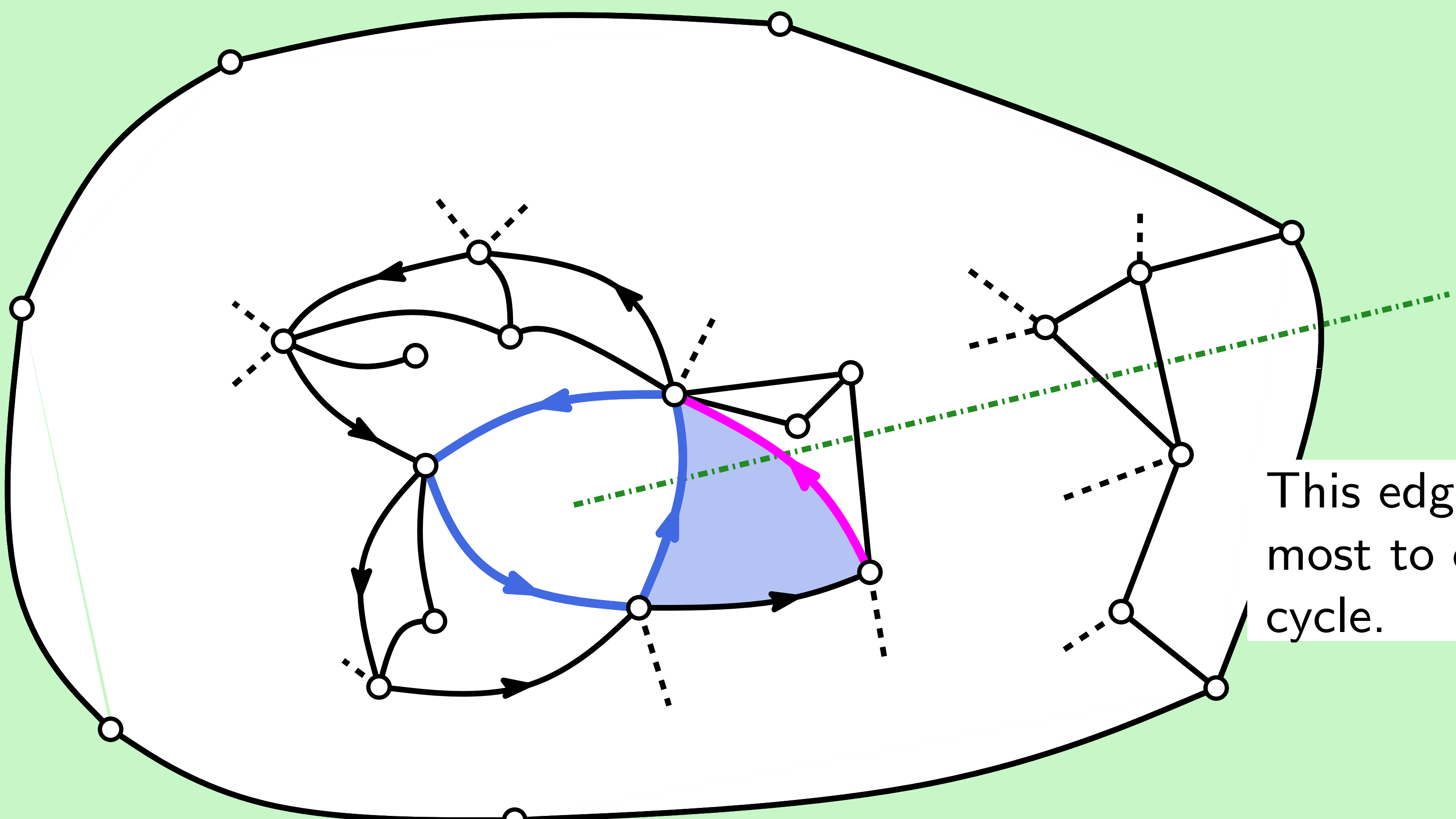

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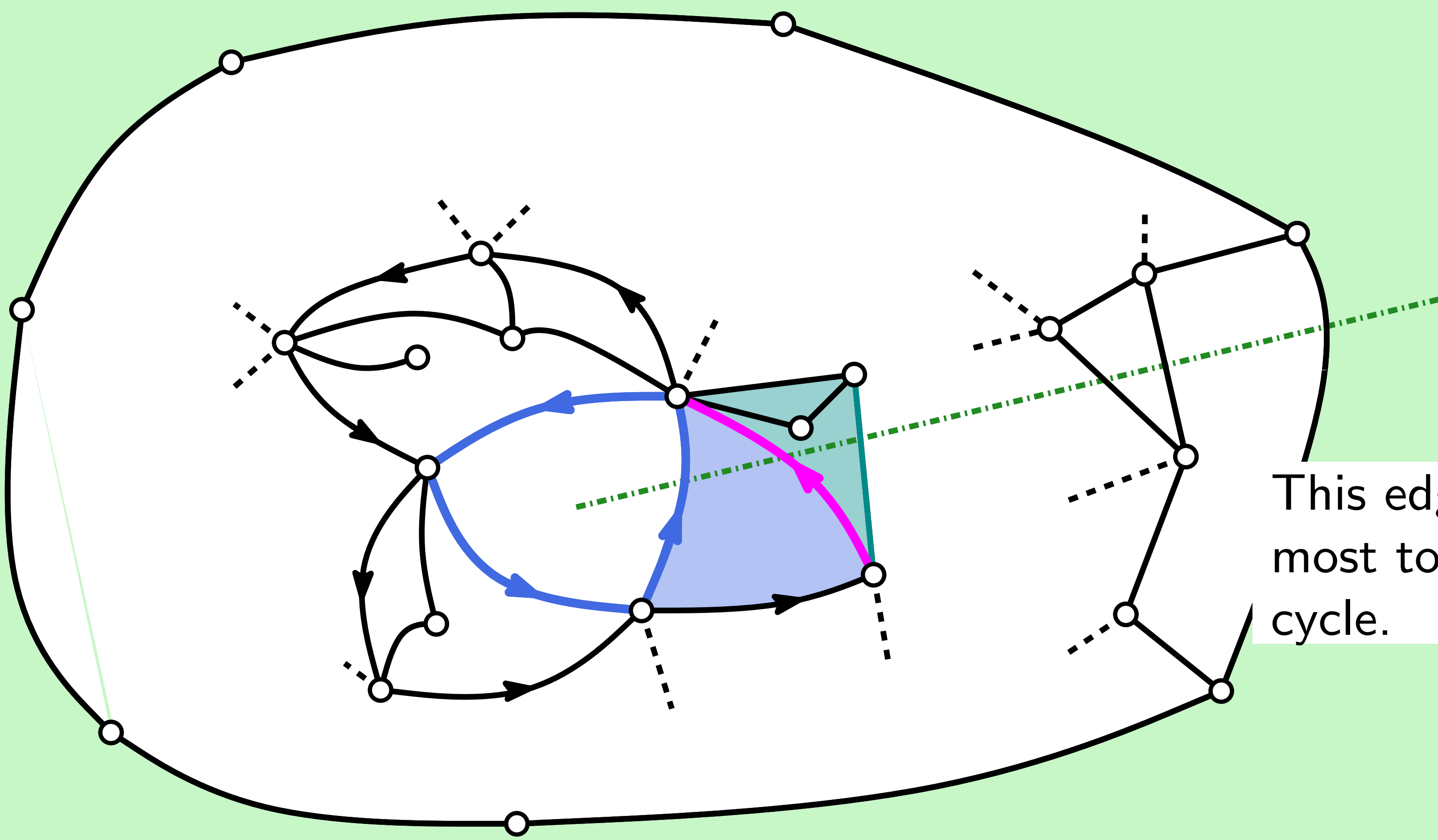

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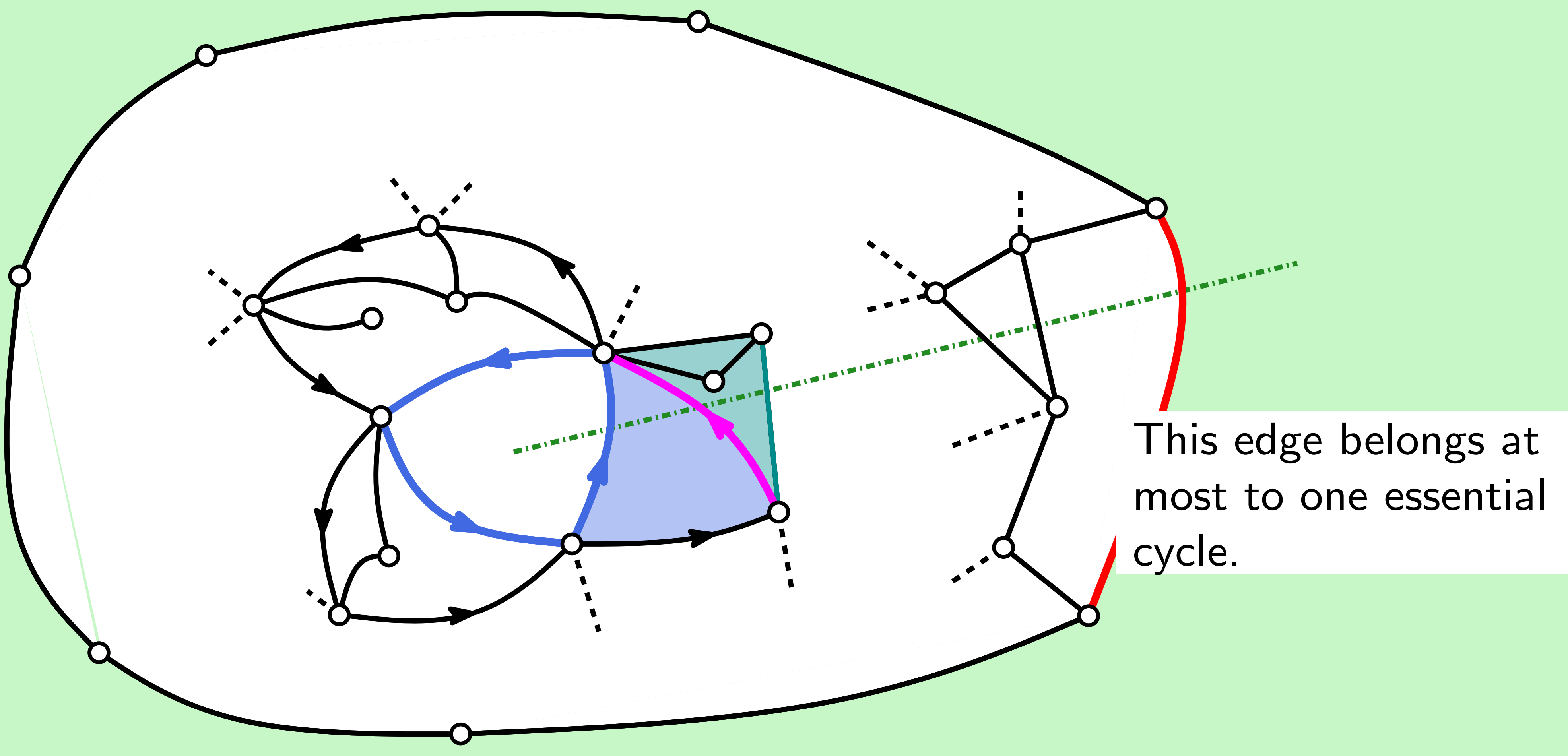

Summary

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- If there exists an α-orientation, α is feasible. Deciding whether α is feasible or not is **easy**.

- We can go from one α-orientation to another by flipping directed cycles and even by flipping only **essential** directed cycles.

- Flipping essential cycles gives a **lattice** structure. In particular, there exists a unique α-orientation without counterclockwise cycles and a unique one without clockwise cycles.

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Exercise : about simple triangulations

• Use Euler formula, to obtain an equation between the number of vertices and edges in a triangulation.

Define $\alpha(v) = 3$ for any inner vertex v, and $\alpha(v) = 1$ otherwise.

• Is α feasible for any triangulation ?

• Give necessary and sufficient conditions for a triangulation to admit an α-orientation.

• For such a triangulation, prove that it admits a unique α-orientation without ccw cycles.

Let T be endowed with its minimal α-orientation. For $e \in E(T)$, let P_e be the left most path started at e and stopped on the outer face.

• Prove that P_e is self avoiding.