Convergence of simple Triangulations

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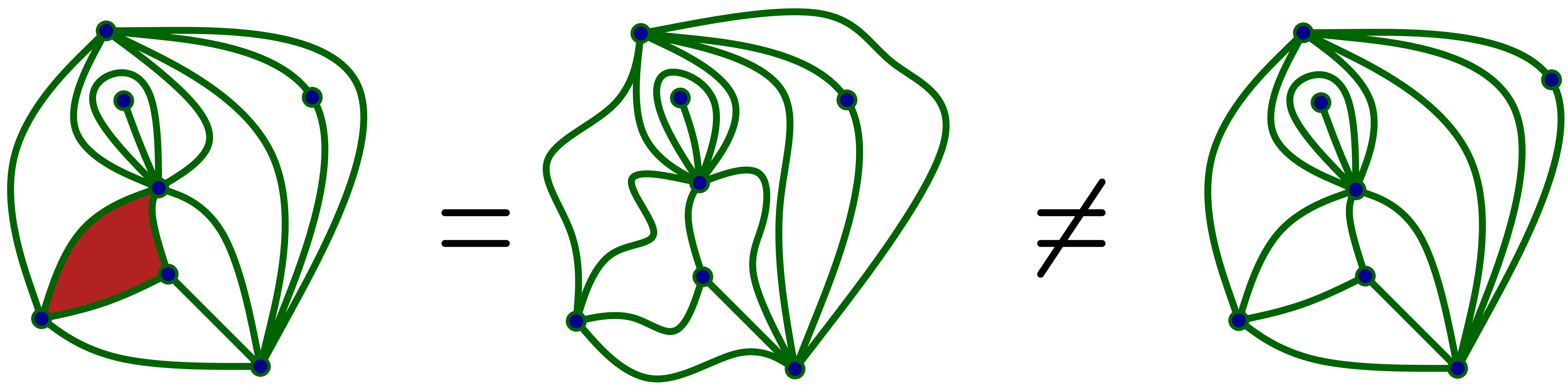
Planar Maps – Triangulations.

A **planar map** is the embedding of a connected graph in the sphere up to continuous deformations.



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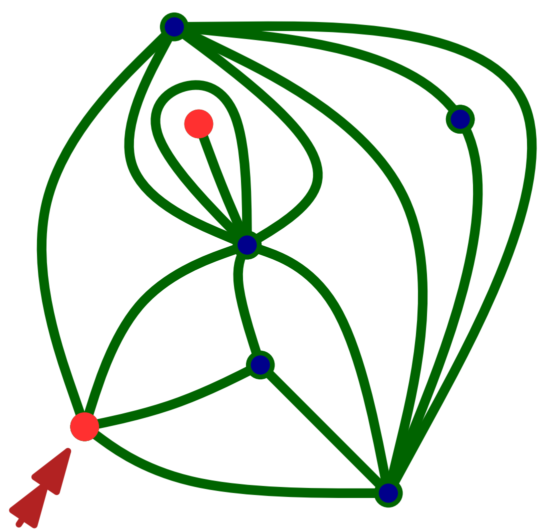

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Plane maps are **rooted**. Face that contains the root = **outer face**

Distance between two vertices = number of edges between them.
Planar map = **Metric space**

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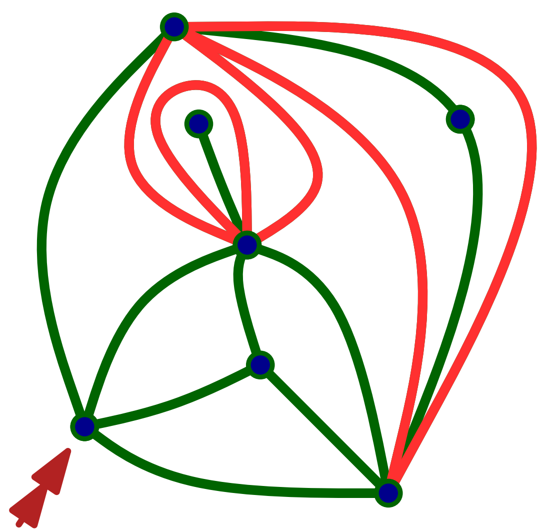
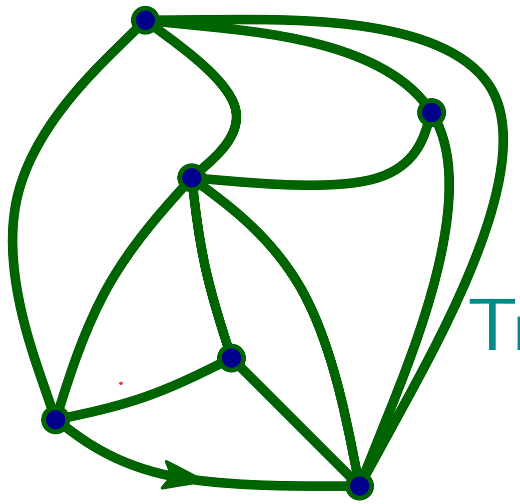


Triangulation = all faces are triangles.

Simple map = no loops nor multiple edges

Model + Motivation



Simple Triangulation

Euler Formula : $v + f = 2 + e$

Triangulation : $2e = 3f$

$\mathcal{M}_n = \{$Simple triangulations of size $n\}$

$= n + 2$ vertices, $2n$ faces, $3n$ edges

$M_n =$ Random element of $\mathcal{M}_n$

What is the behavior of M_n when n goes to infinity ?
typical distances ? convergence towards a continuous object ?

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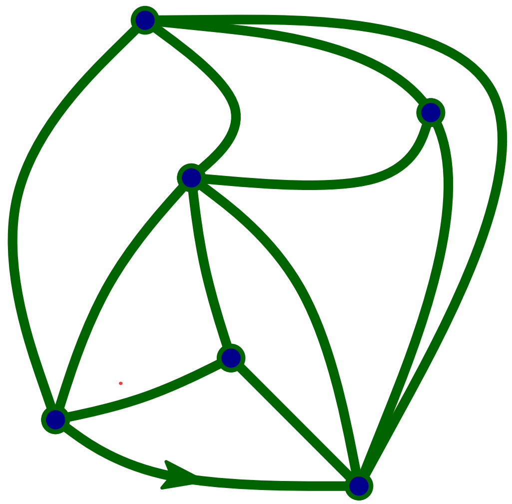

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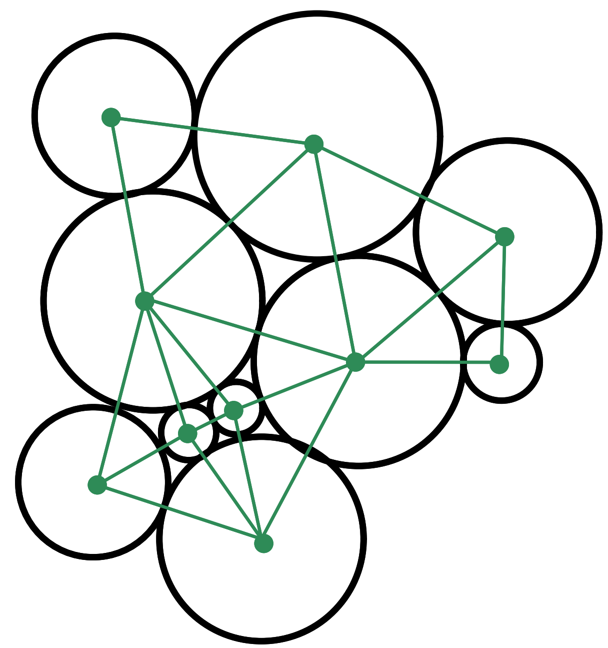
One motivation : Circle-packing theorem

Each simple triangulation M has a unique (up to Möbius transformations and reflections) circle packing whose tangency graph is M.
[Koebe-Andreev-Thurston]

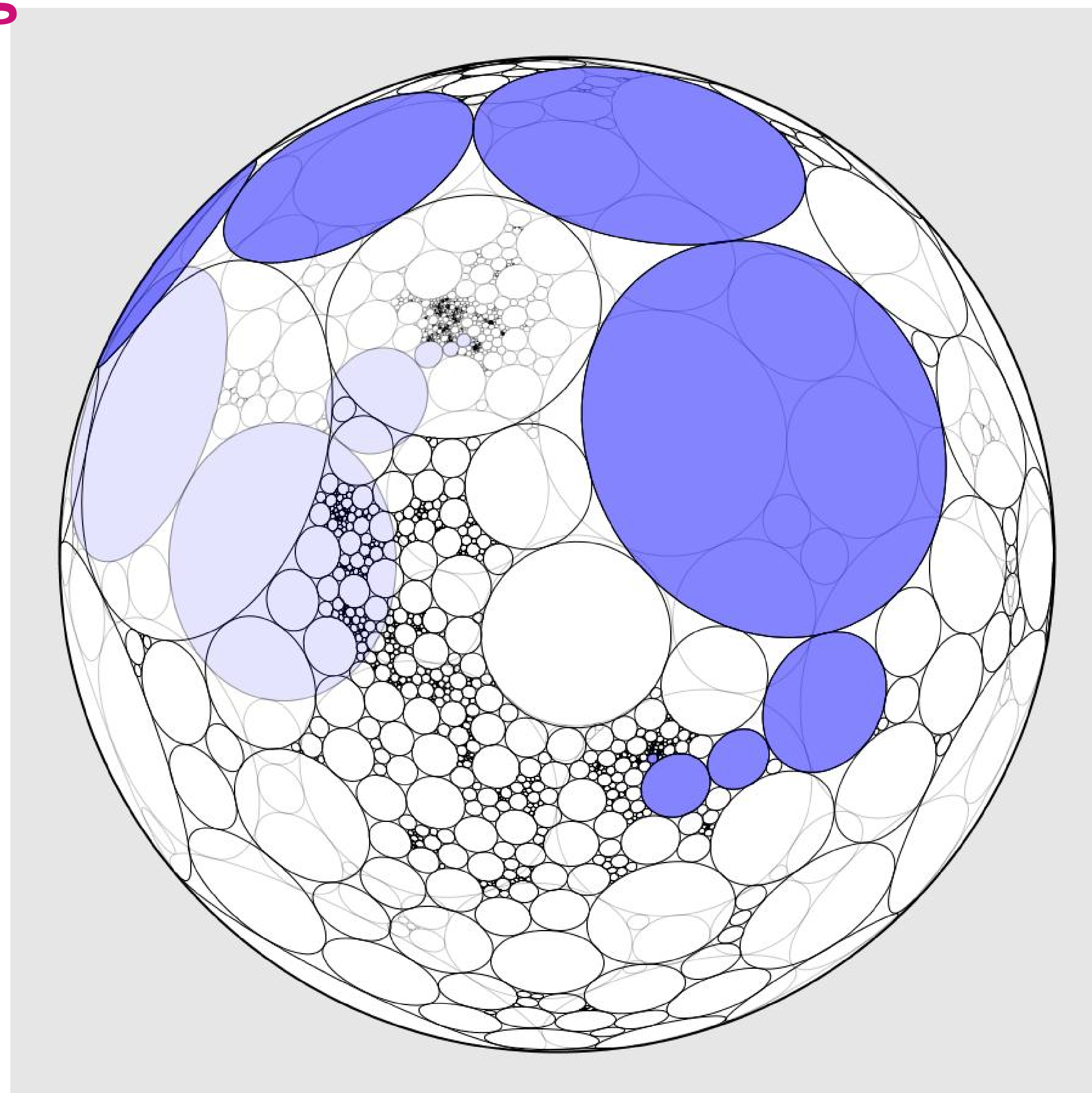


Gives a canonical embedding of simple triangulations in the sphere and possibly of their limit.

Random circle packing

Random circle packing = canonical embedding of random simple triangulation in the sphere.

Gives a way to define a canonical embedding of their limit ?



Team effort : code by Kenneth Stephenson, Eric Fusy and our own.

Convergence of uniform quadrangulations

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  Typical distance is $n^{1/4}$ + convergence of the profile

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  Hausdorff dimension of the Brownian map is 4.

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  Convergence towards the Brownian map
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general maps
NOT simple maps

Problem : These results relie on nice bijections between maps and labeled trees [Schaeffer '98], [Bouttier-Di Francesco-Guitter '04].

The result

Theorem : [Addario-Berry, A.]

$(M_n) =$ sequence of random **simple** triangulations, then:

$$\left(M_n, \left(\frac{3}{4n}\right)^{1/4} d_{M_n} \right) \xrightarrow{(d)} (M, D^{\star}),$$

for the distance of Gromov-Hausdorff on the isometry classes of compact metric spaces.

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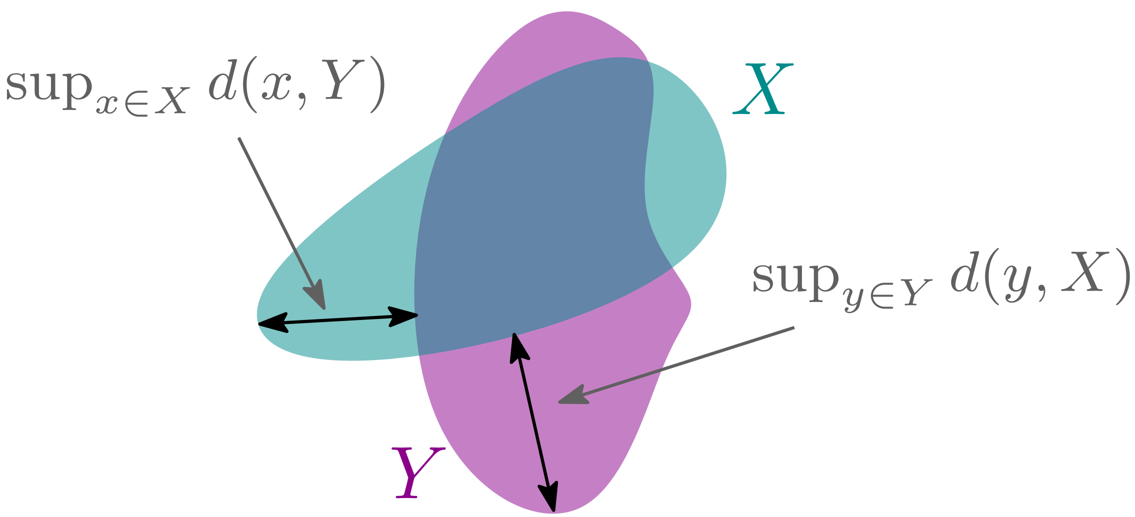
Exactly the same kind of result as Le Gall and Miermont's.

Gromov-Hausdorff distance

Hausdorff distance between X and Y two compact sets of (E, d) :

$$d_H(X,Y) = \max\{\sup_{x\in X} d(x,Y), \sup_{y\in Y} d(y,X)\}$$



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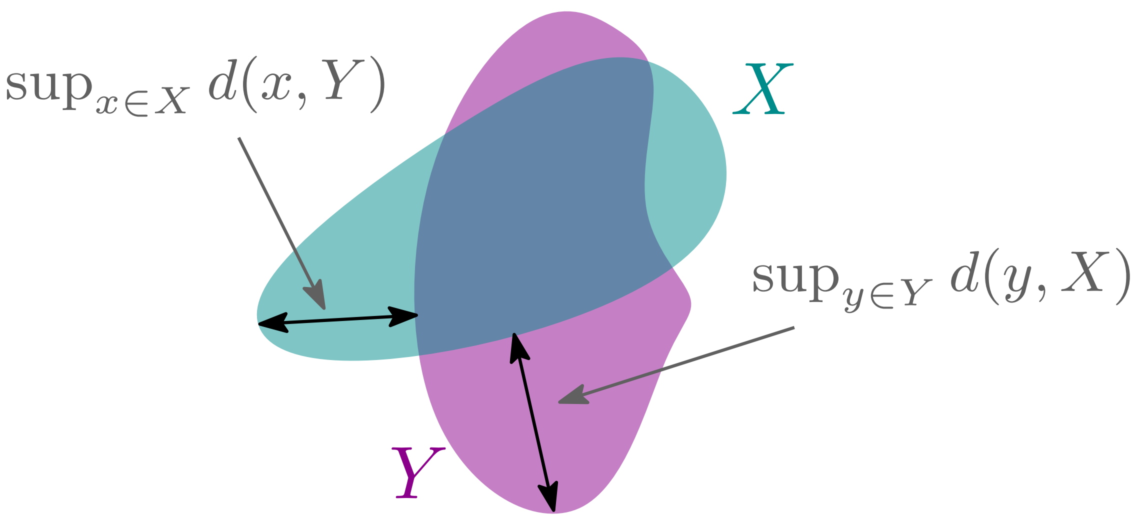

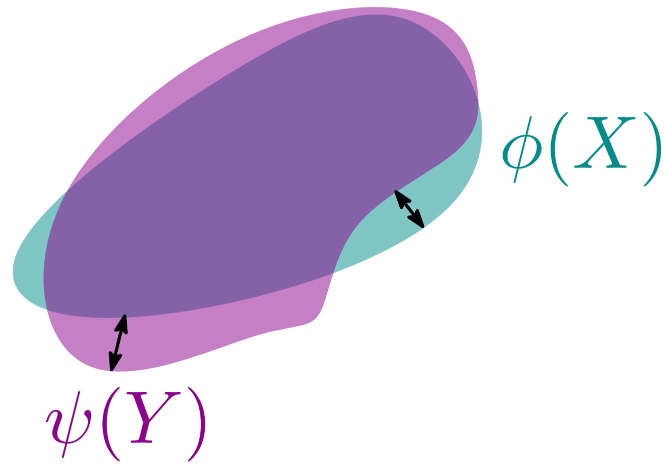

Gromov-Hausdorff distance btw two compact metric spaces E and F:

$$\mathbf{d_{GH}(E,F)} = \inf \mathbf{d_H}(\phi(\mathbf{E}), \psi(\mathbf{F}))$$

Infimum taken on :

- all the metric spaces M
- all the isometric embeddings $\phi, \psi : E, F \to M$.

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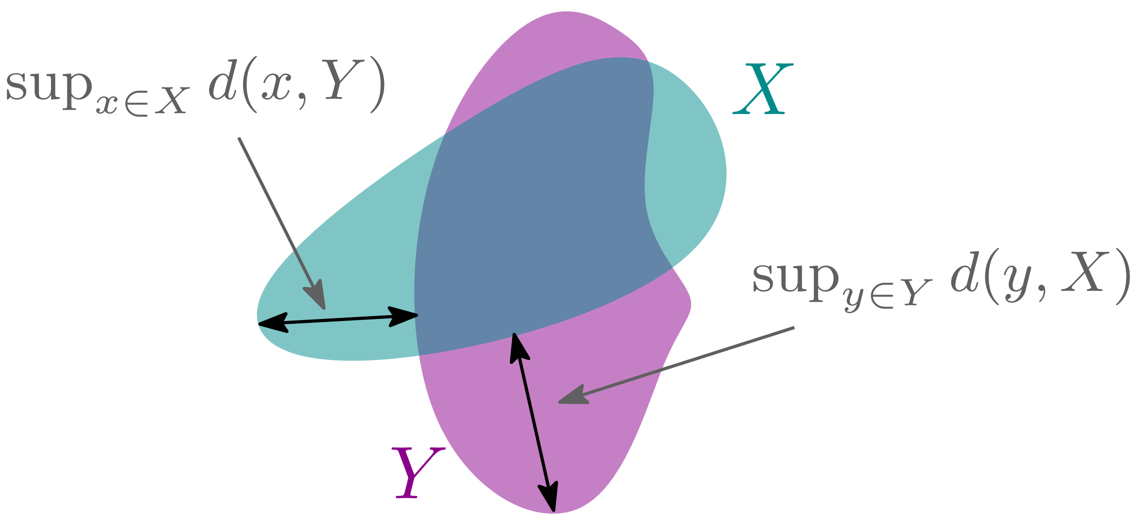

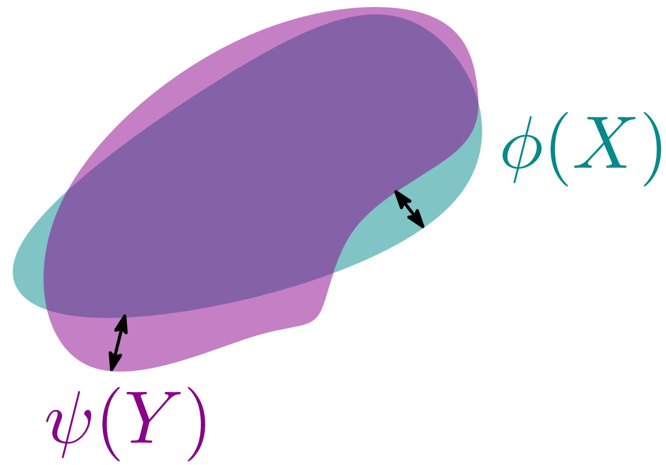

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{isometry classes of compact metric spaces with GH distance}
= complete and separable (= "Polish") space.

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Idea of proof :

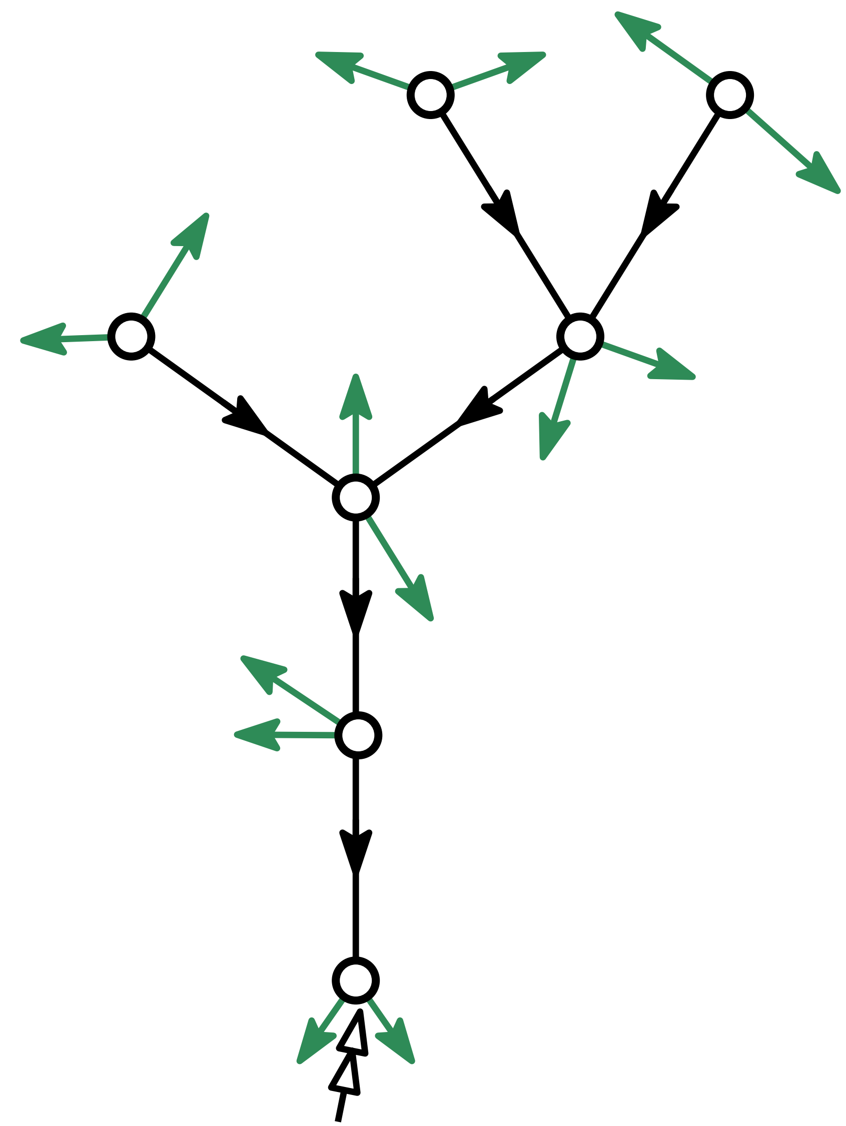
- encode the simple triangulations by some trees,
- **study the limits of trees**,
- interpret the **distance in the maps by some function of the tree**.

From blossoming trees to simple triangulations

plane tree:
plane map that is a tree

rooted plane tree:
one corner is distinguished

2-blossoming tree:
planted plane tree such that each vertex carries two leaves



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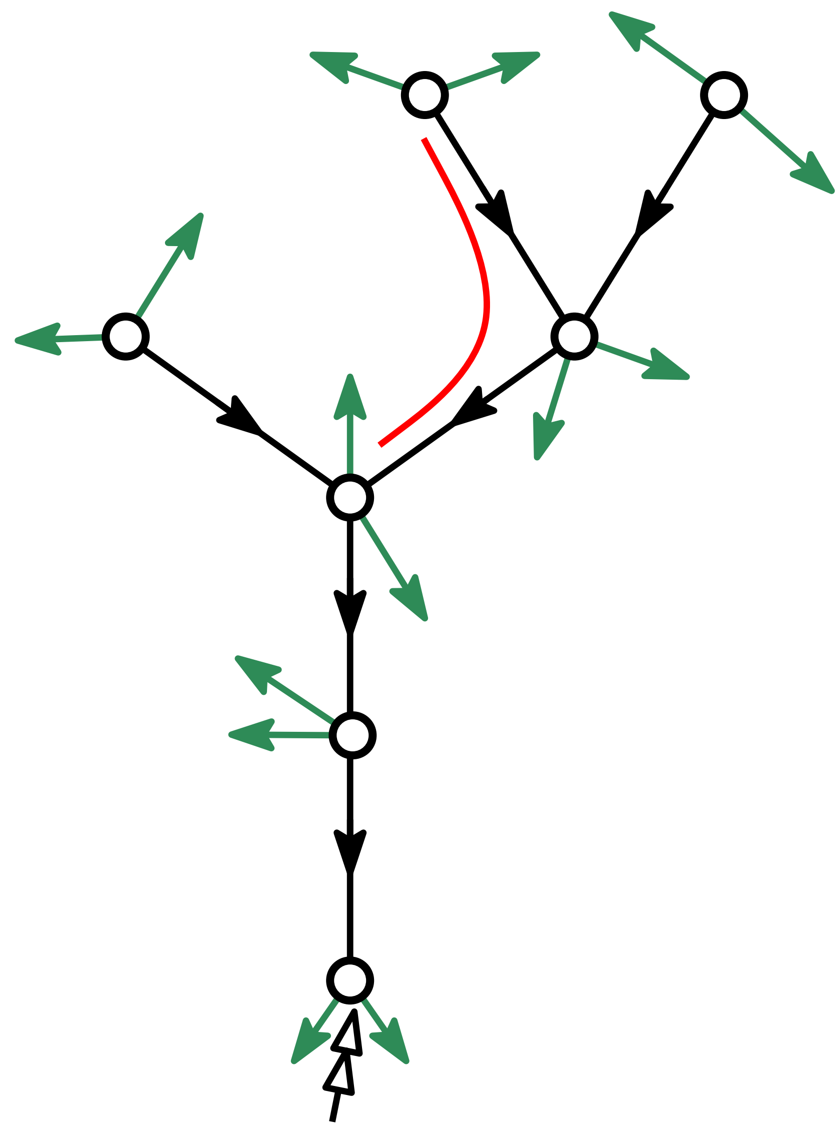
- If a leaf is followed by two internal edges,



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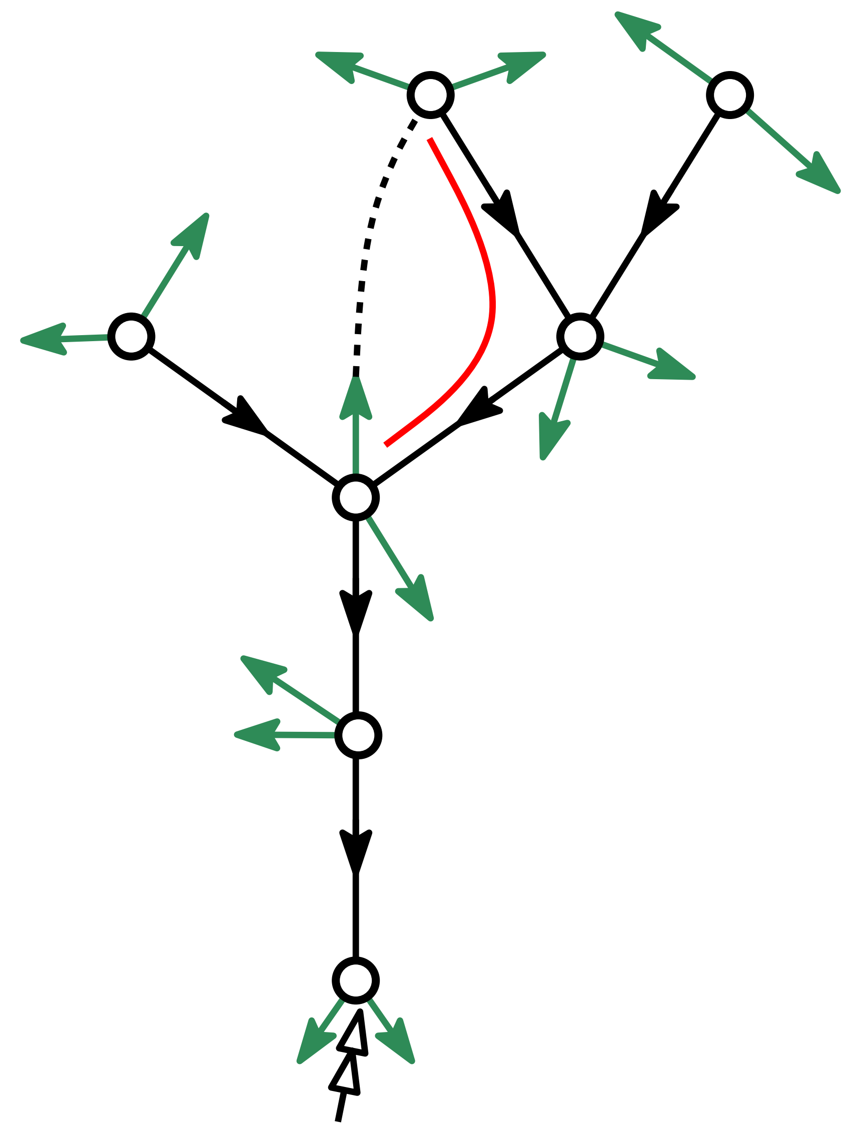

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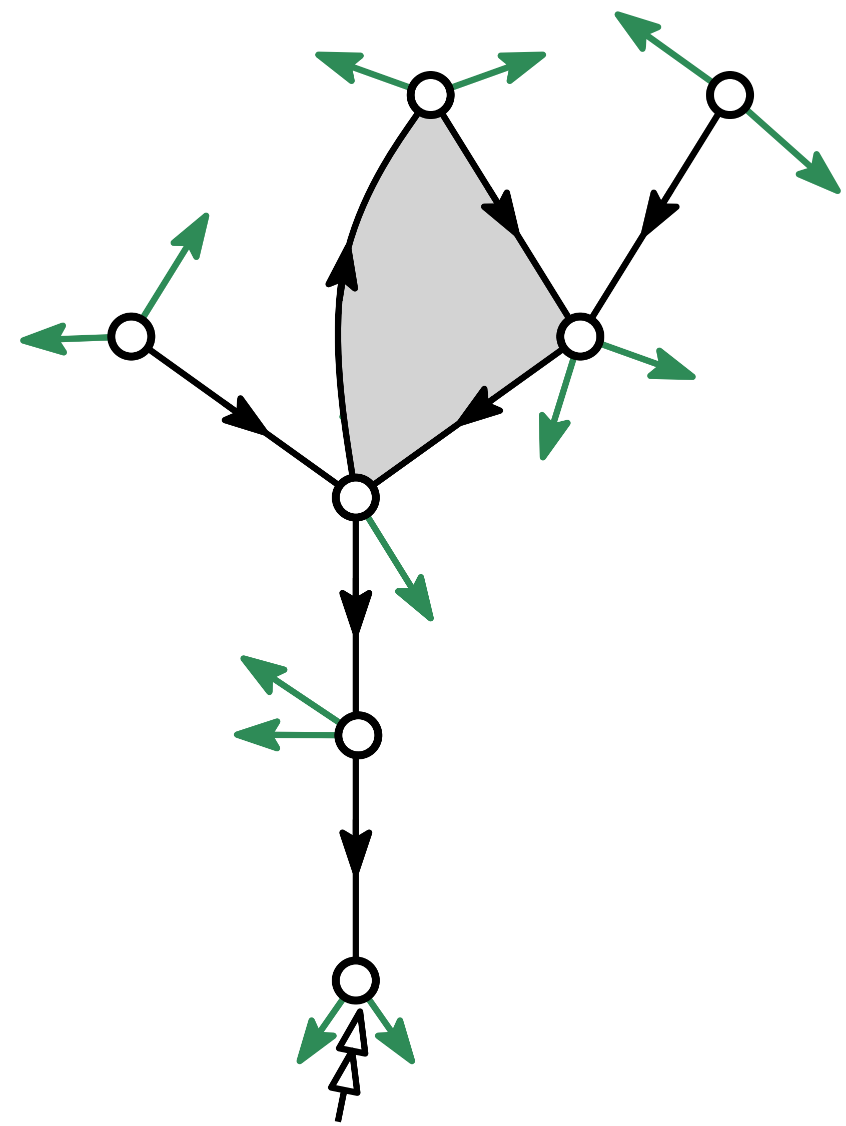

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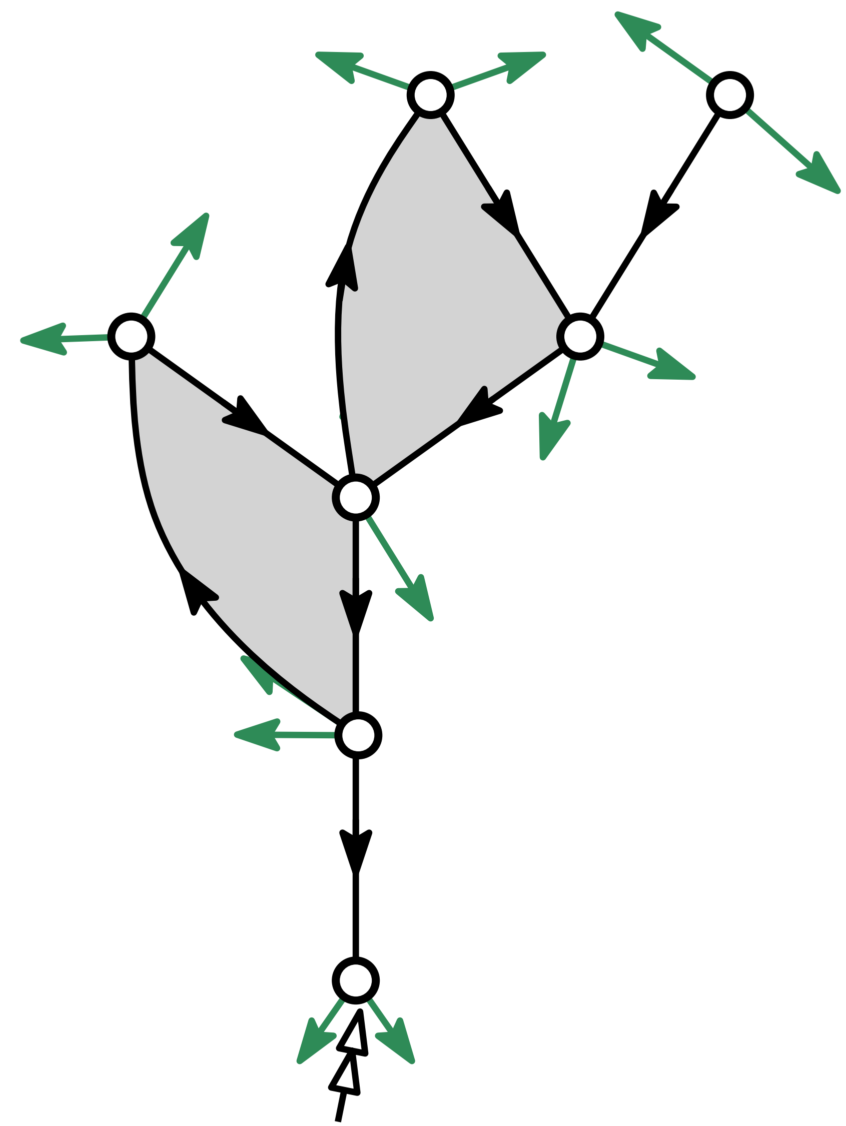

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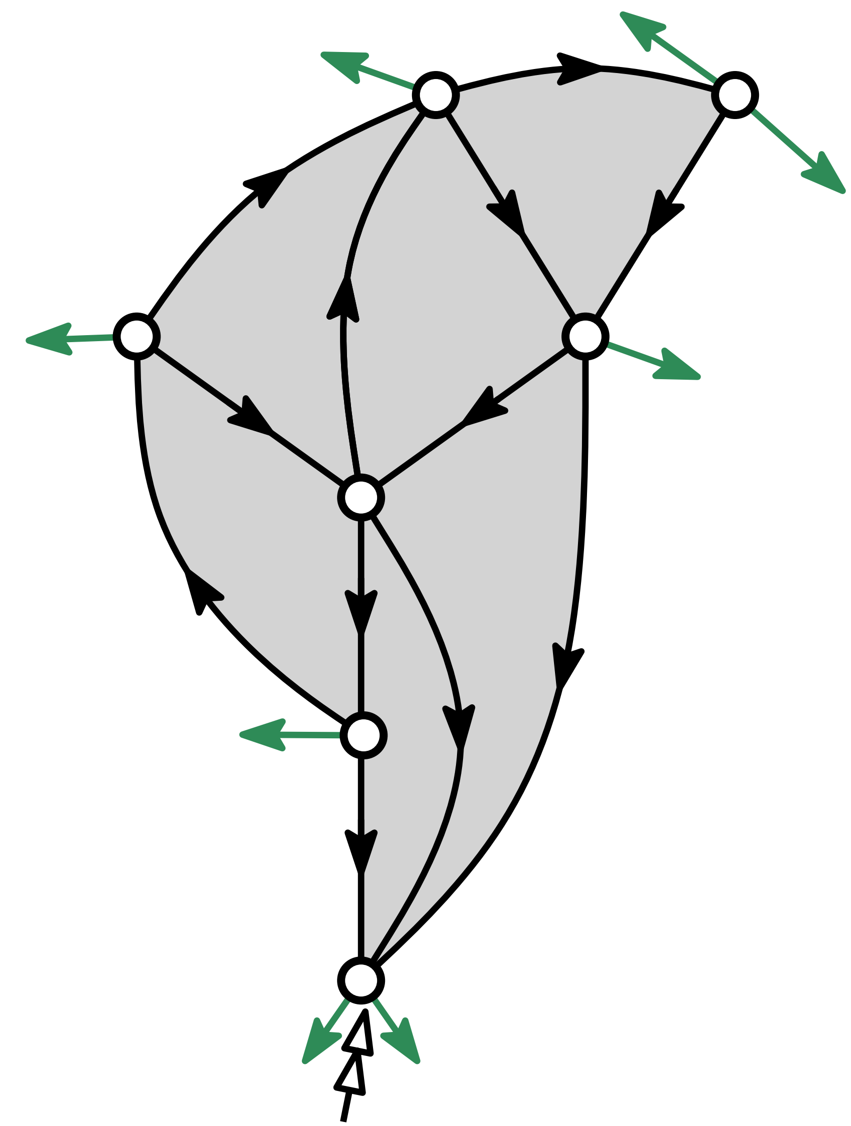
When finished two vertices have still two leaves and others have one.

Tree **balanced** = root corner has two leaves



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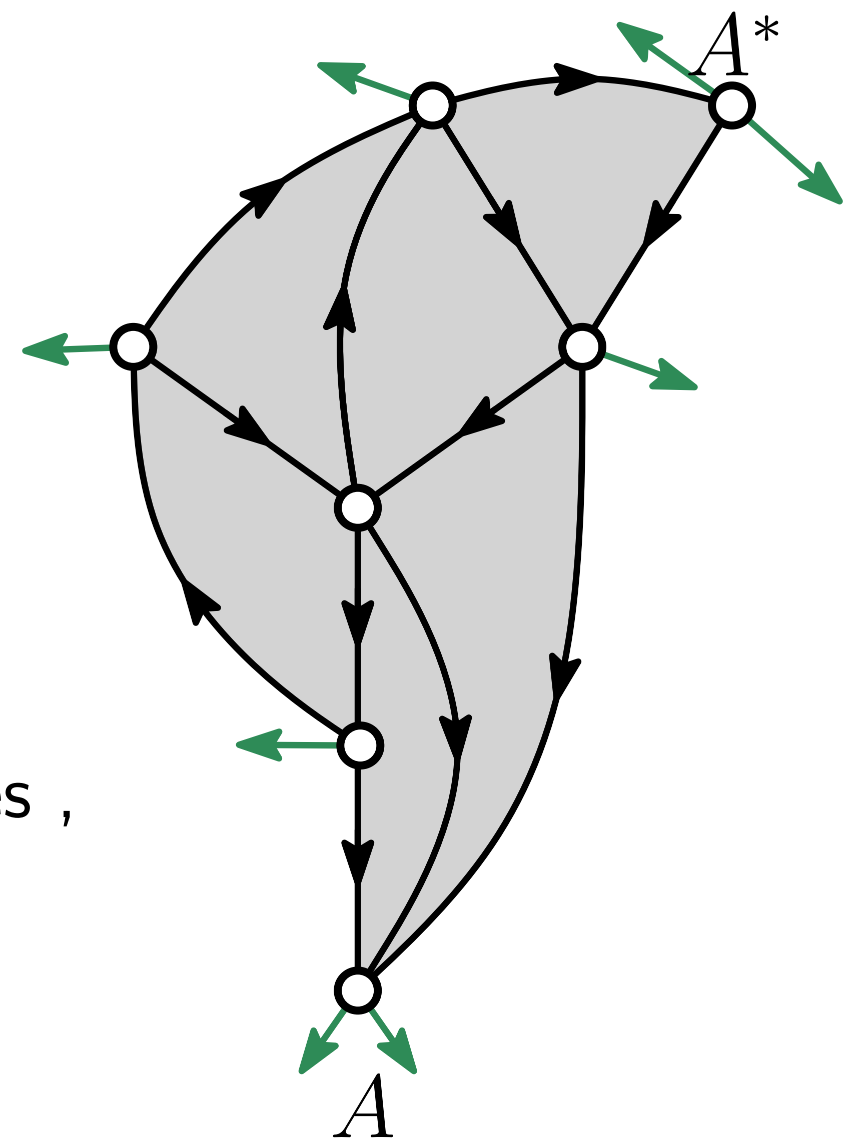

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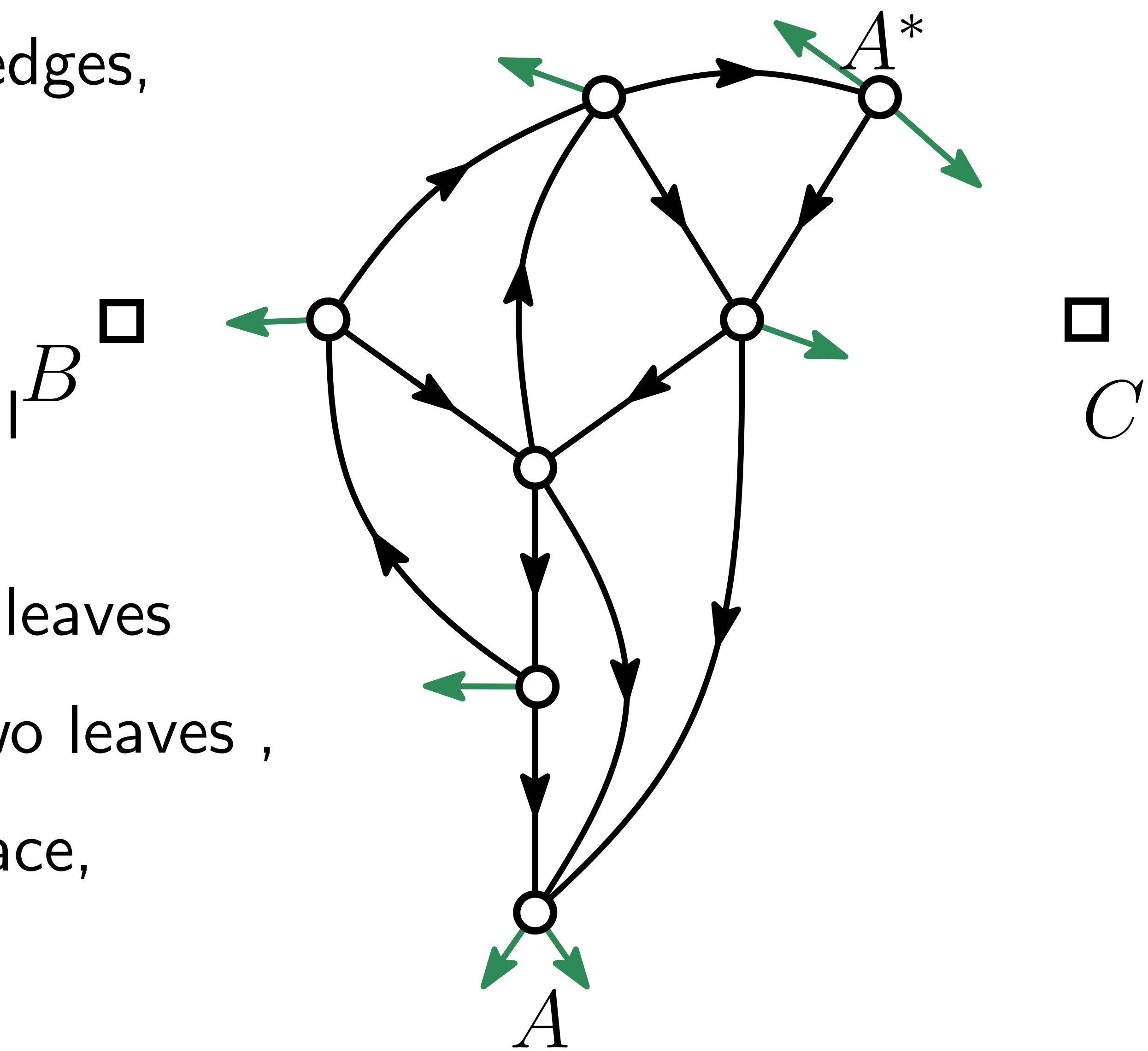

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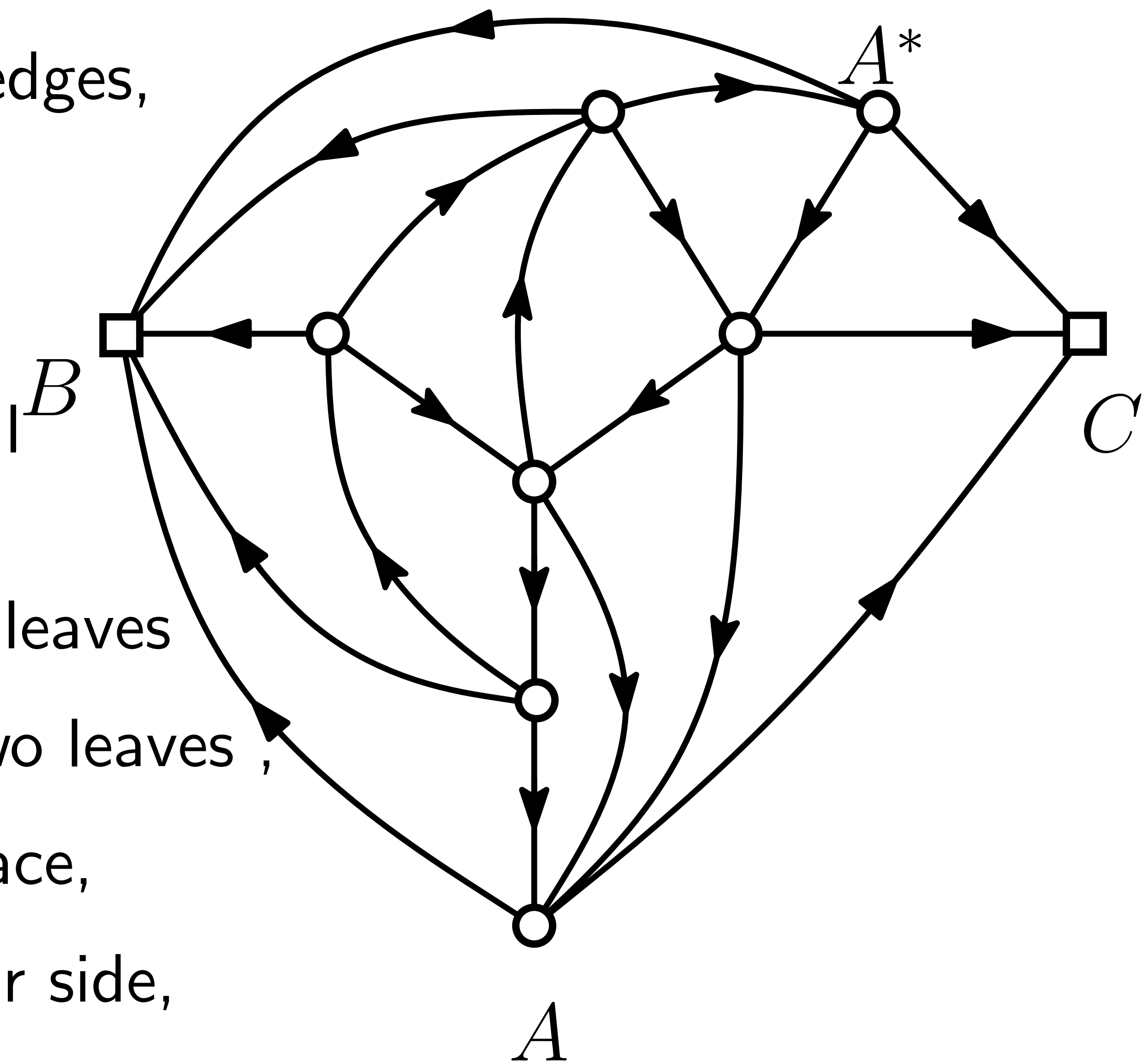

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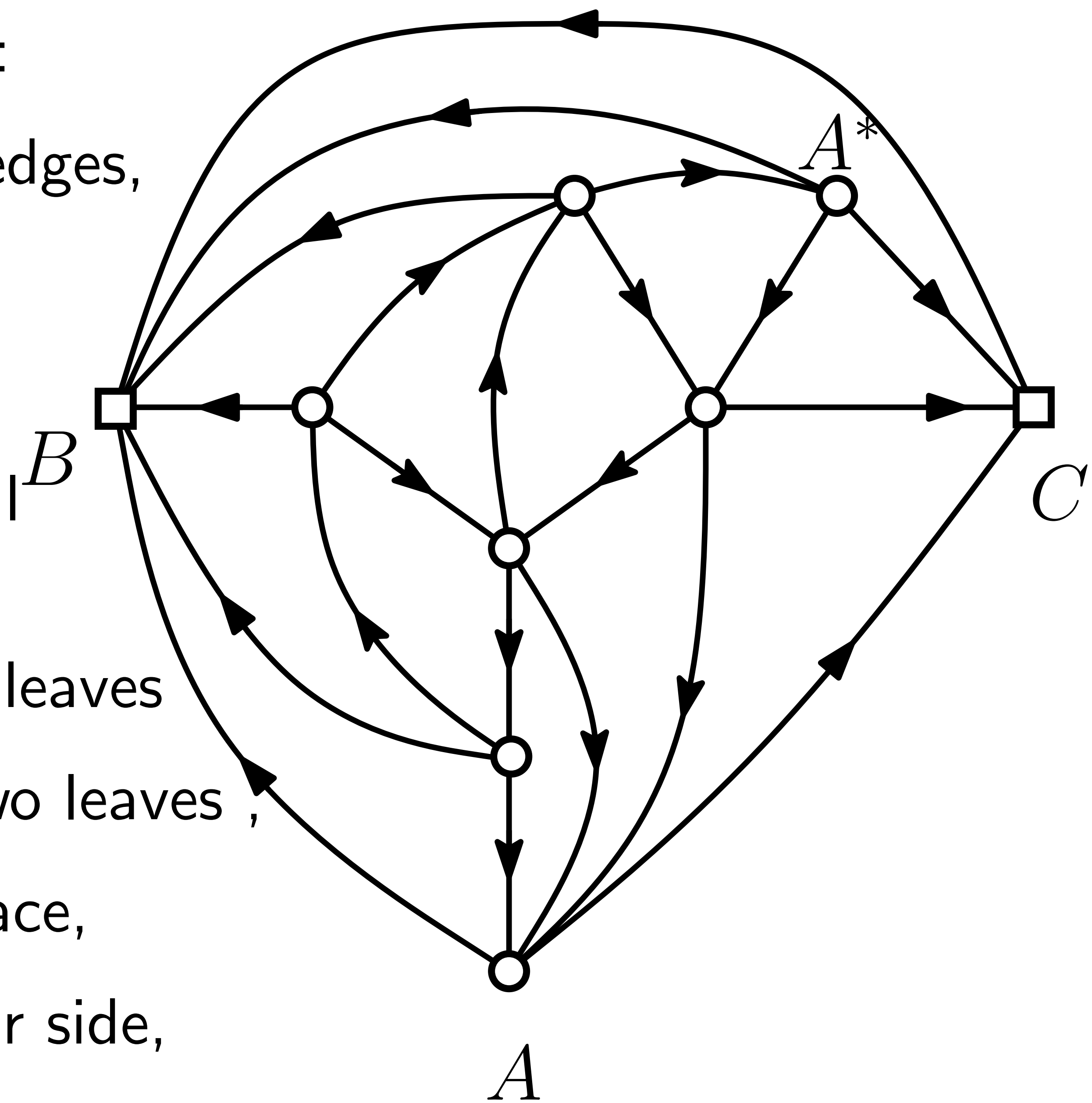
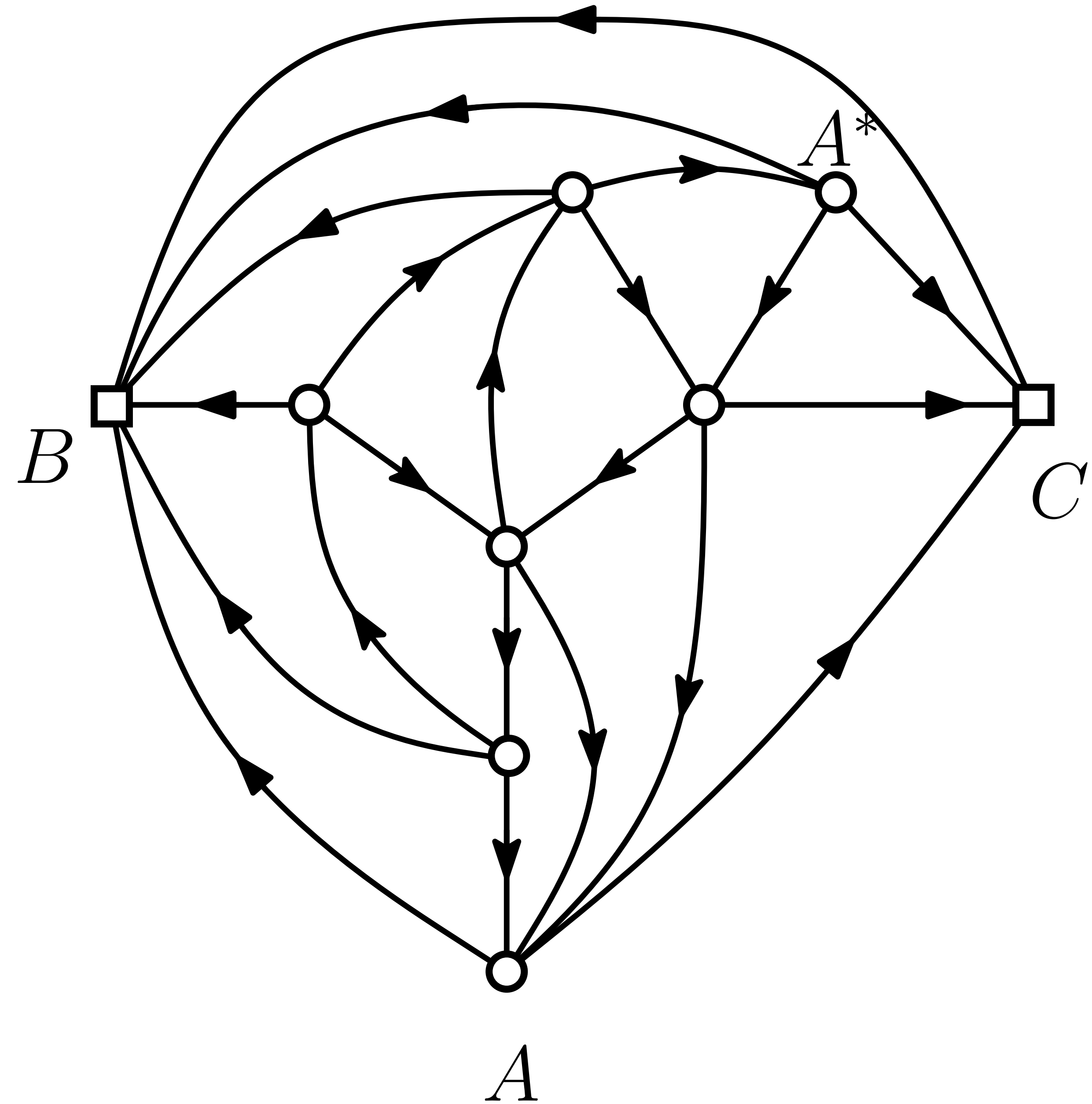
- Connect B and C.



From blossoming trees to simple triangulations

Simple triangulation endowed with its unique orientation such that :

- out$(v) = 3$ for v an inner vertex
- out$(A) = 2$, out$(B) = 1$ and out$(C) = 0$
- no counterclockwise cycle



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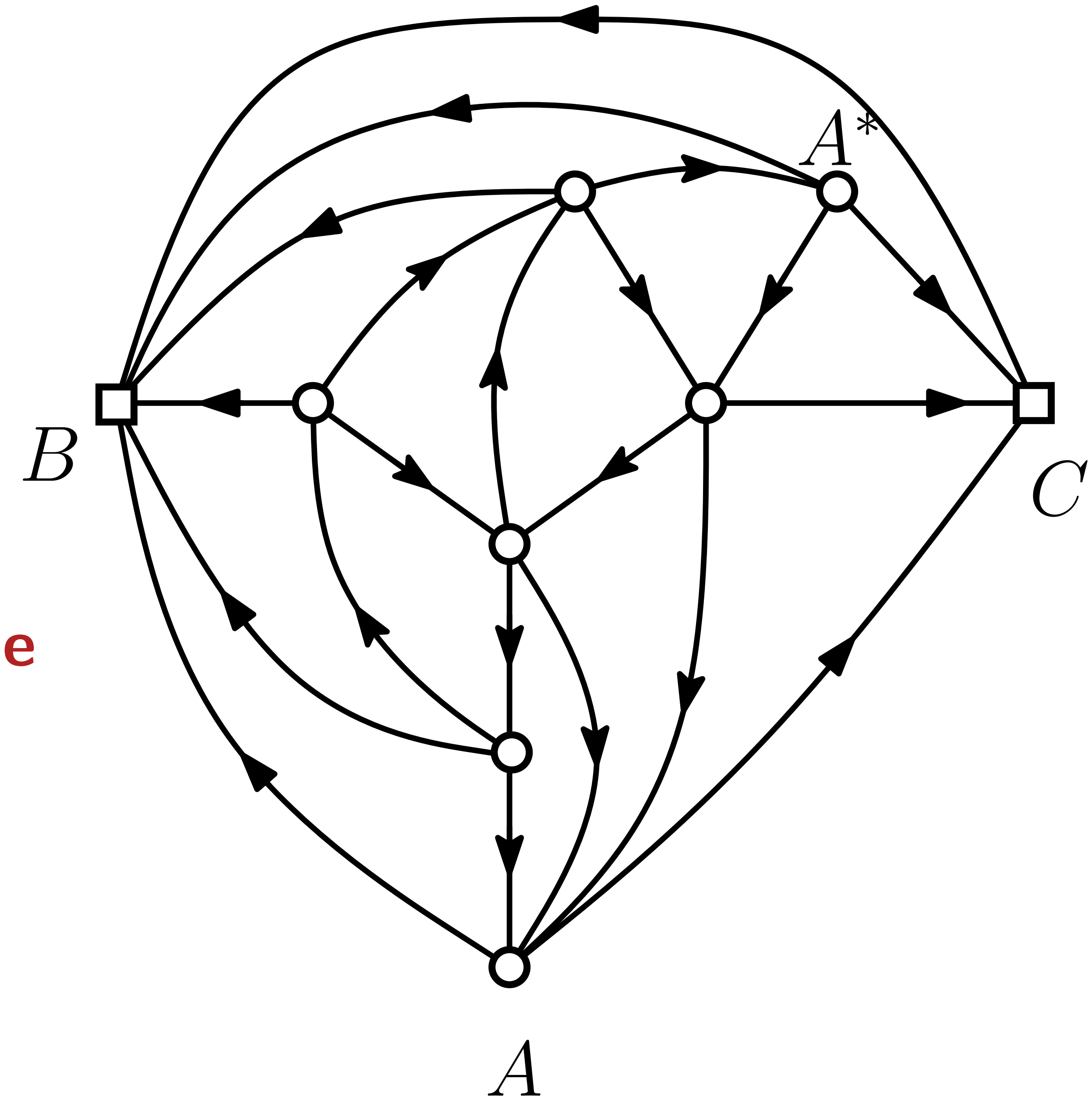
The orientations characterize simple triangulations [Schnyder]



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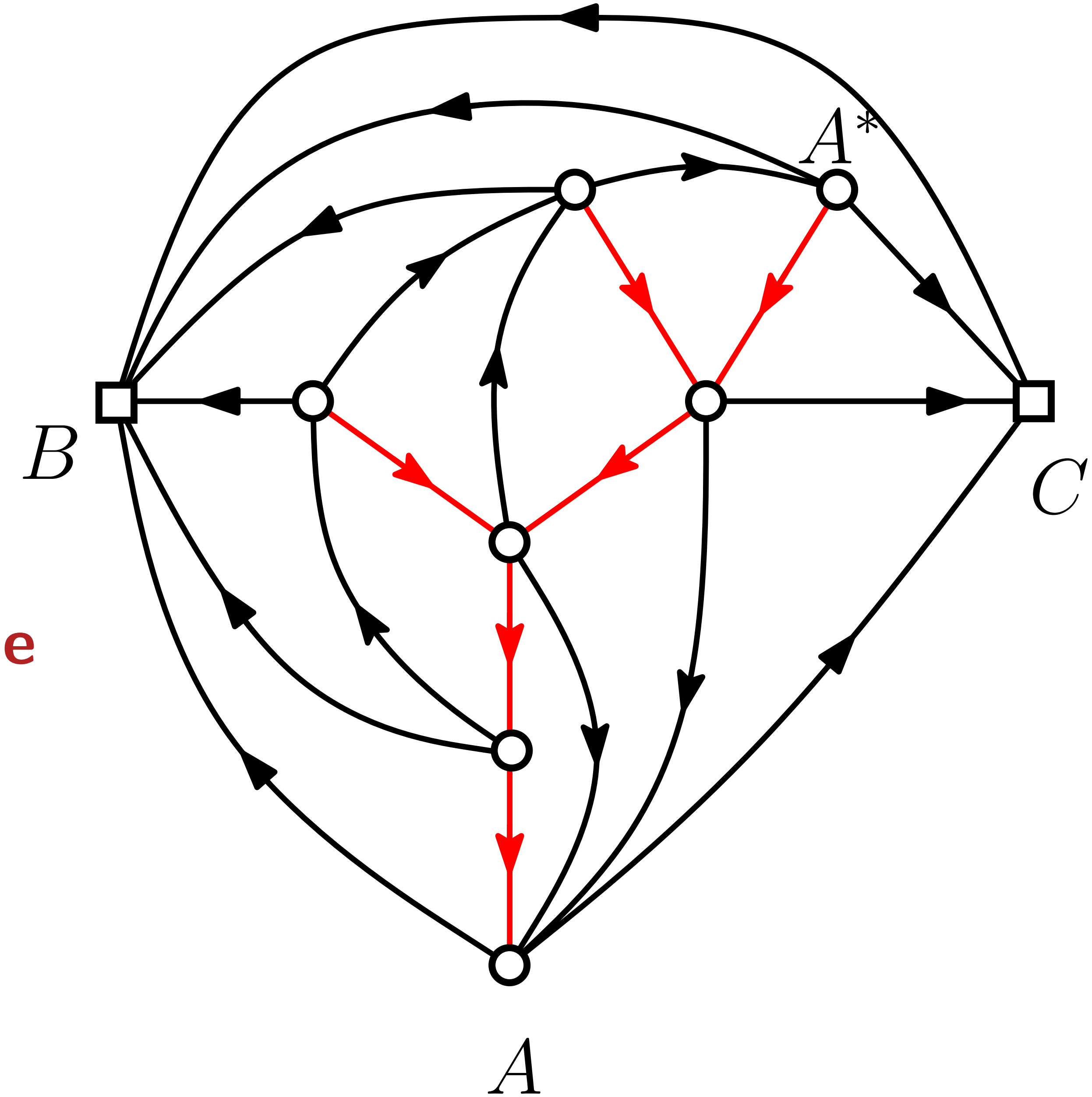
Given the orientation the blossoming tree is the leftmost spanning tree of the map (after removing B and C).



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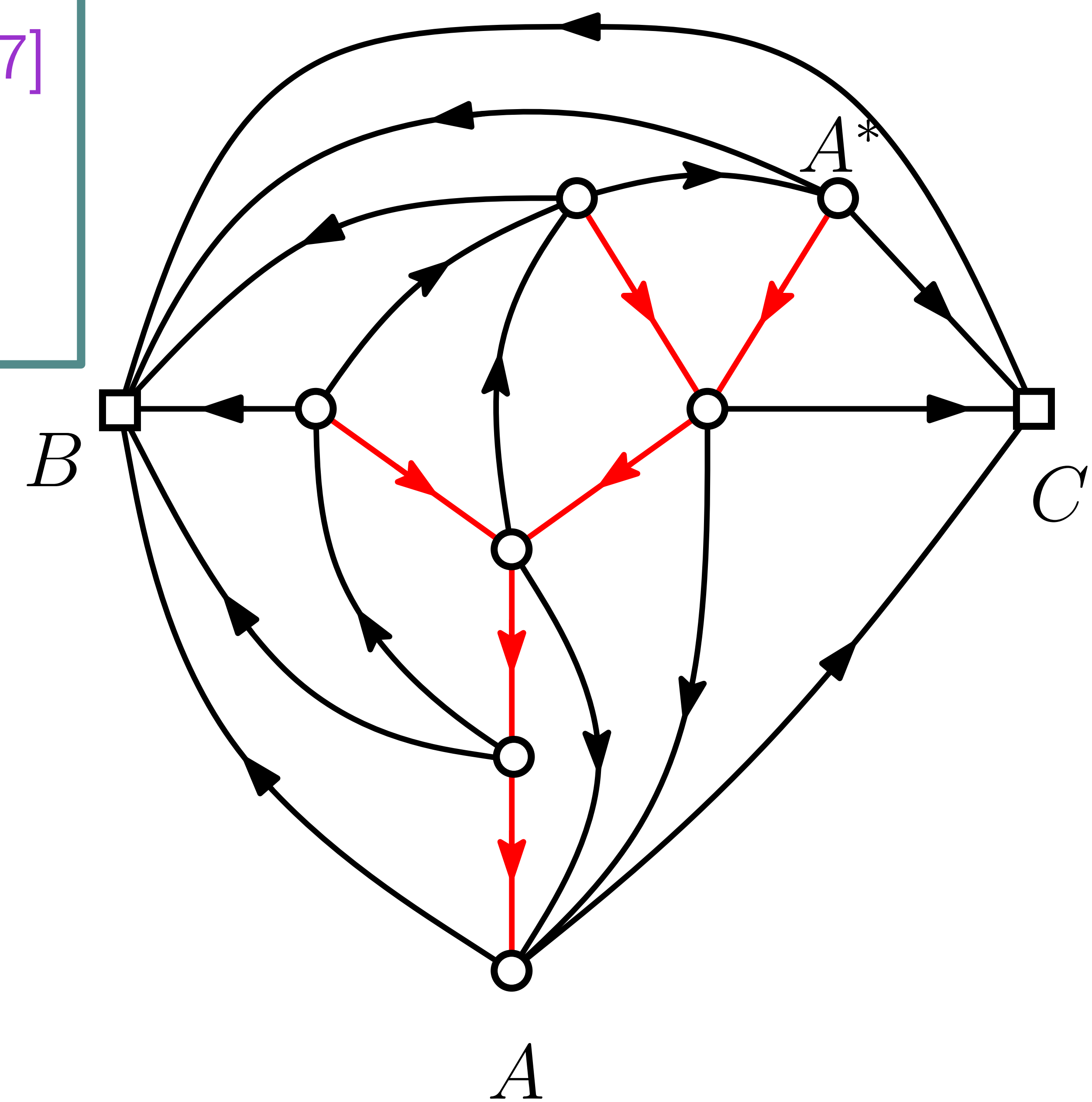
Proposition: [Poulalhon, Schaeffer '07]

The closure operation is a bijection between balanced 2-blossoming trees and simple triangulations.



Same bijection with corner labels

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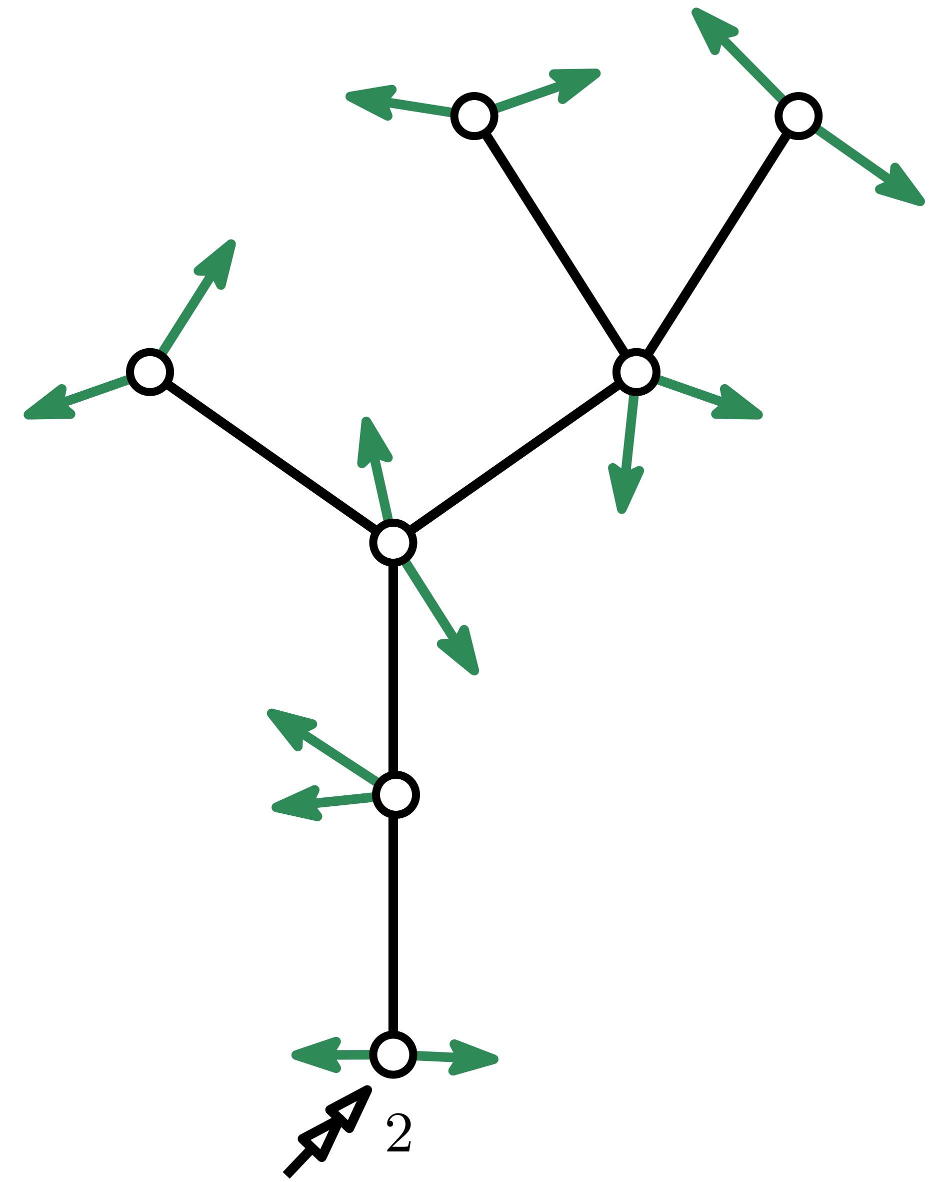

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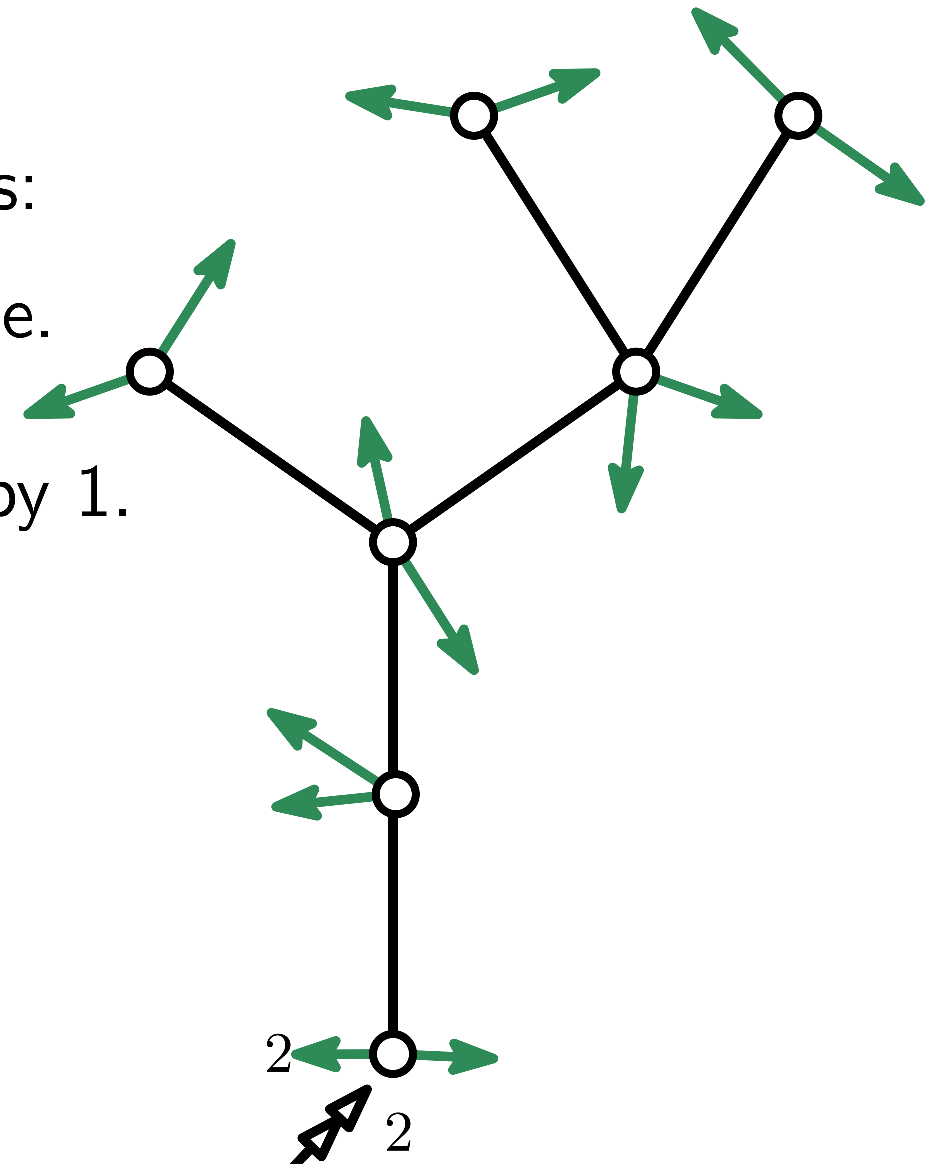

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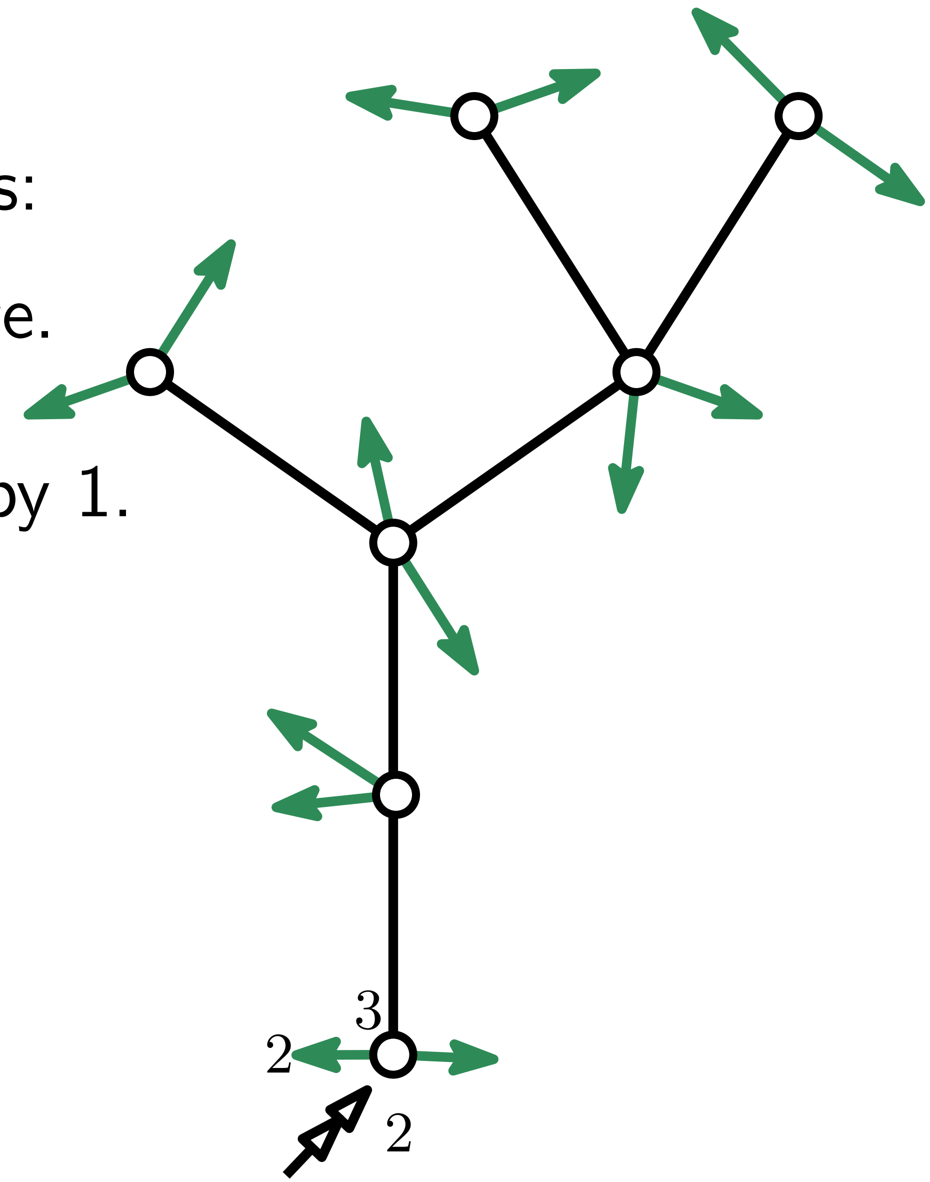

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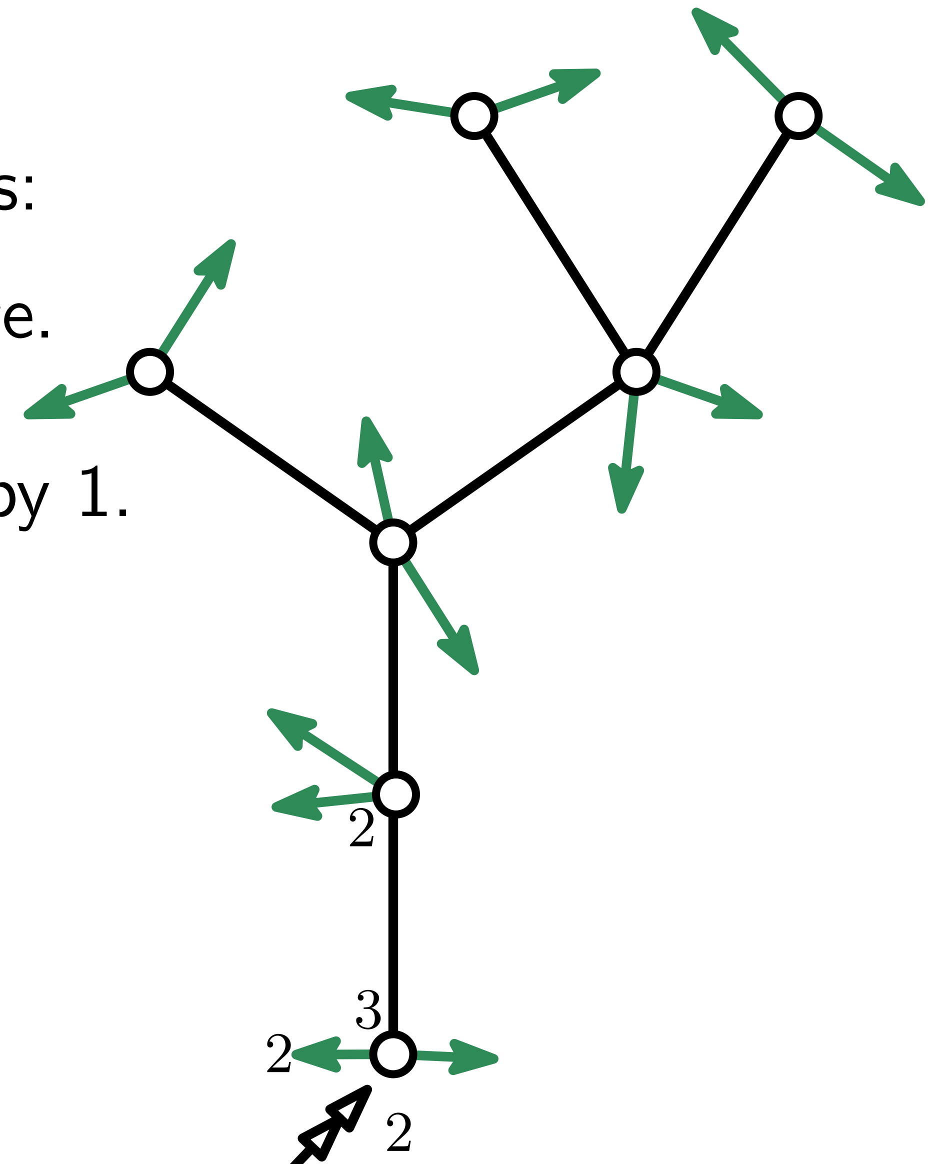

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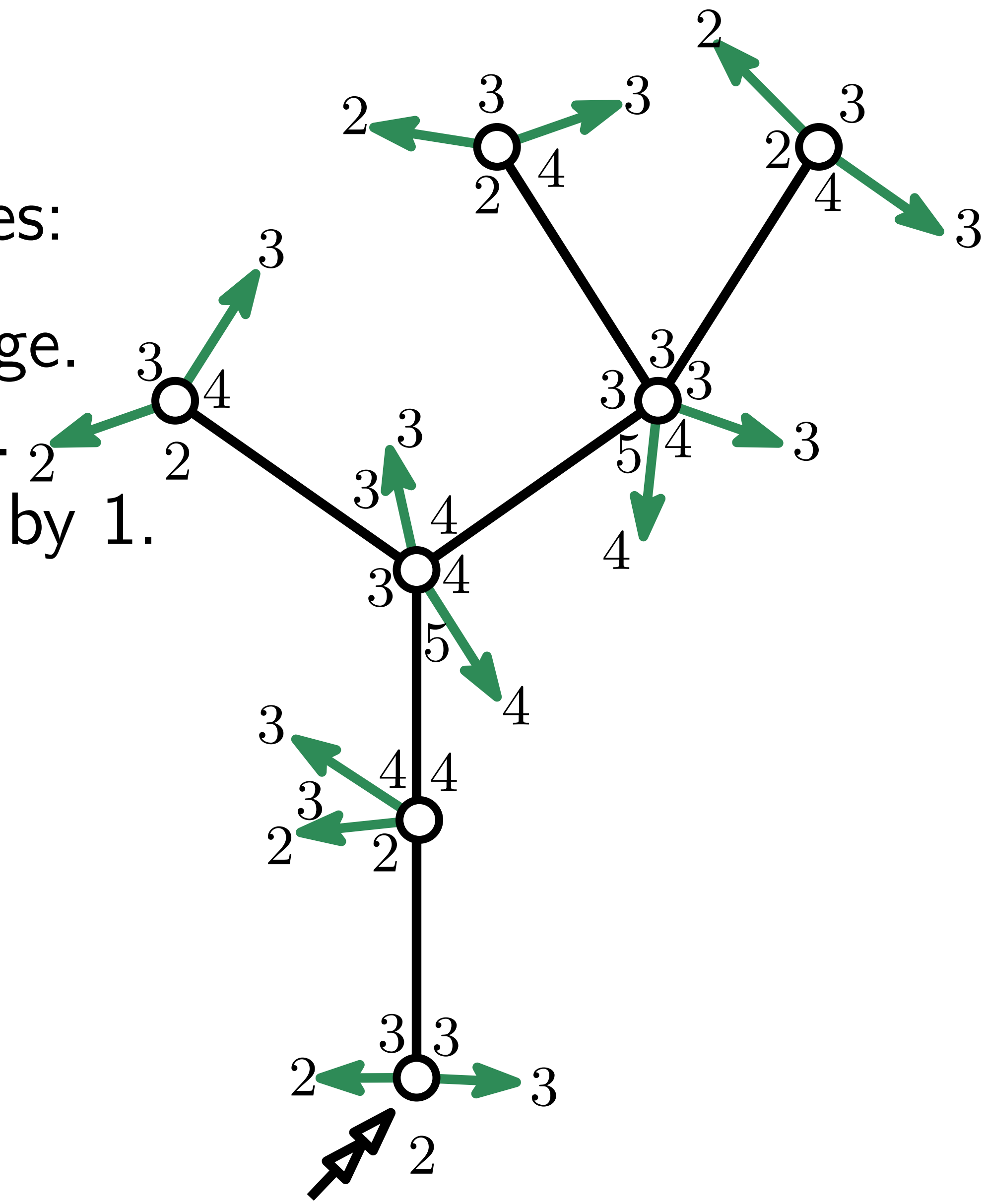

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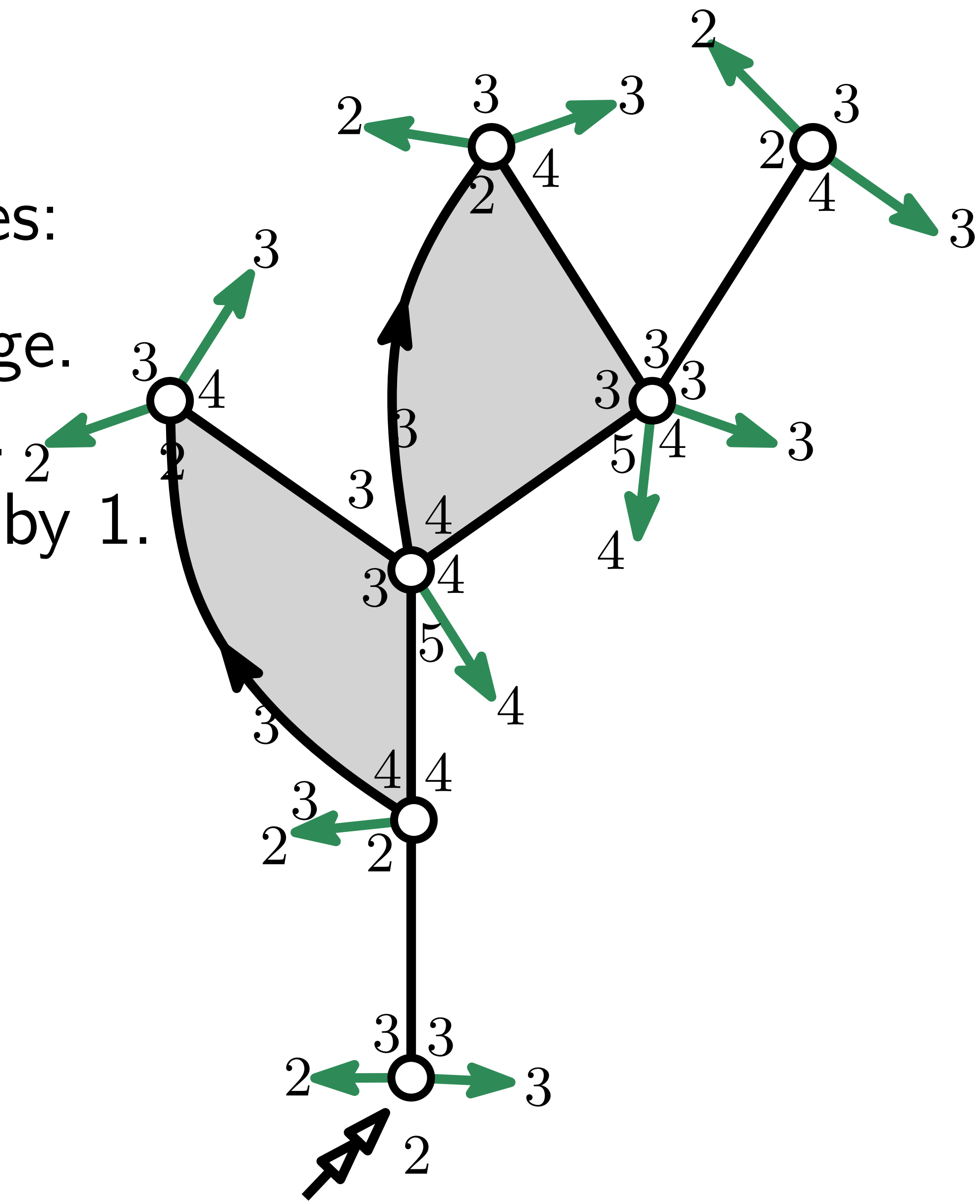
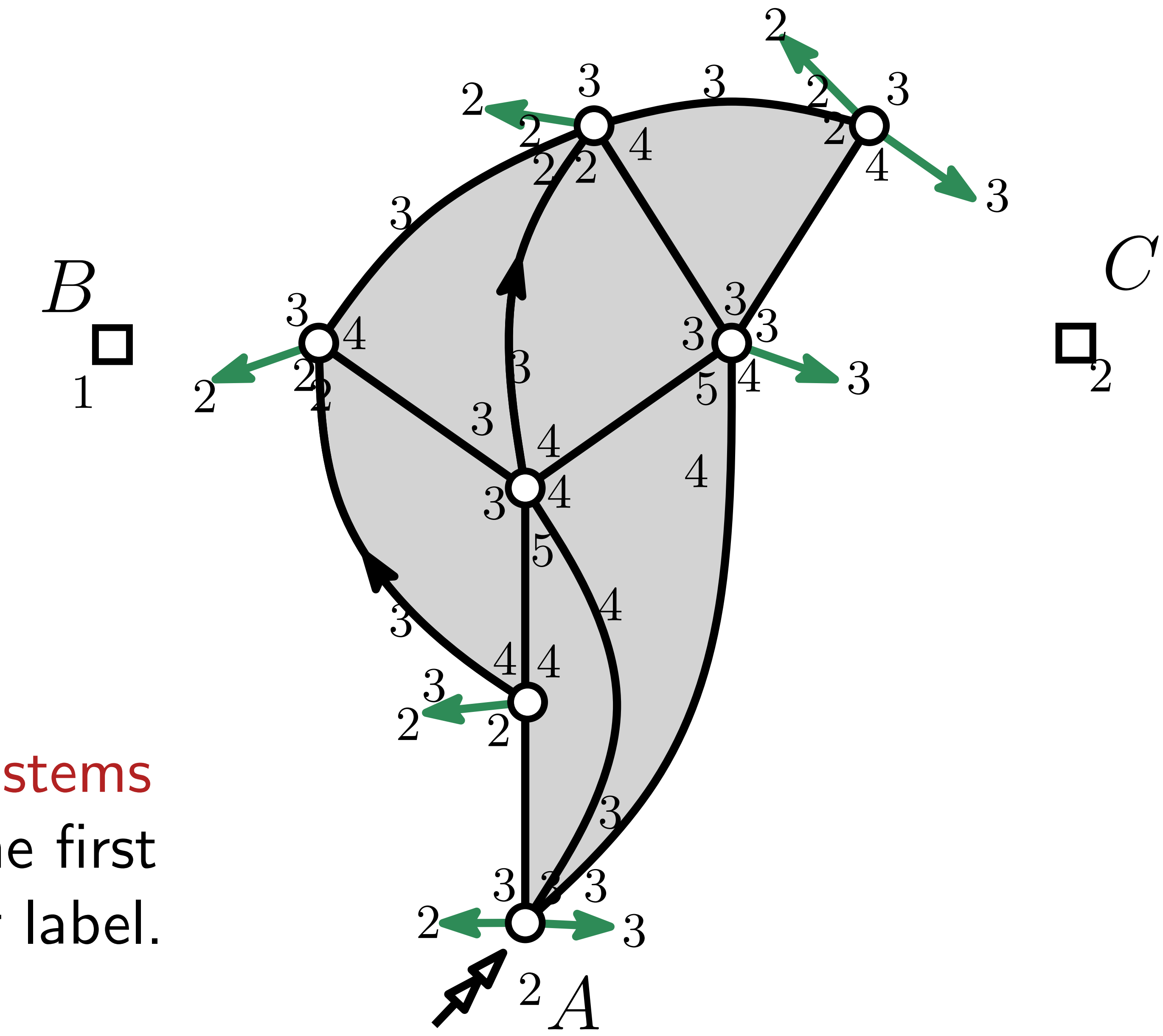


Aside: Tree is balanced $\Leftrightarrow$
all labels ≥ 2
$+$root corner incident to two stems

Closure: Merge each leaf with the first subsequent corner with a smaller label.

Same bijection with corner labels



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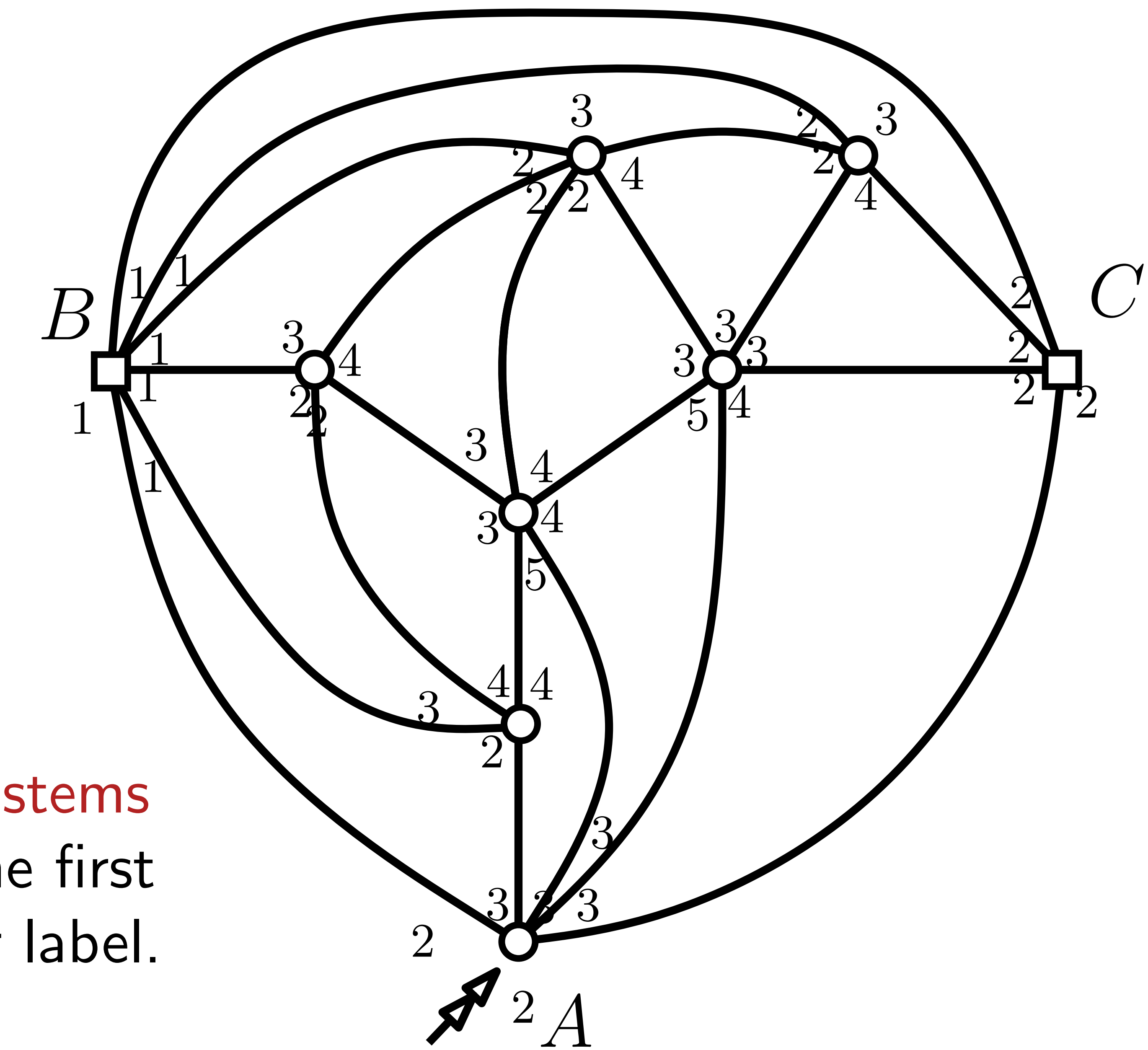

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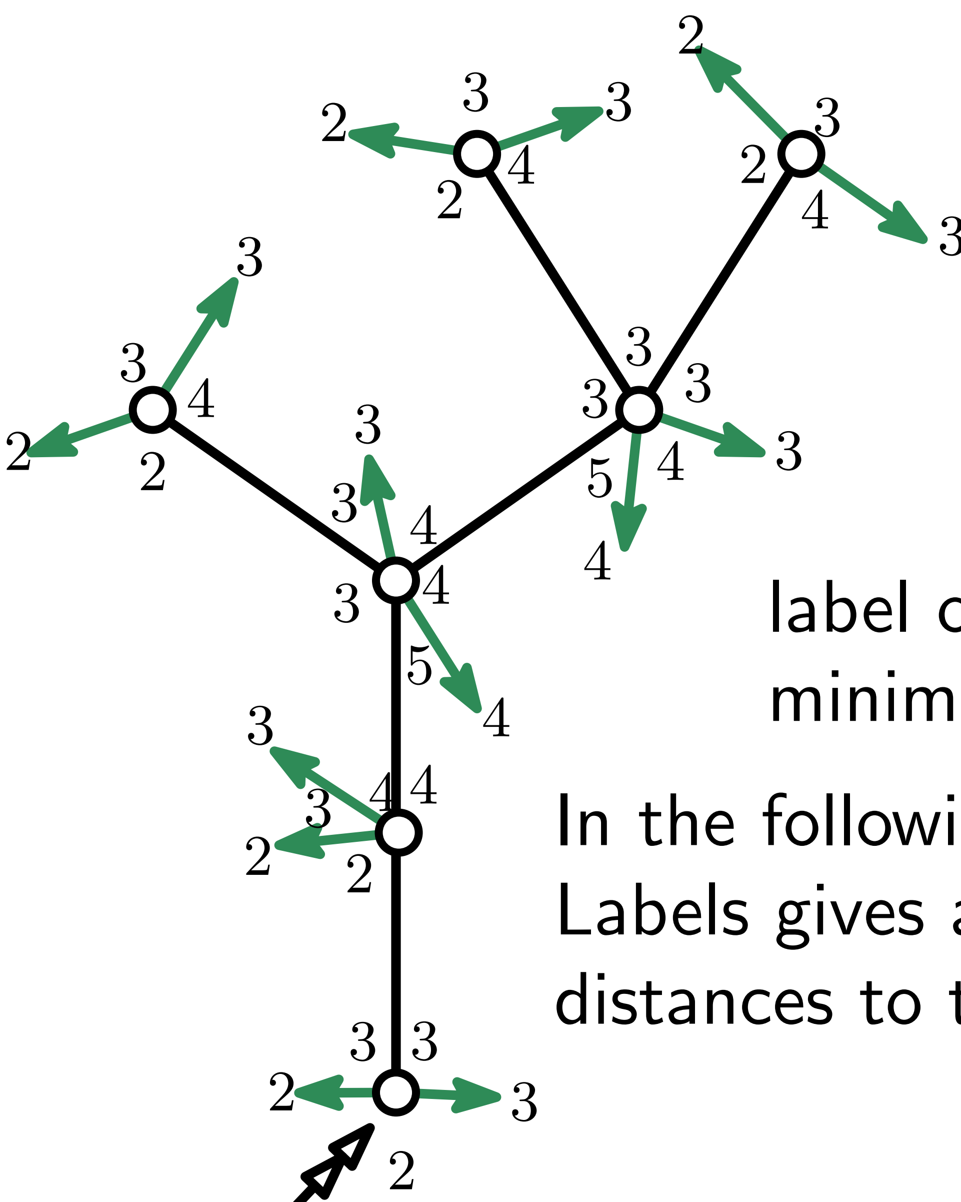
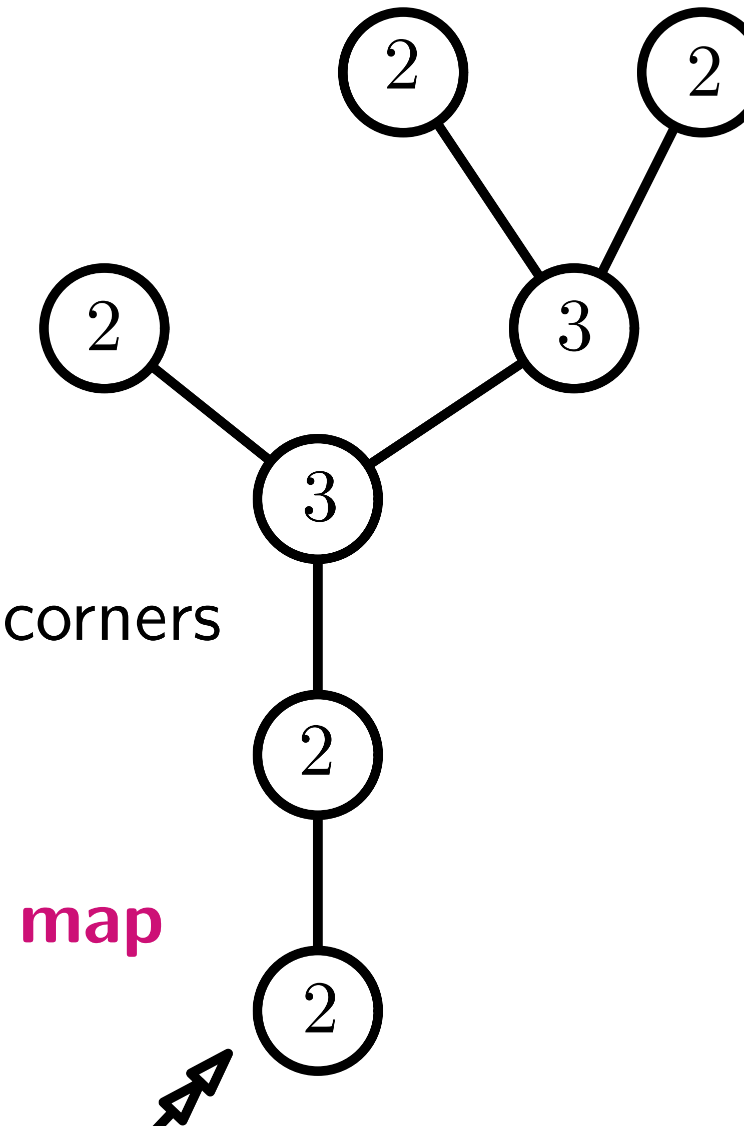
From blossoming trees to labeled trees



label of a vertex =
minimum label of its corners

In the following:
Labels gives approximate
distances to the root **in the map**



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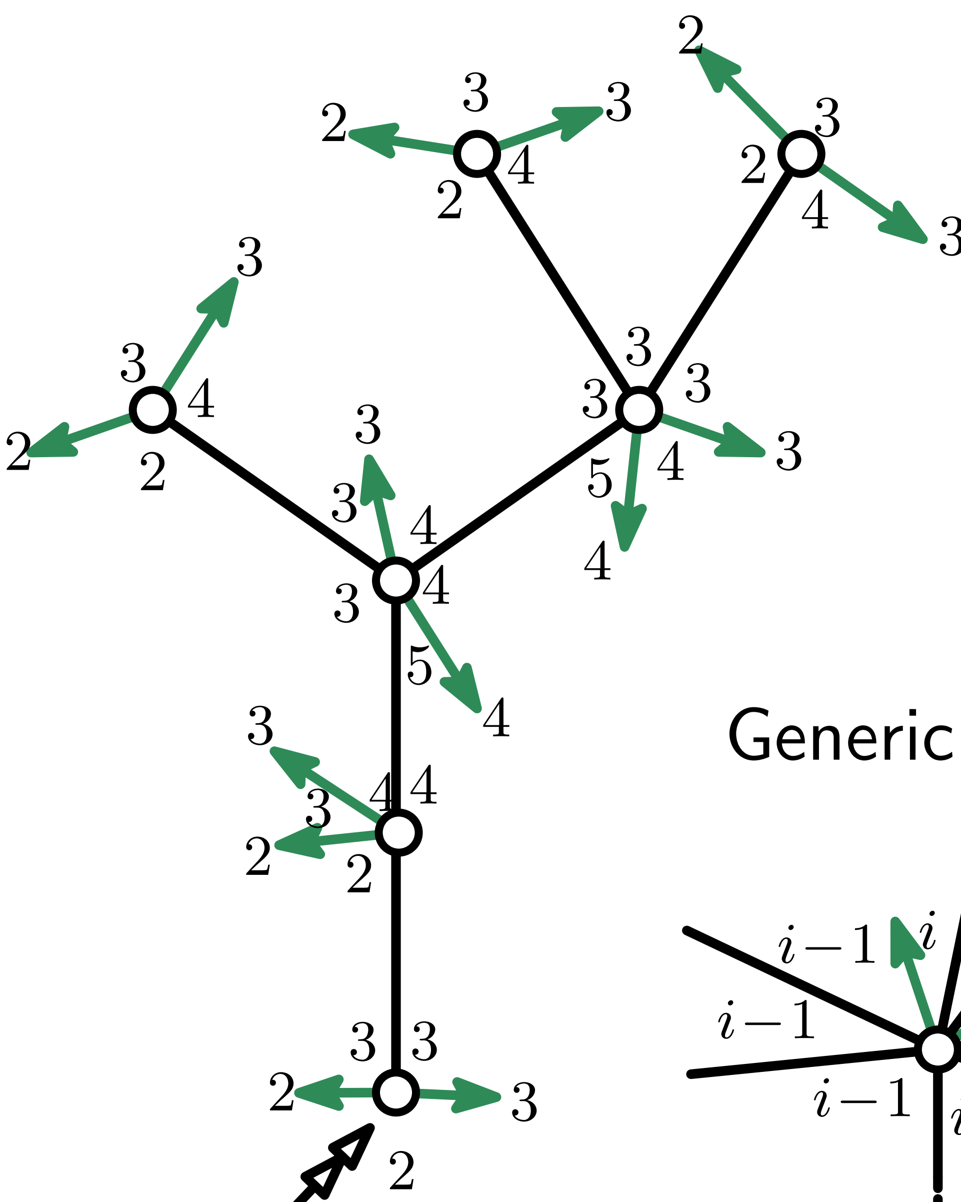
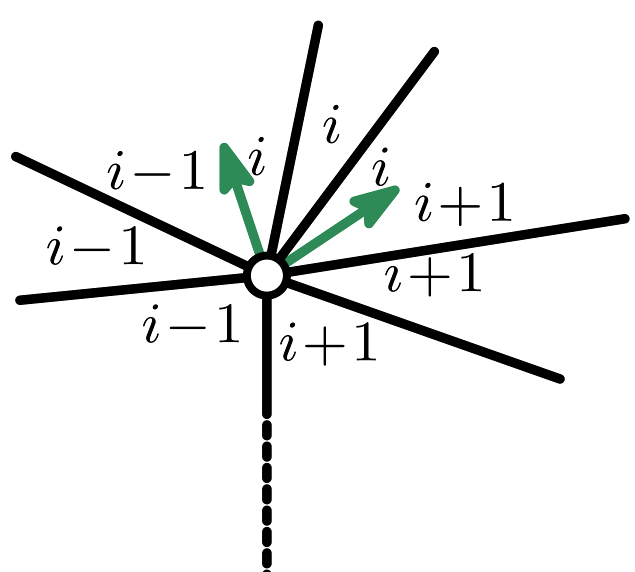
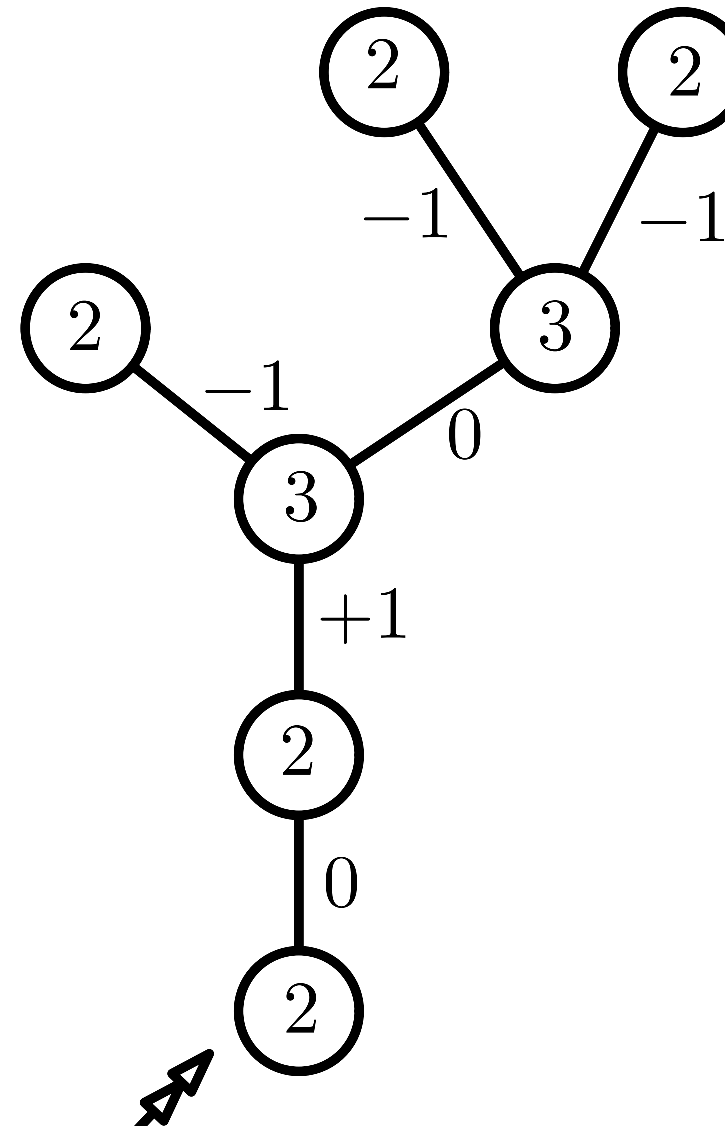


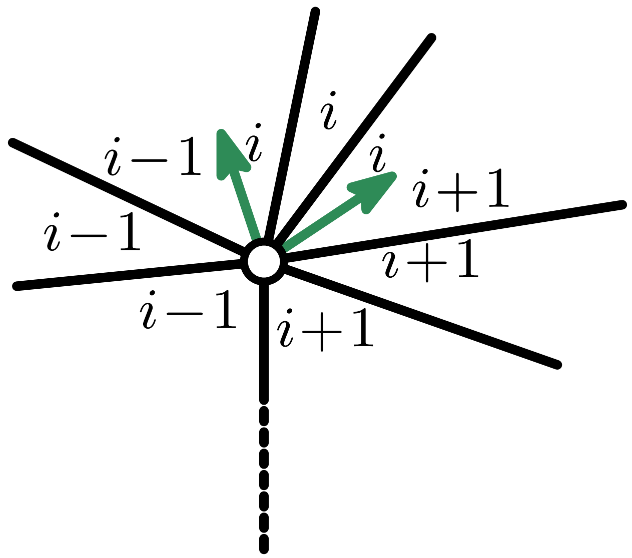
Generic vertex :

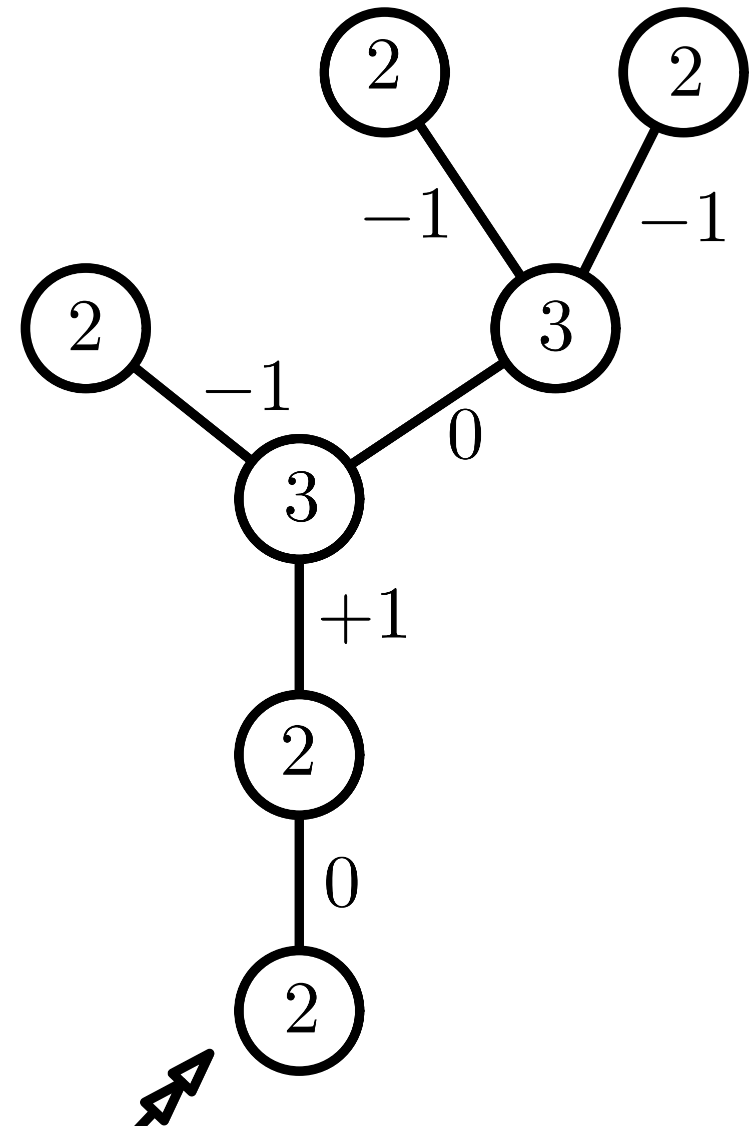




From blossoming trees to labeled trees

Generic vertex :





- Can retrieve the blossoming tree from the labeled tree.
- Labeled tree = GW trees + random displacements on edges uniform on

$$\{(-1, -1, \dots, -1, 0, 0, \dots, 0, 1, 1 \dots, 1)\}.$$

almost the setting of [Janson-Marckert] and [Marckert-Miermont] but r.v are not "locally centered" $\Rightarrow$ symmetrization required

Convergence of labeled trees

Theorem : [Addario-Berry, A.]
For a sequence of simple random triangulations (M_n), the contour and label process of the associated labeled tree satisfie:

$$\left((3n)^{-1/2}C_{\lfloor nt\rfloor},(4n/3)^{-1/4}\tilde{Z}_{\lfloor nt\rfloor}\right)_{0\leq t\leq 1}\underset{n\to\infty}{\overset{(d)}{\to}}(e_t,Z_t)_{0\leq t\leq 1},$$

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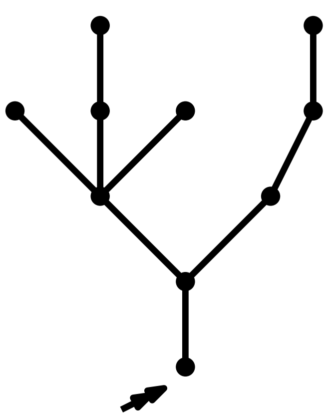

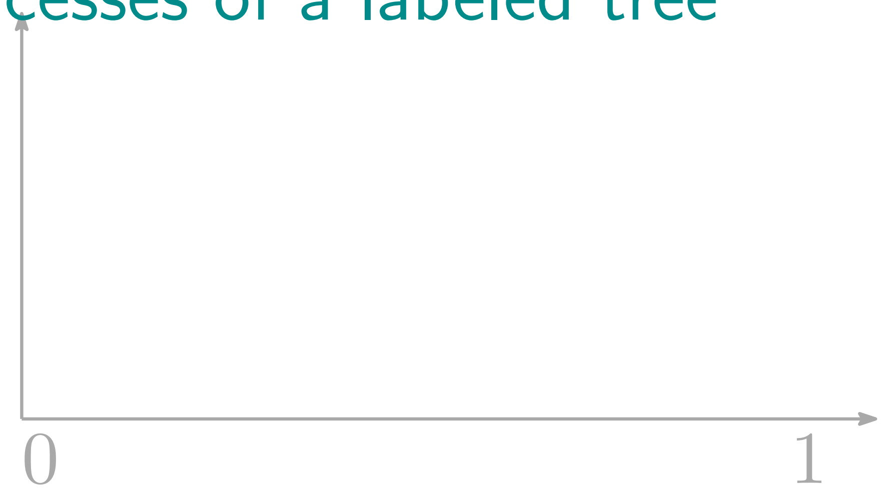

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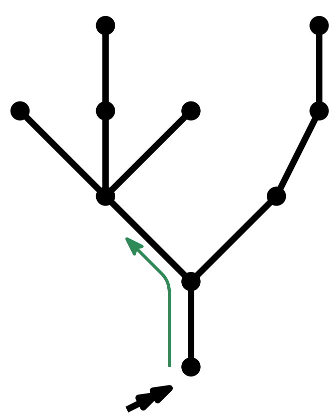

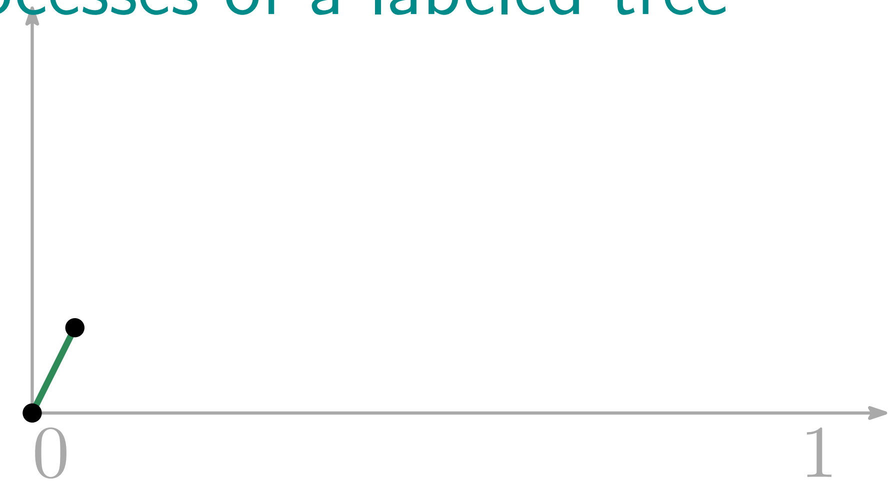

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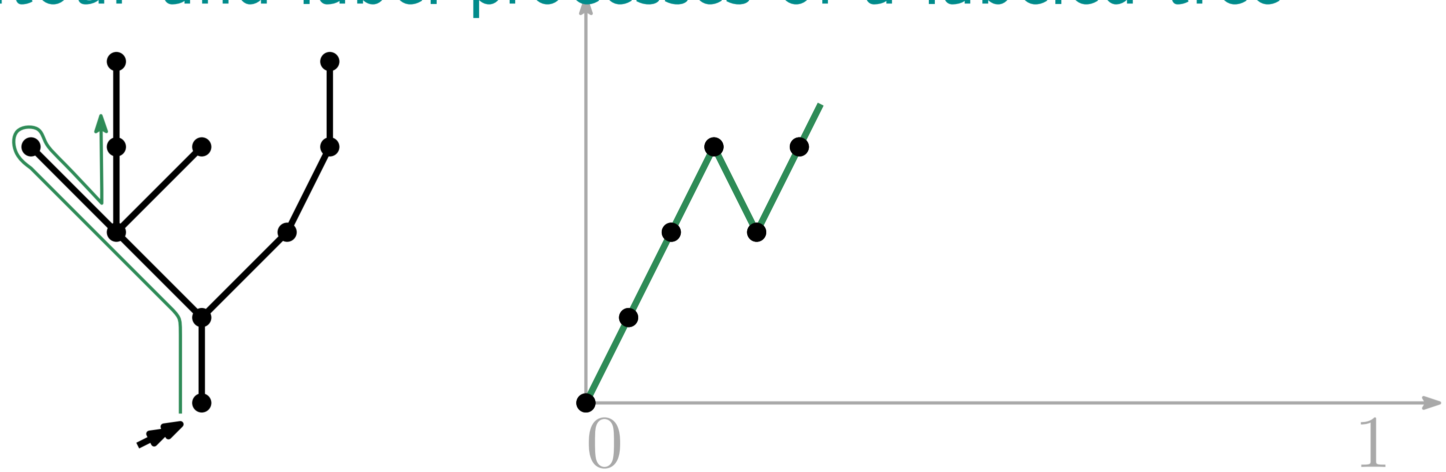

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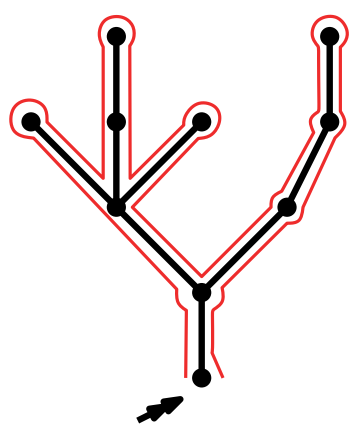

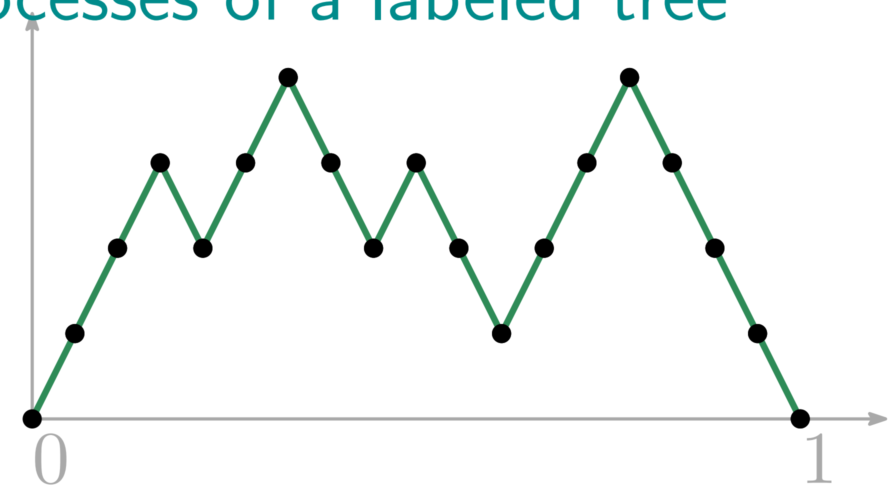

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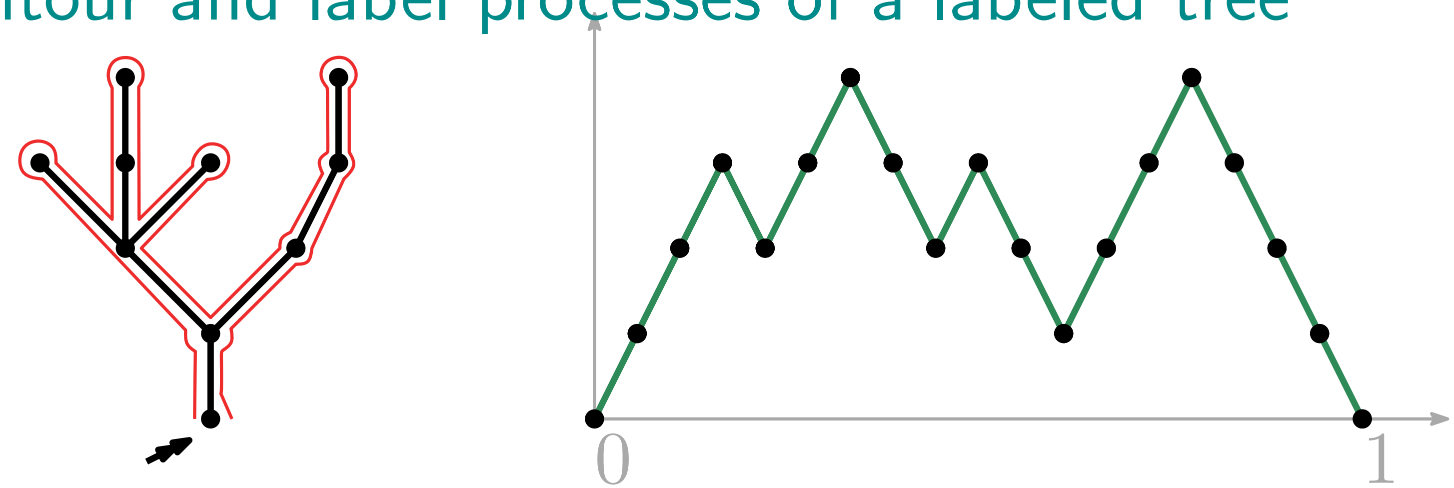

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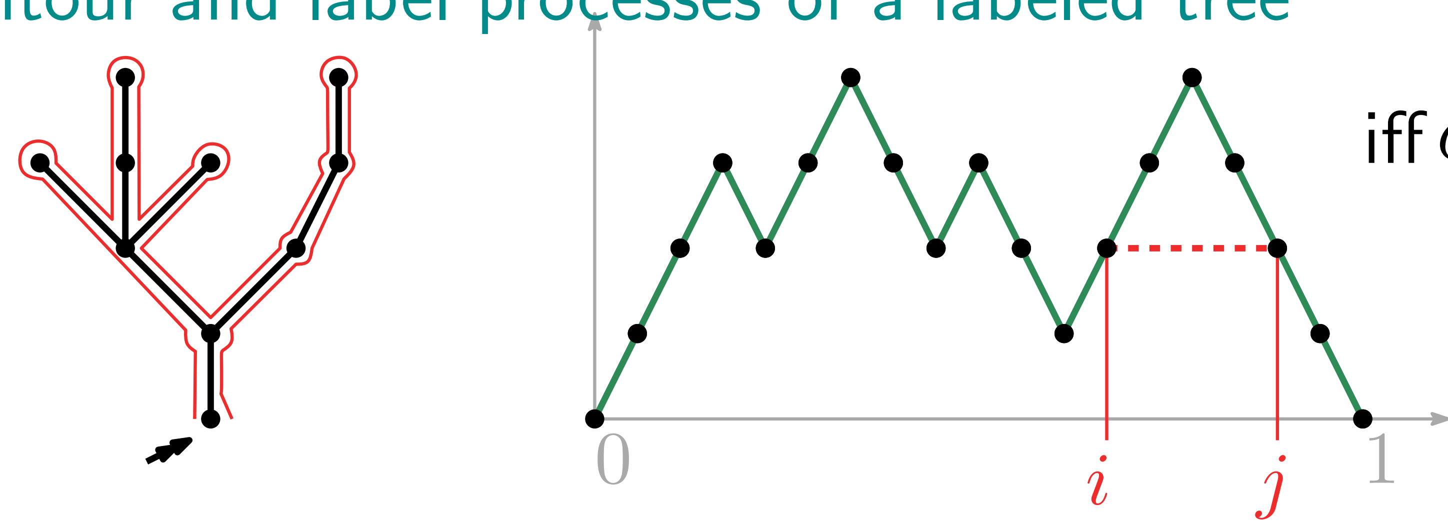

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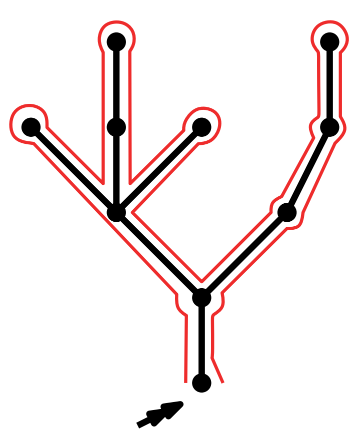

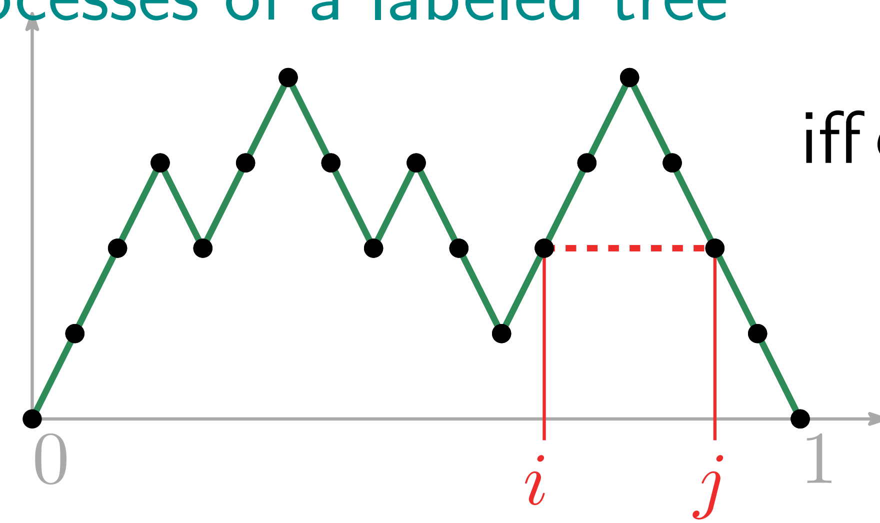

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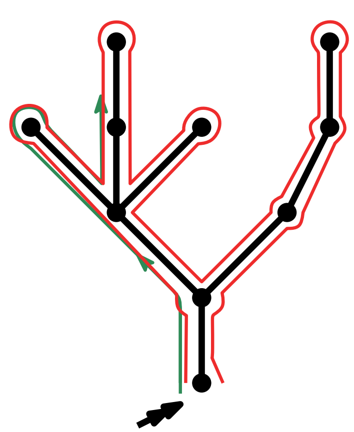

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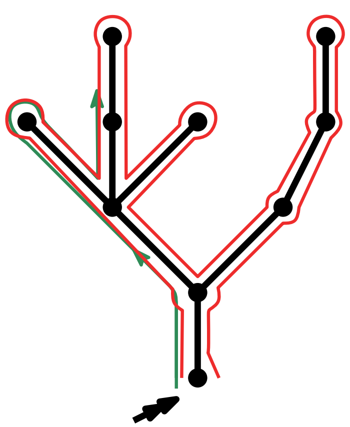

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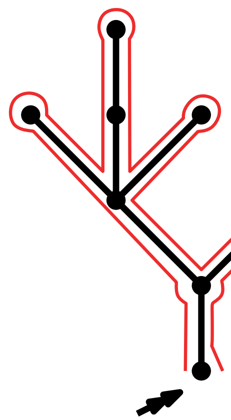
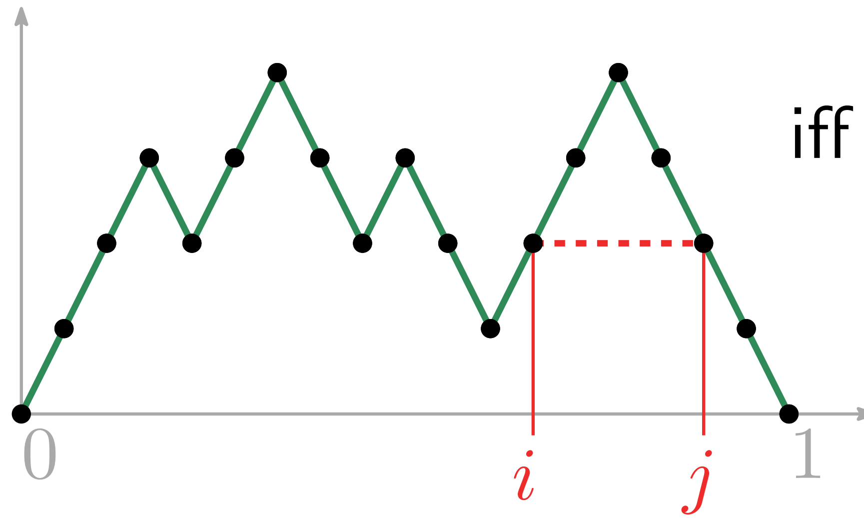
1st step : the Brownian tree [Aldous]





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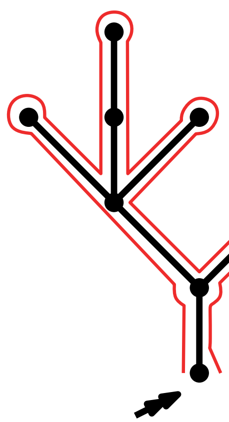

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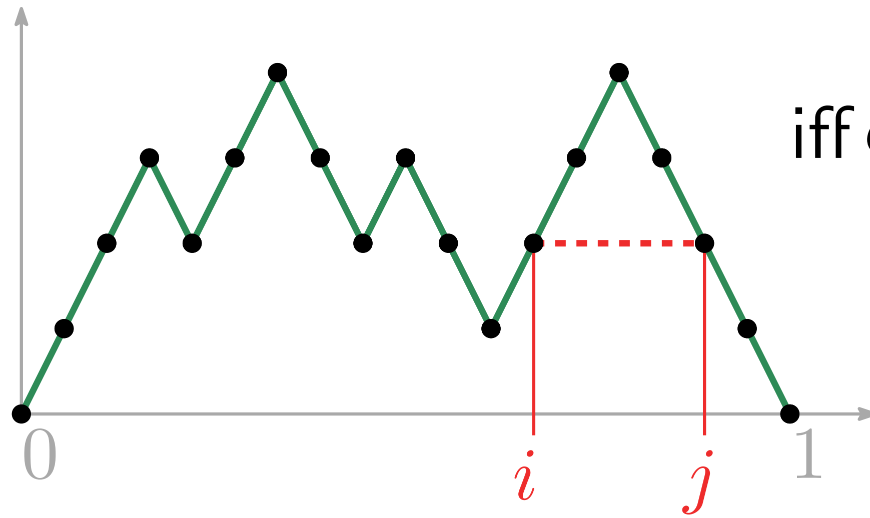

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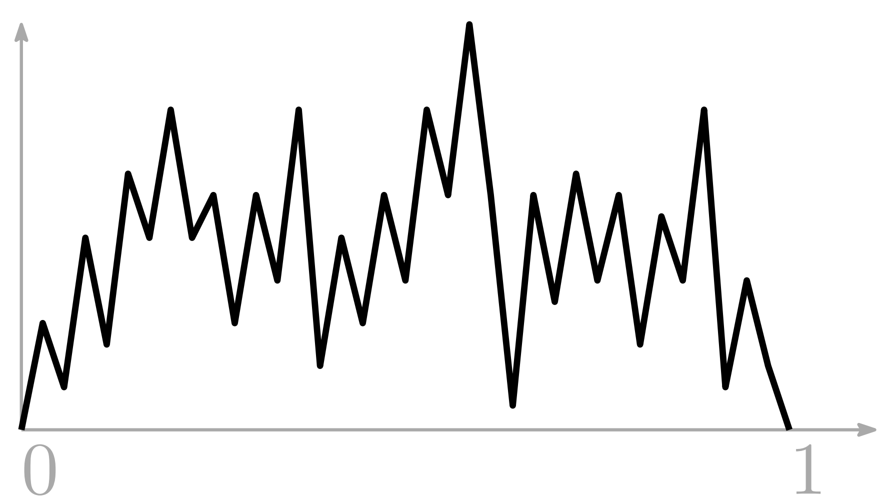

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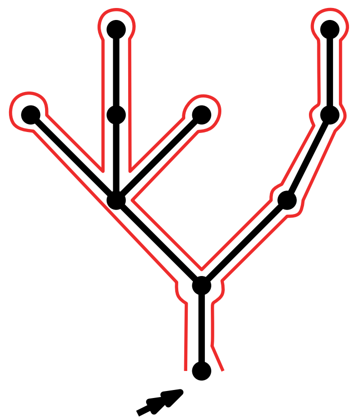

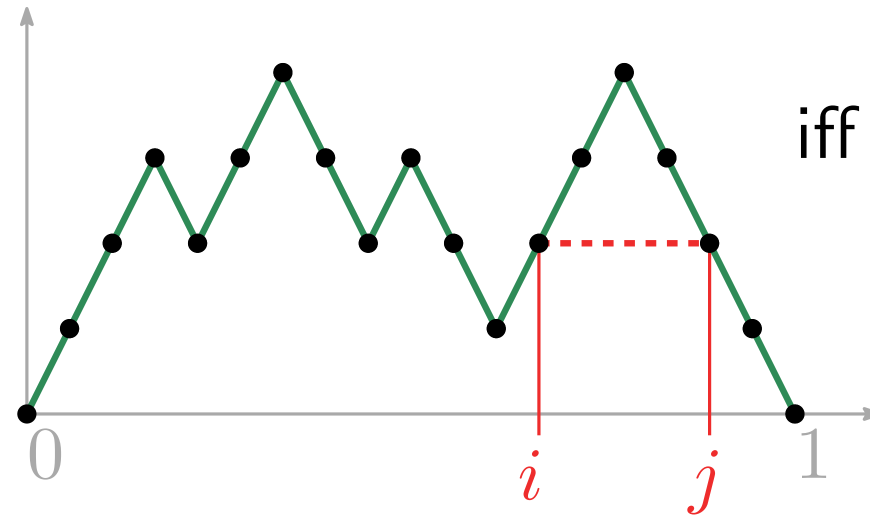

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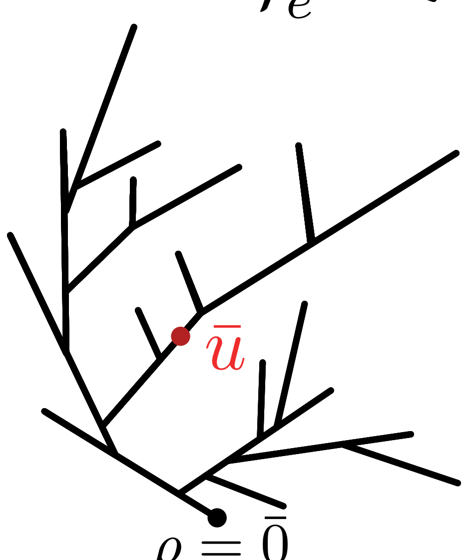

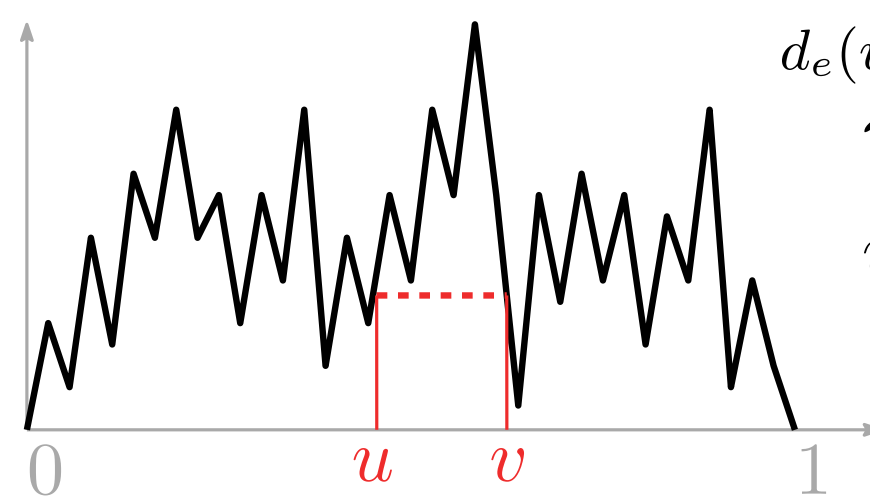

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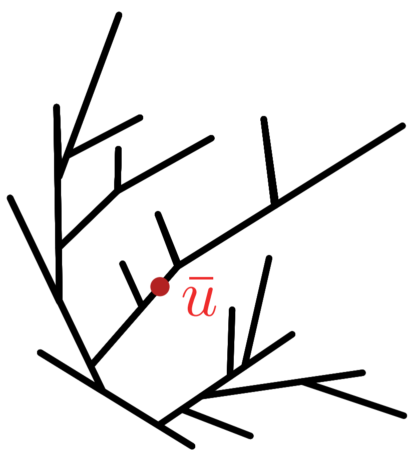

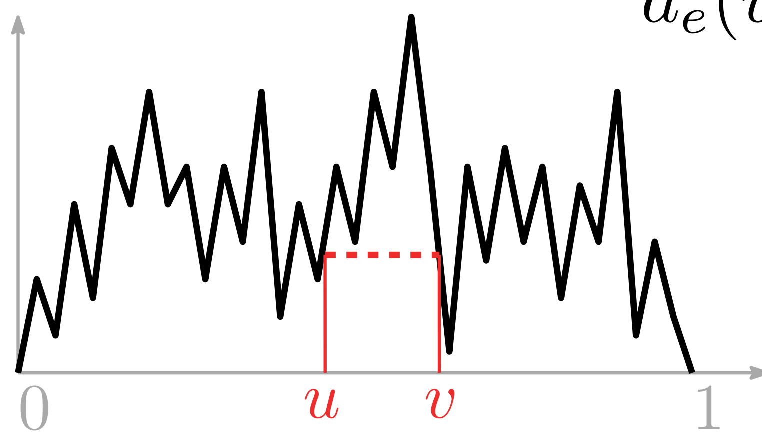

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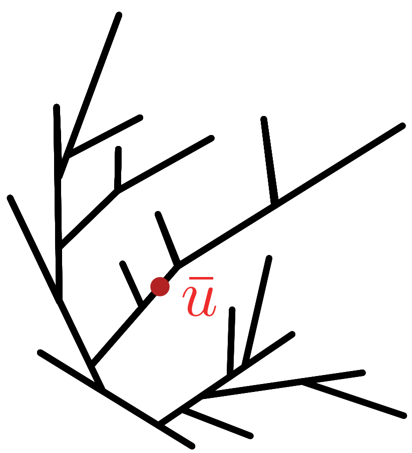

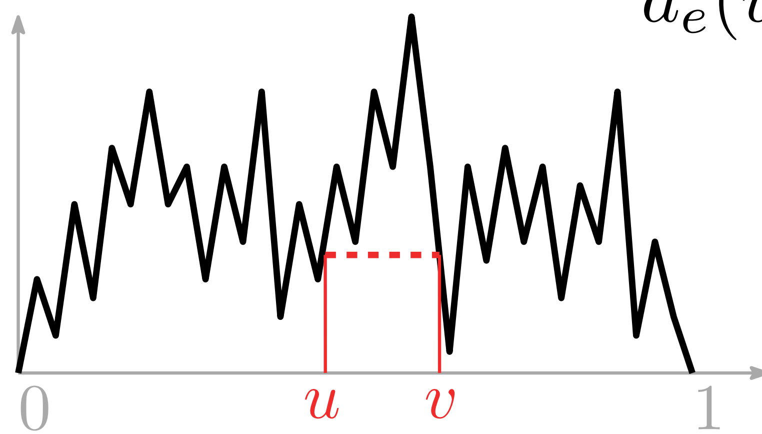

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Conditional on $\mathcal{T}_e$, Z a centered Gaussian process with $Z_\rho = 0$ and $E[(Z_s - Z_t)^2] = d_e(s,t)$

$Z \sim$ **Brownian motion on the tree**

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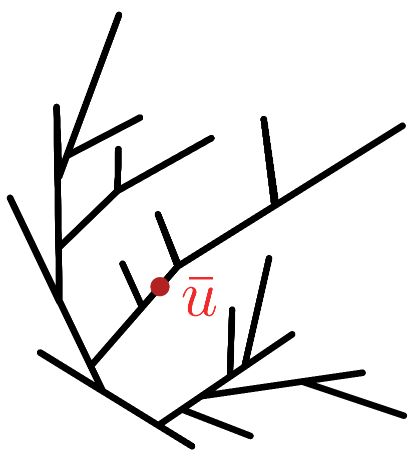

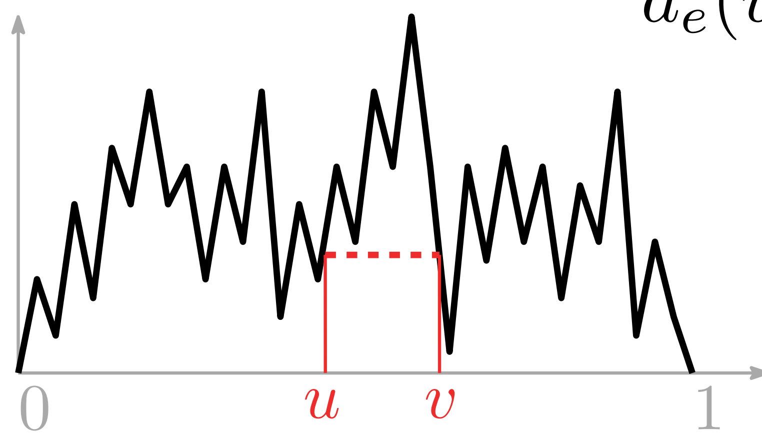

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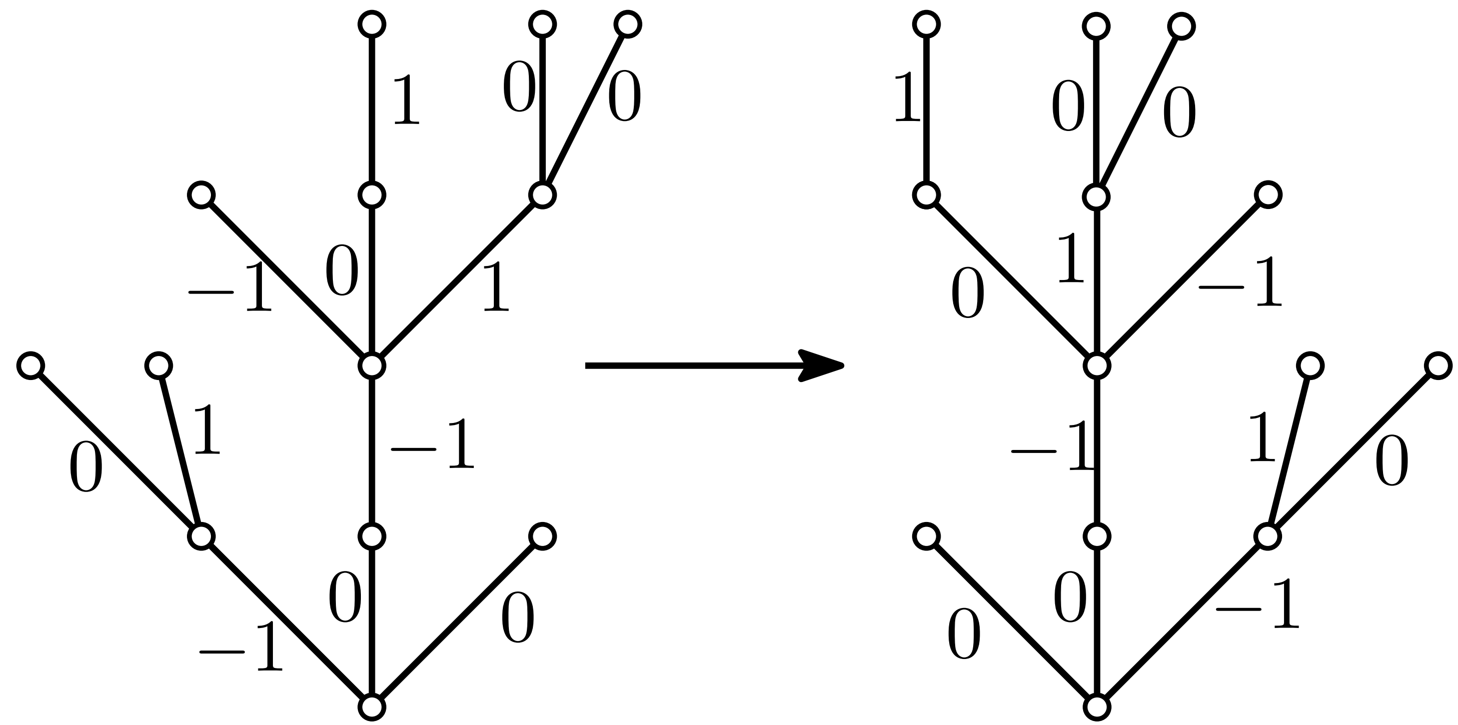
Idea of proof :

Start with one of “our” tree and apply a random permutation at each vertex



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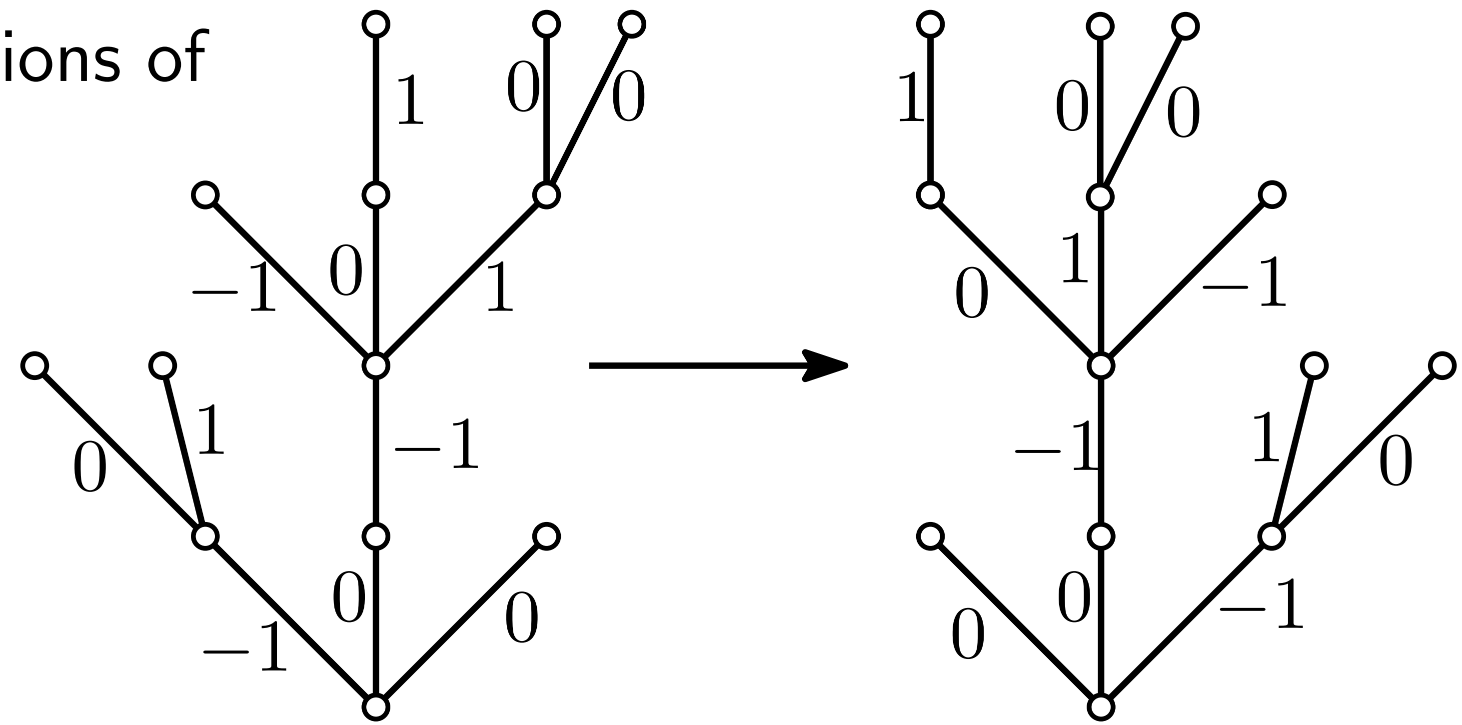

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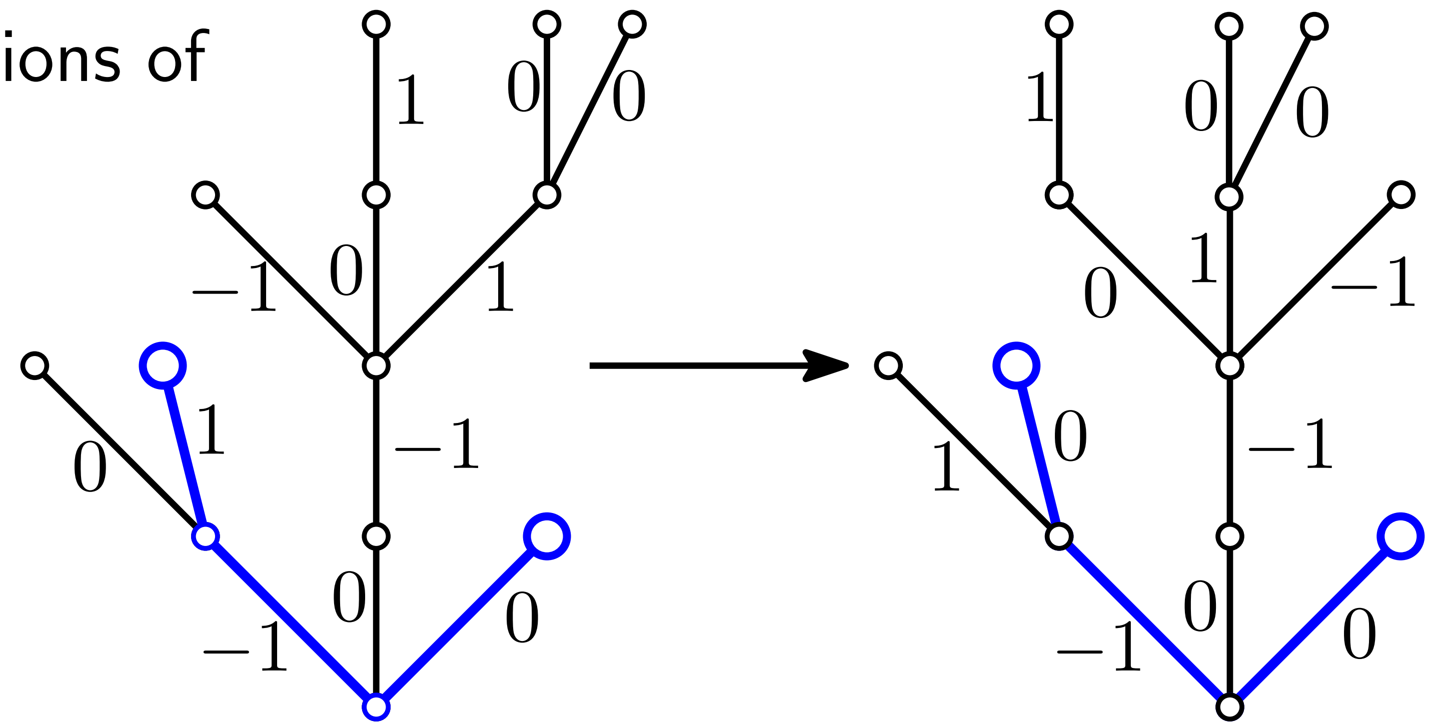

Solution:
- consider subtree $T\langle k\rangle$ spanned by k random vertices
- permute displacements and edges only outside $\langle T\rangle$.
- permute only displacements on $\langle T\rangle$.

Gives a coupling between “our” model and the fully permuted model: sufficient control to prove convergence for the true model.

Distances in simple triangulations

M_n = simple triangulation

$(C_{\lfloor nt \rfloor}, \tilde{Z}_{\lfloor nt \rfloor})$ = contour and label process of the associated tree

$Z_{\lfloor nt \rfloor}$ = distance **in the map** between vertex "$\lfloor nt \rfloor$" and the root.

Theorem : [Addario-Berry, A.]
M_n= random simple triangulation, then for all $\varepsilon > 0$:

$$\mathbb{P}\left(\sup_{0 \leq t \leq 1} \left\{ \left| \tilde{Z}_{\lfloor nt \rfloor} - Z_{\lfloor nt \rfloor} \right| \right\} \geq \varepsilon n^{1/4} \right) \to 0.$$

i.e. the label process of the tree gives the distance to the root in the map.

Distances in simple triangulations

Claim 1: $3d_{M_n}(root, u_i) \geq L_n(u_i)$

First observation : In the tree, the labels of two adjacent vertices differ by at most 1. **What can go wrong with closures ?**

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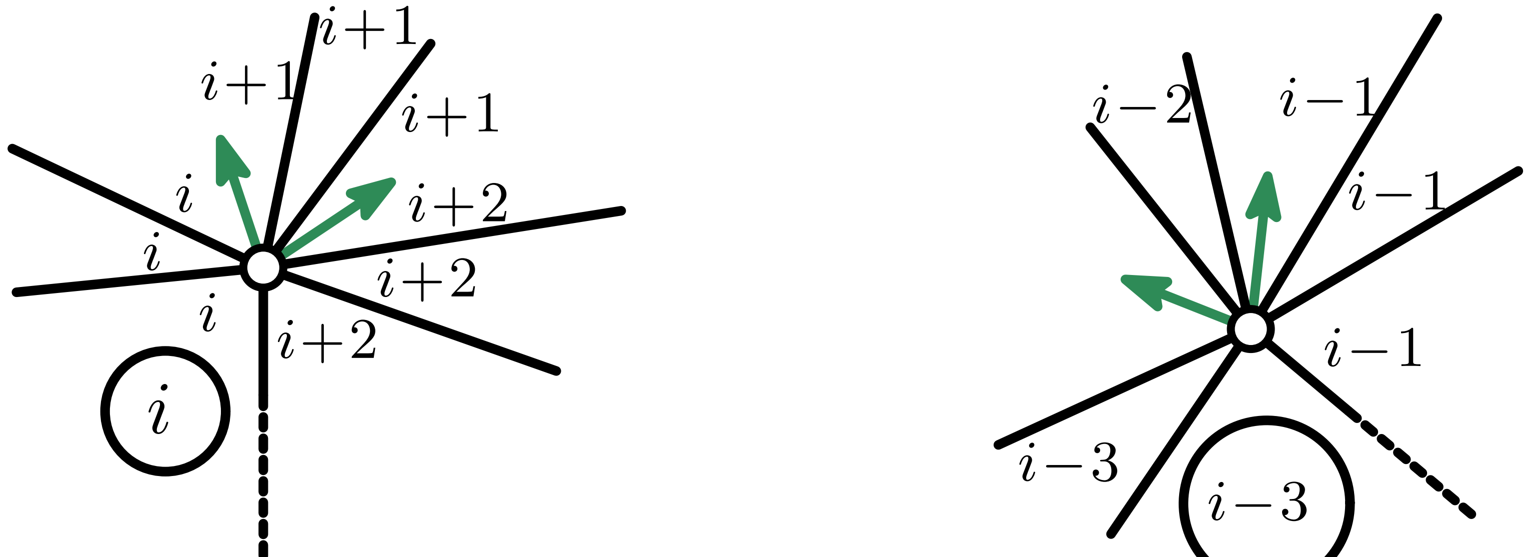

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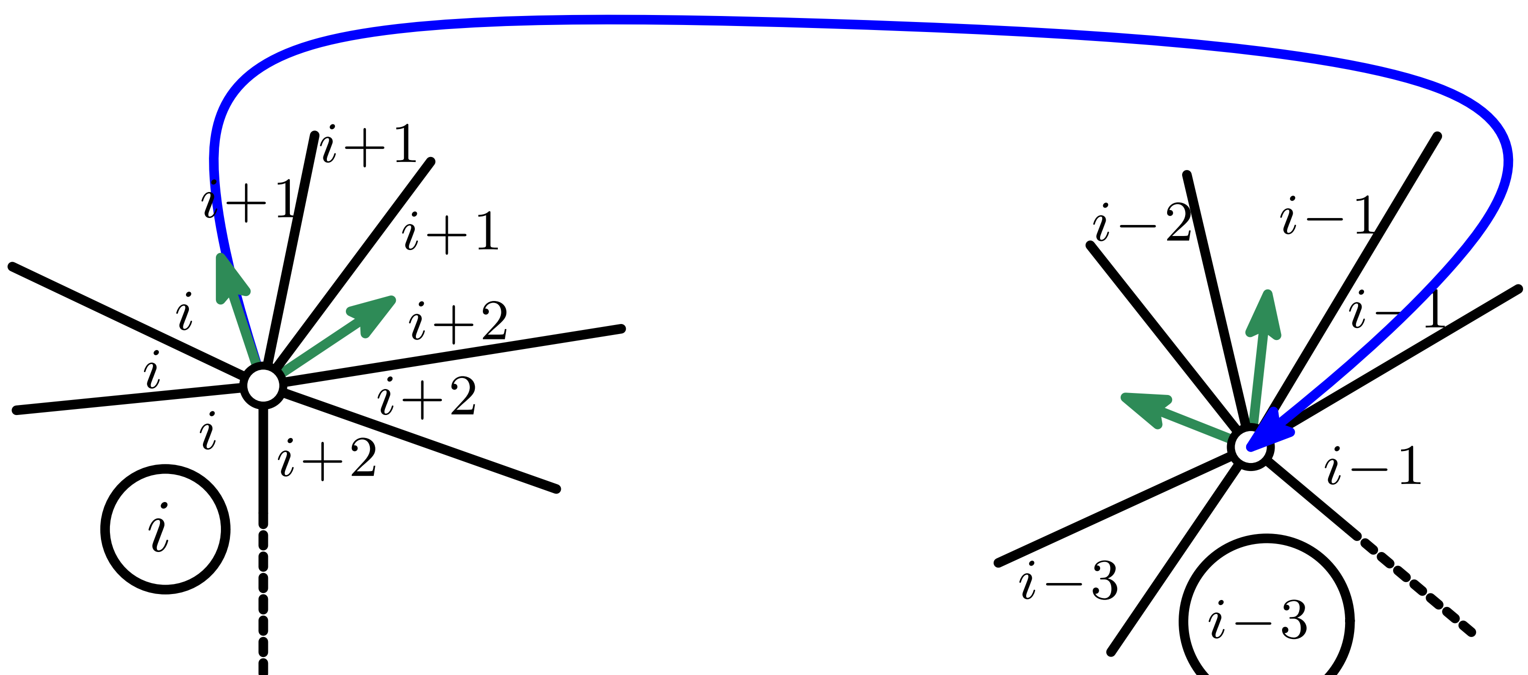

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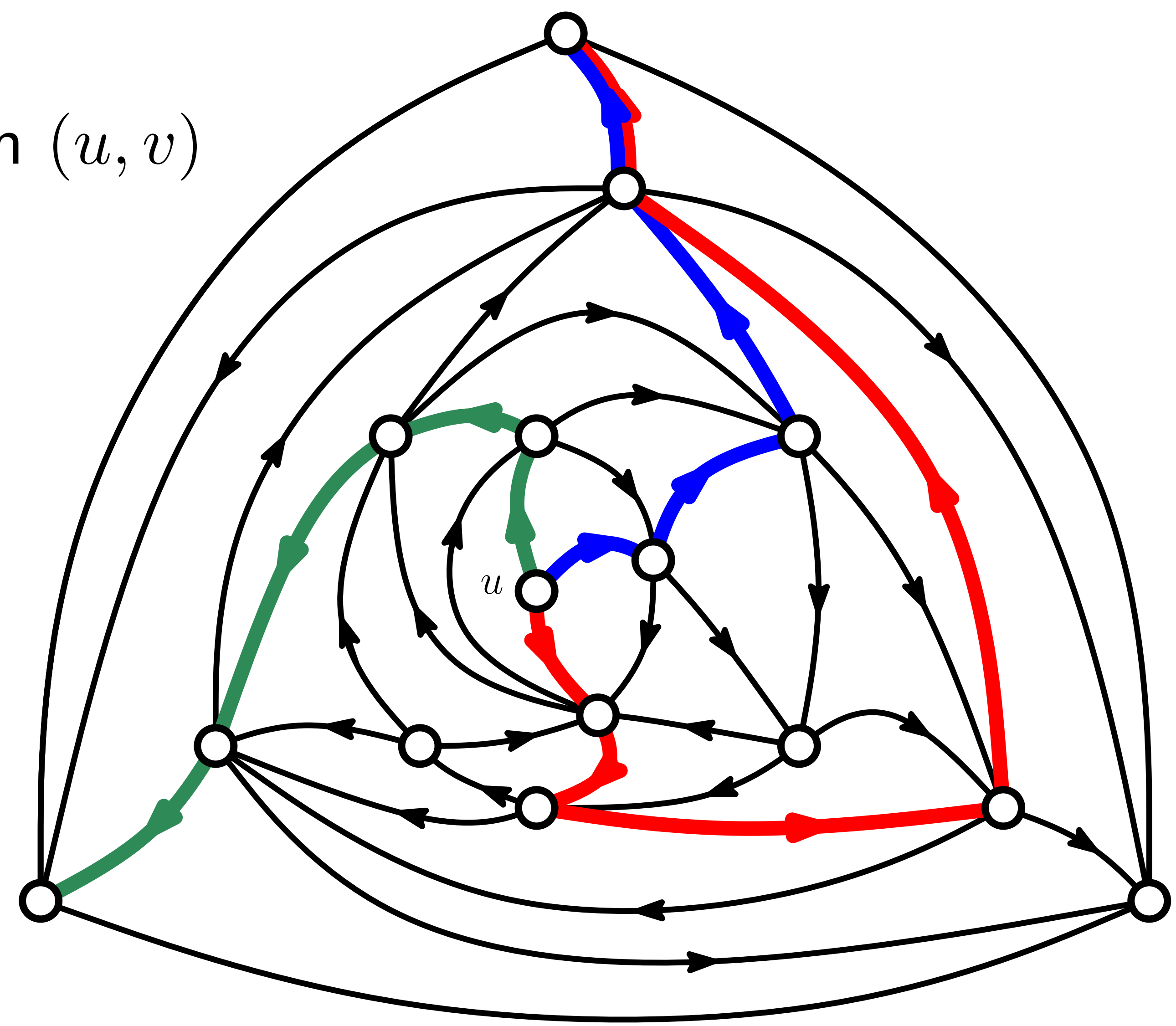
- For each inner vertex : 3 LMP



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- LMP are not self-intersecting
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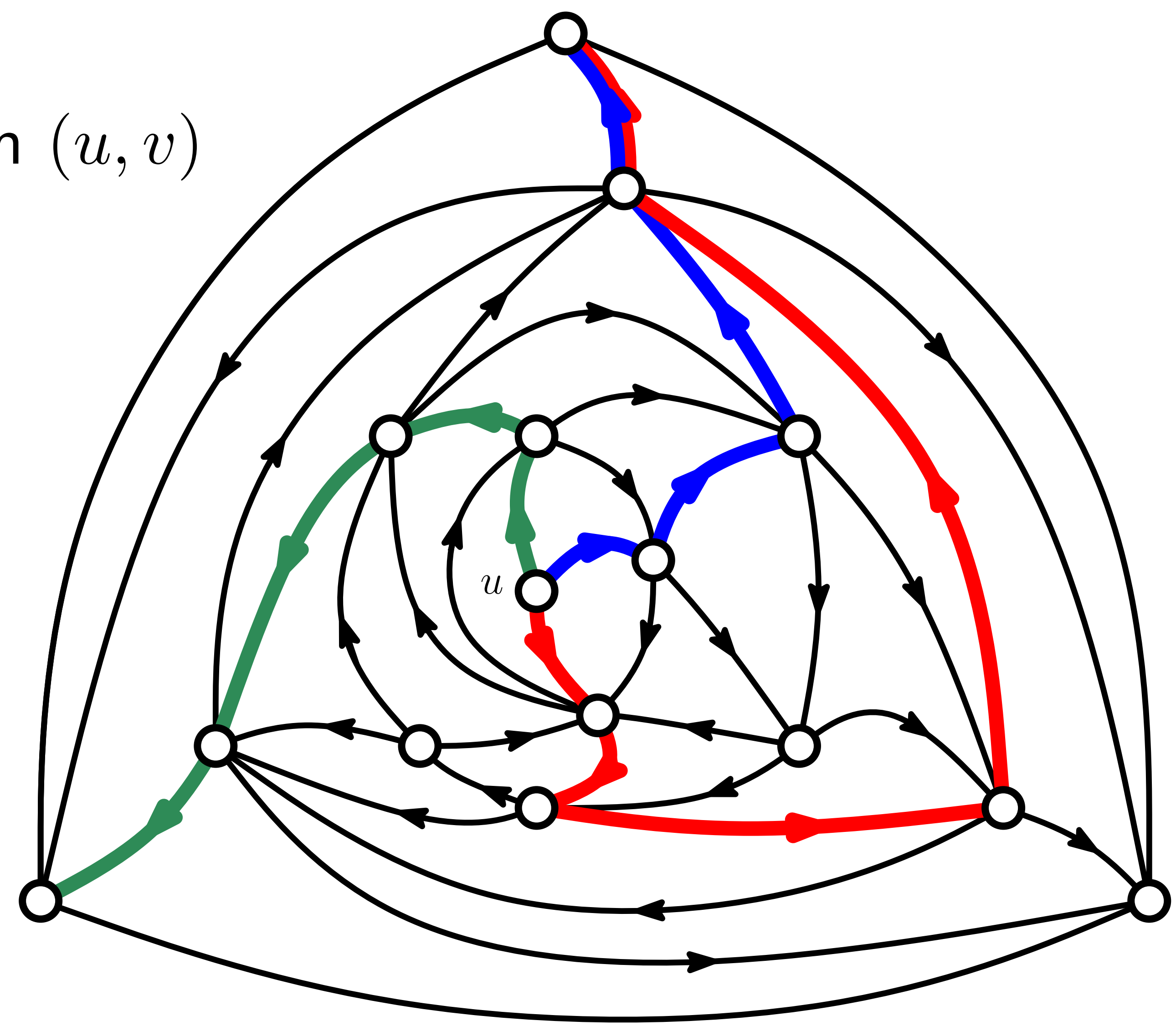

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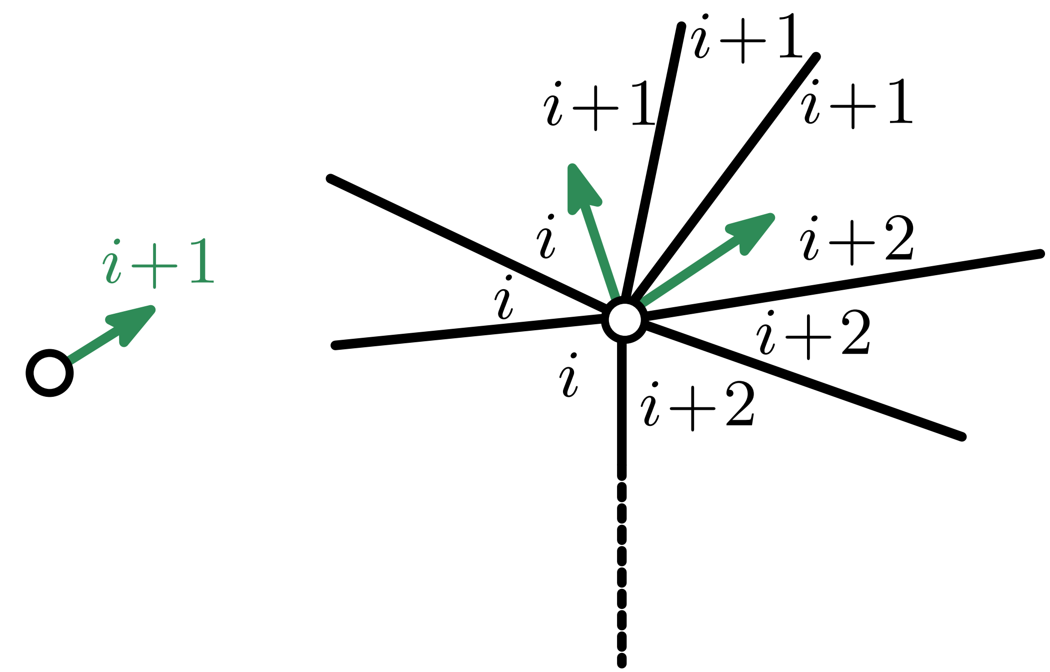
- On the left of a LMP, corner labels decrease exactly by one.

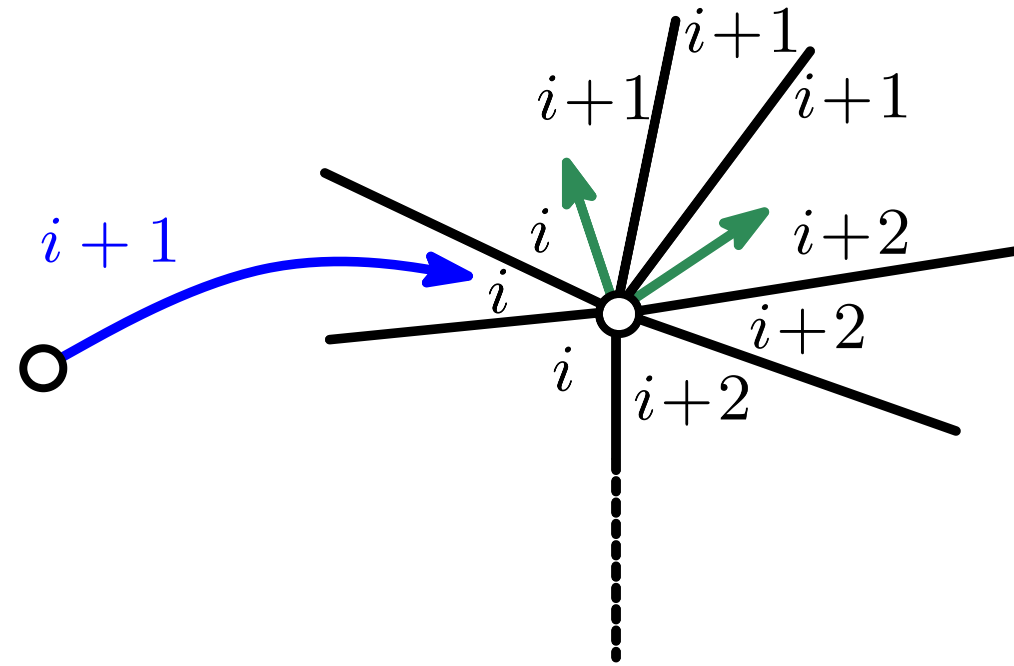


Distances in simple triangulations

Claim 1: $3d_{M_n}(root, u_i) \geq L_n(u_i)$

Claim 2 : $d_{M_n}(root, u_i) \leq L_n(u_i)$

- Consider the **Left Most Path** from (u, v) to the root face.
- For each inner vertex : 3 LMP
- LMP are not self-intersecting
  $\Rightarrow$ they reach the outer face
- On the left of a LMP, corner labels decrease exactly by one.



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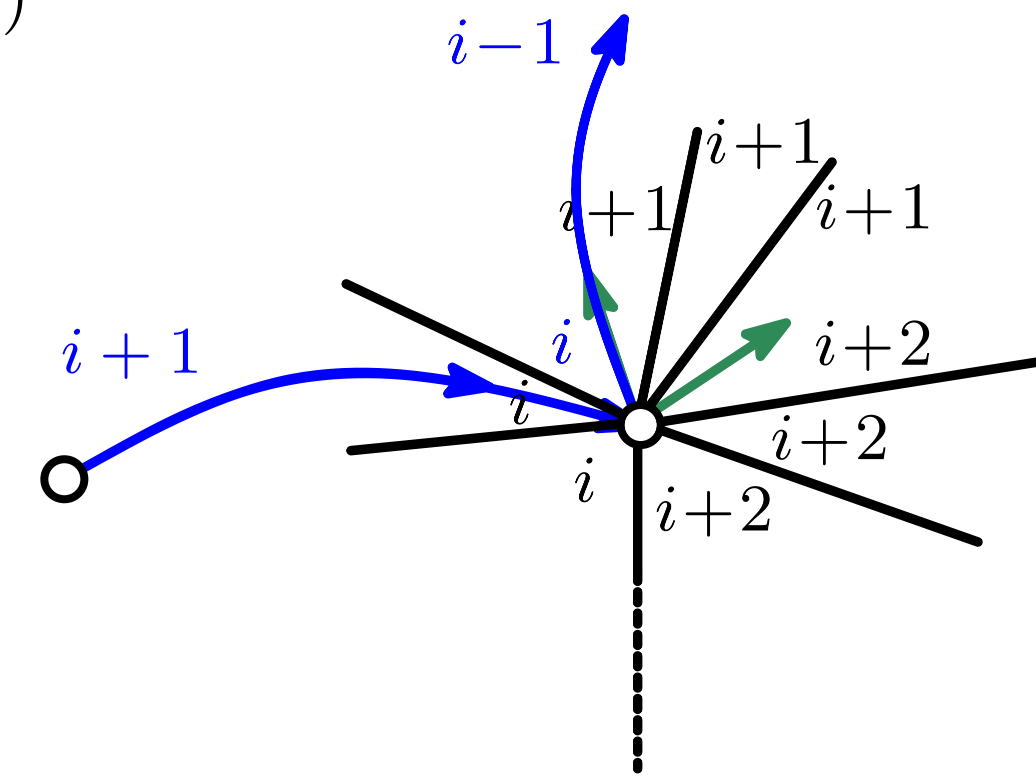

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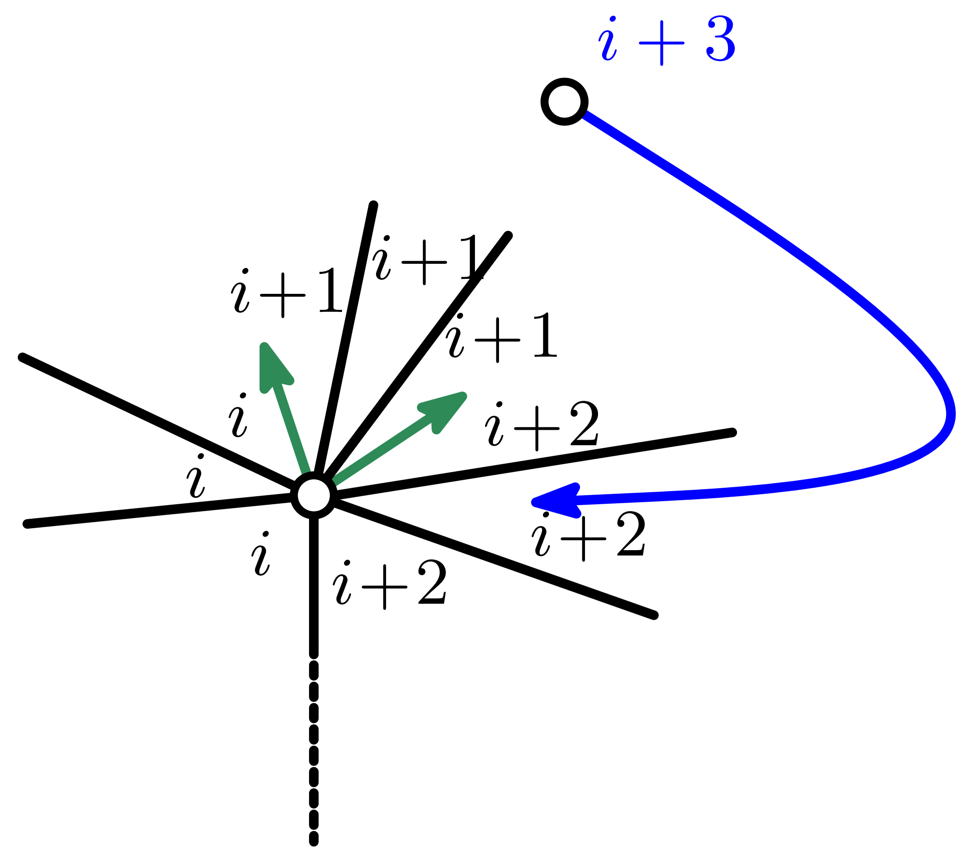

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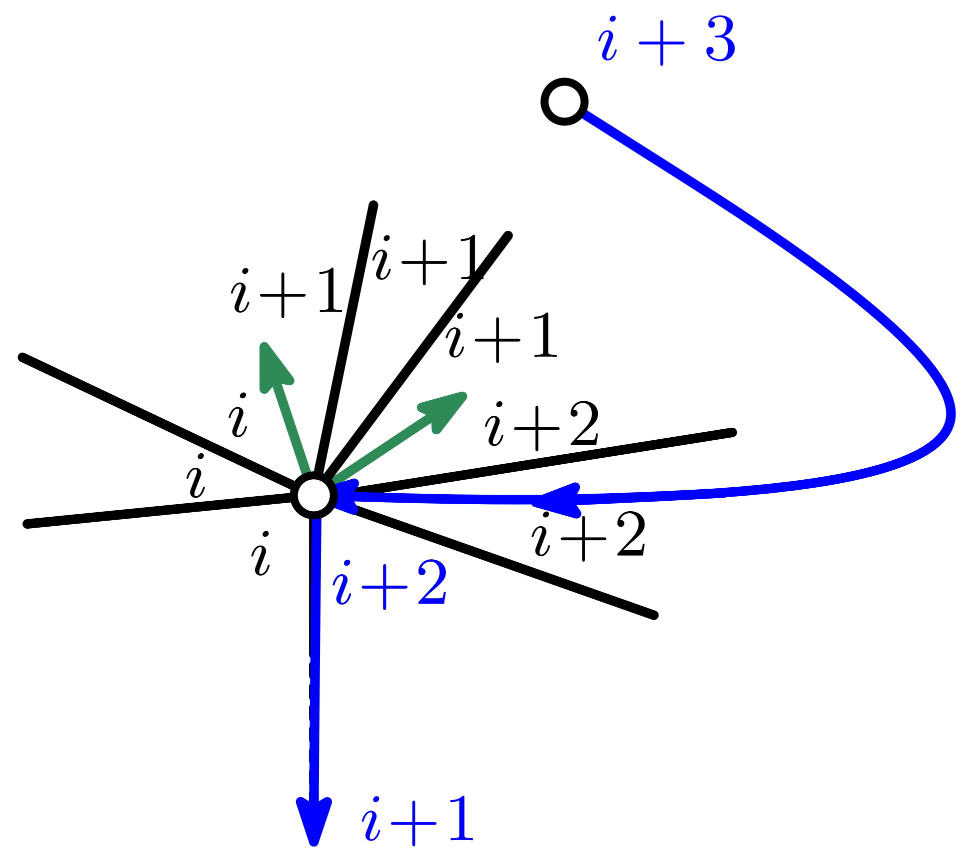
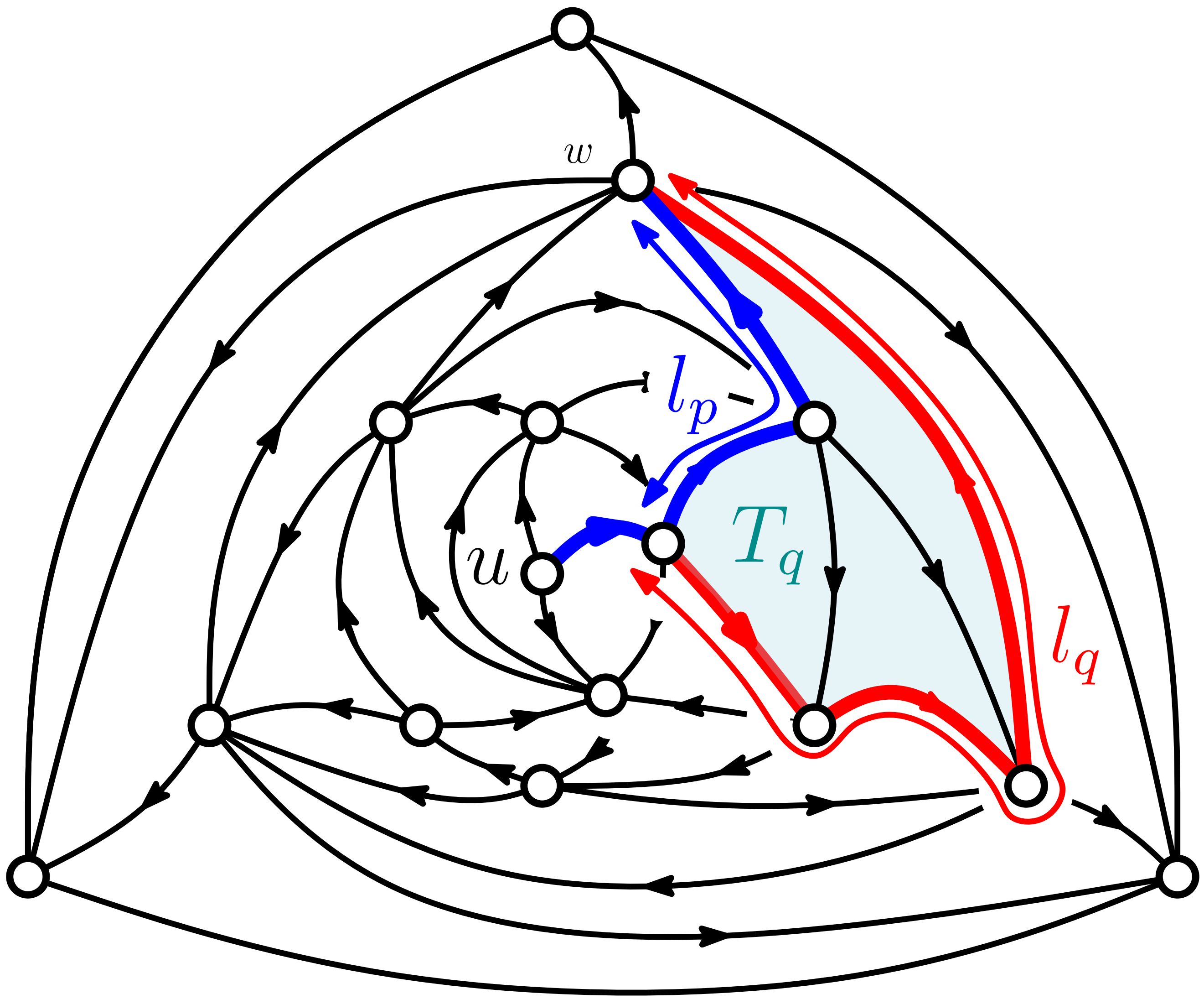


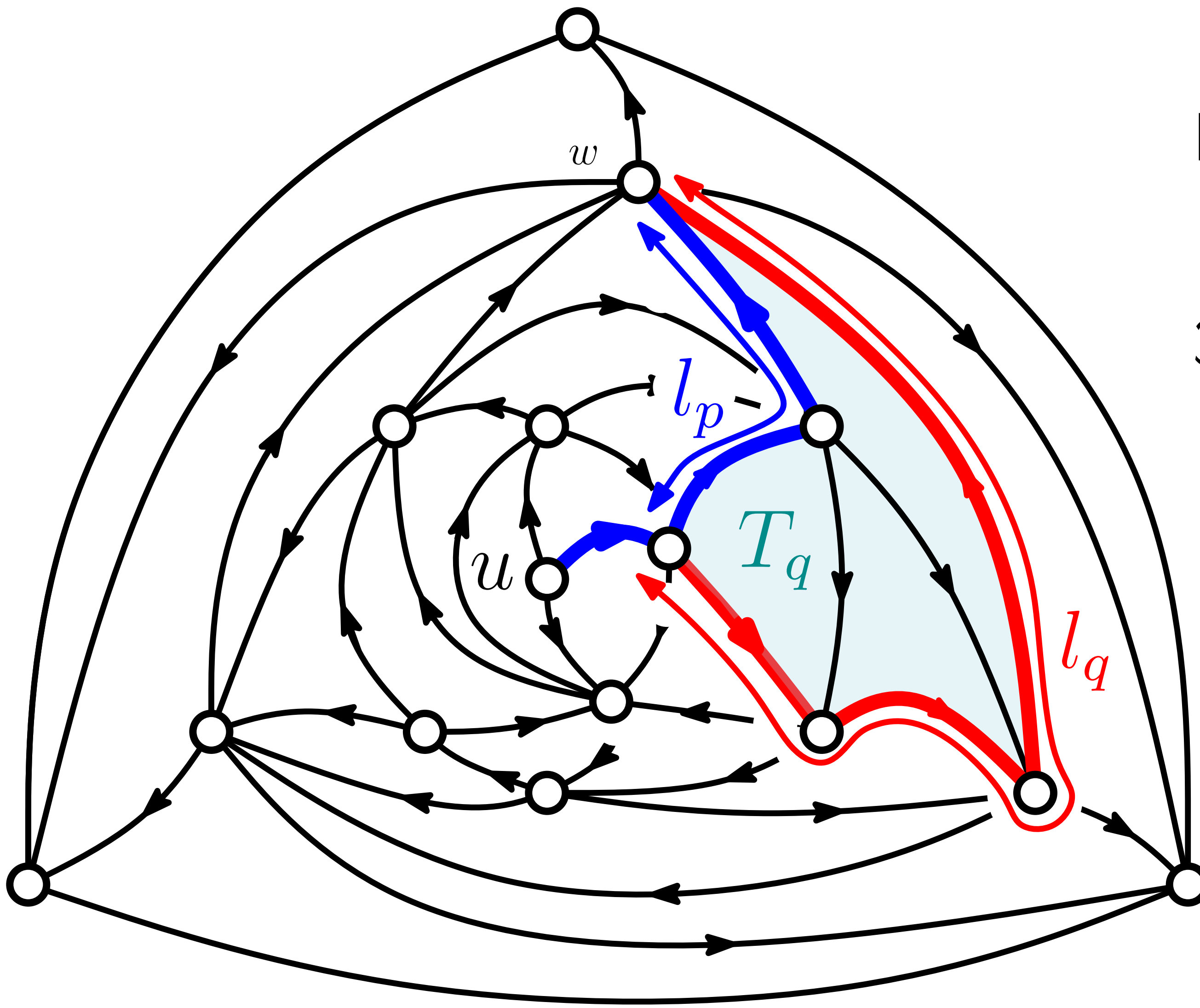
LMP are almost geodesic



Leftmost path

Another path: can it be shorter ?

LMP are almost geodesic



Euler Formula :

$$|E(T_q)| = 3|V(T_q)| - 3 - (\ell_p + \ell_q)$$

3-orientation + LMP :

$$|E(T_q)| \geq 3|V(T_q)| - 2\ell_q - 2$$

$$\implies \ell_q \geq \ell_p + 1$$

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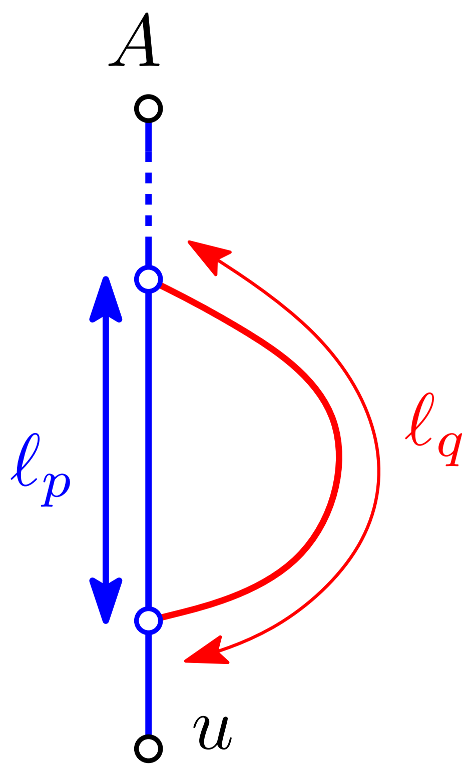

$\ell_q \geq \ell_p + 1$

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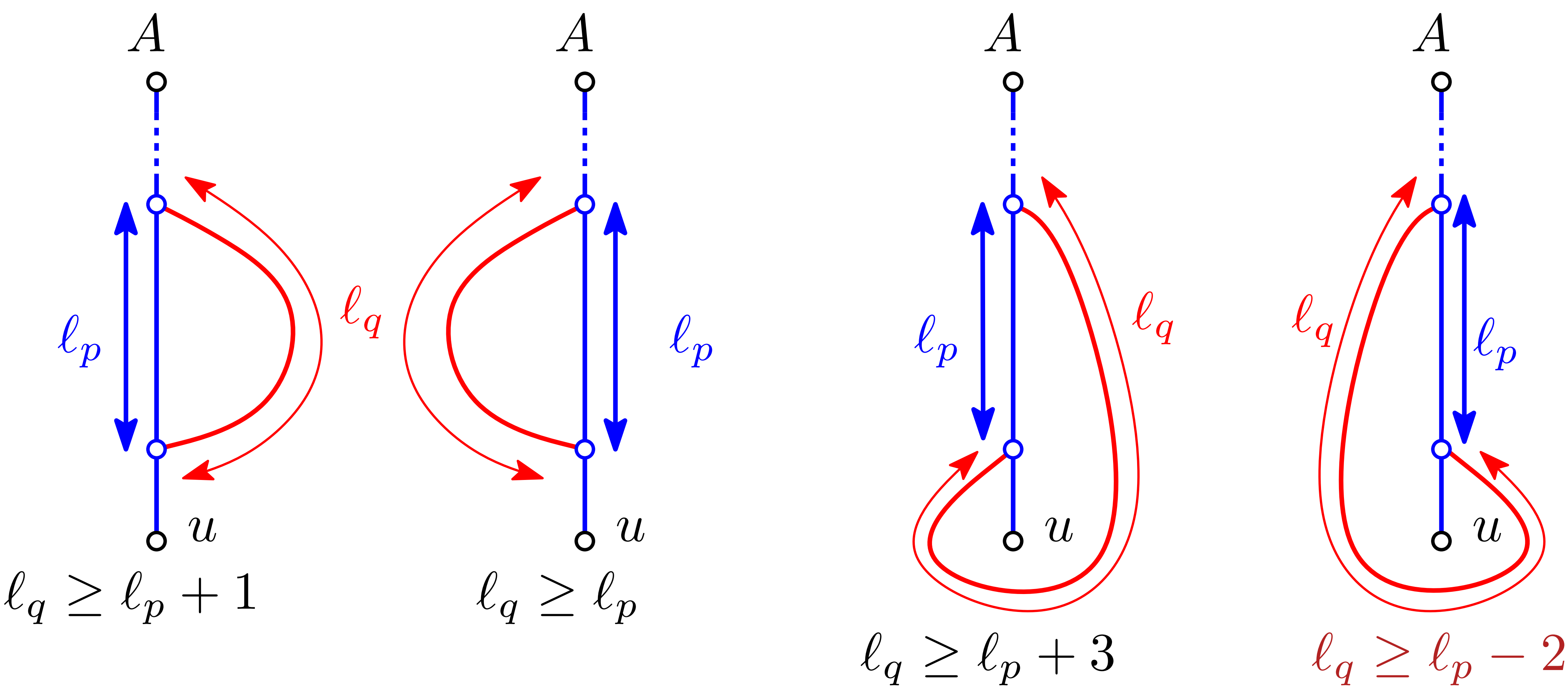
Another path: can it be shorter ? YES



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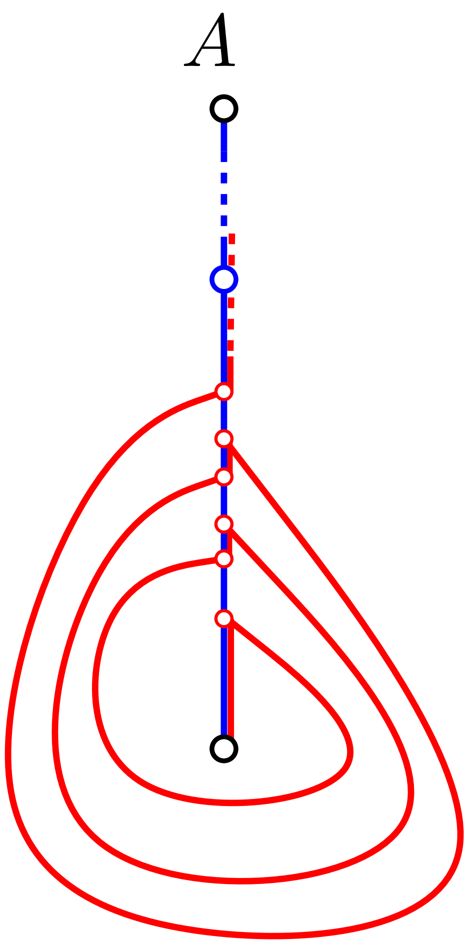
Another path: can it be shorter ? YES ... but not too often



Bad configuration =
too many **windings** around the LMP

But w.h.p a winding cannot be too short.

$\Longrightarrow$ w.h.p the number of windings is $o(n^{1/4})$.

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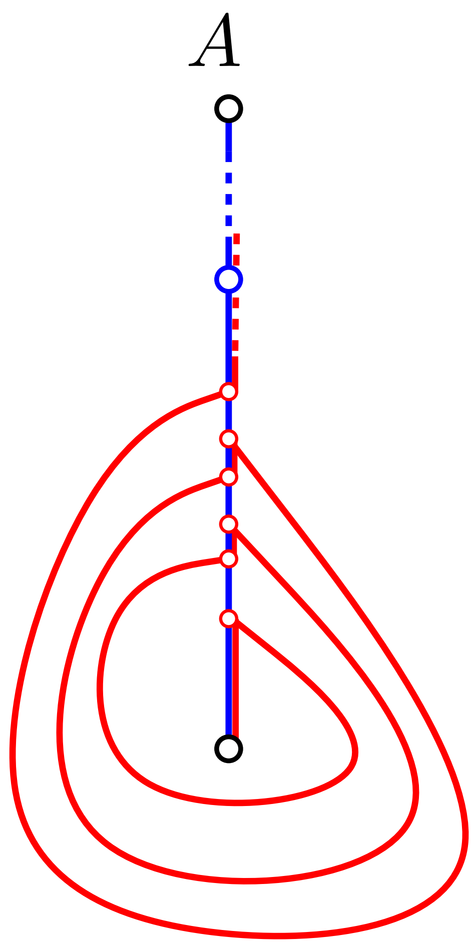

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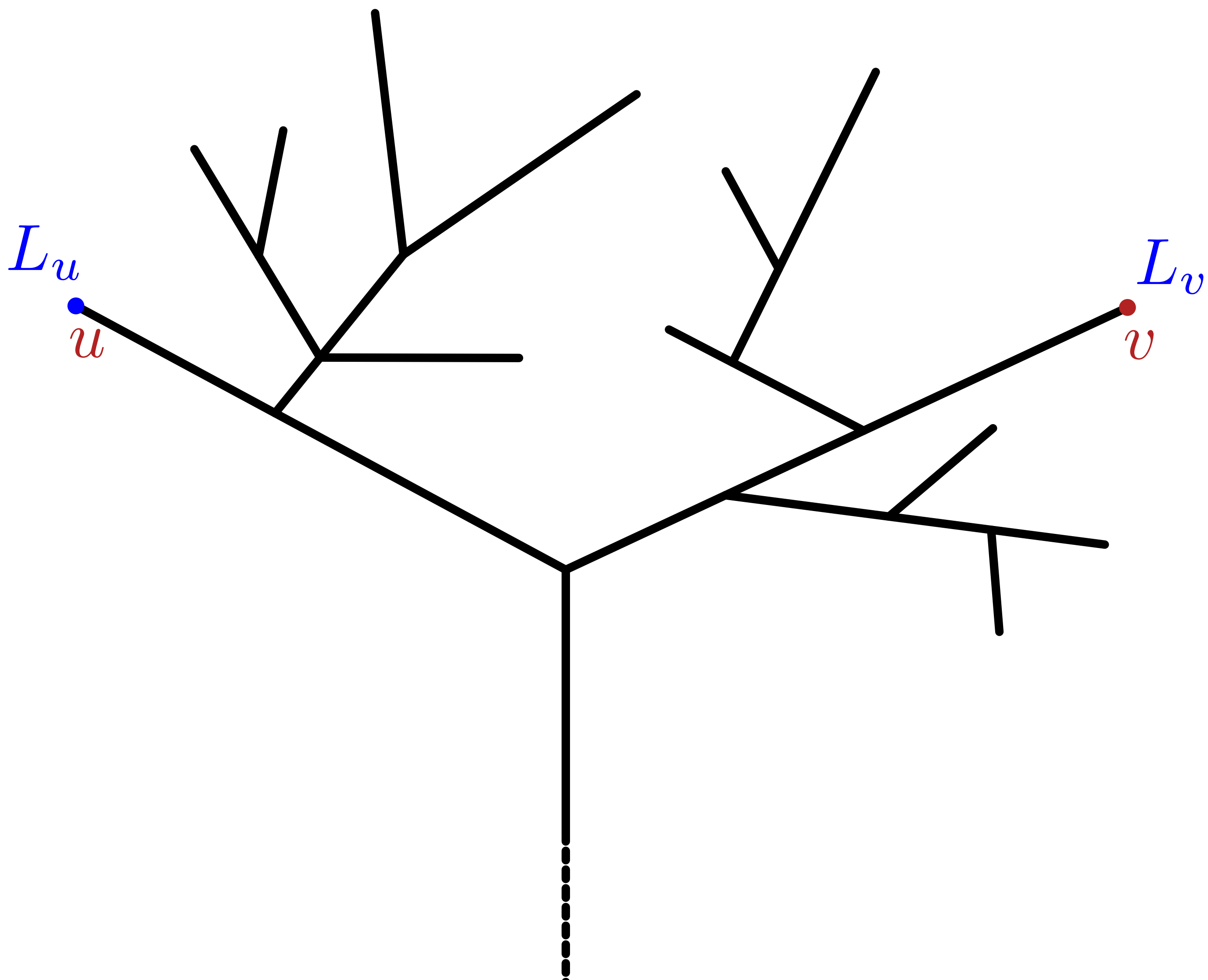
Proposition:

For $\varepsilon > 0$, let $A_{n,\varepsilon}$ be the event that there exists $u \in M_n$ such that $L_n(u) \geq d_{M_n}(u, root) + \varepsilon n^{1/4}$.
Then under the uniform law on $\mathcal{M}_n$, for all $\varepsilon > 0$:

$$\mathbb{P}\left(A_{n,\varepsilon}\right) \to 0.$$

Distances are tight



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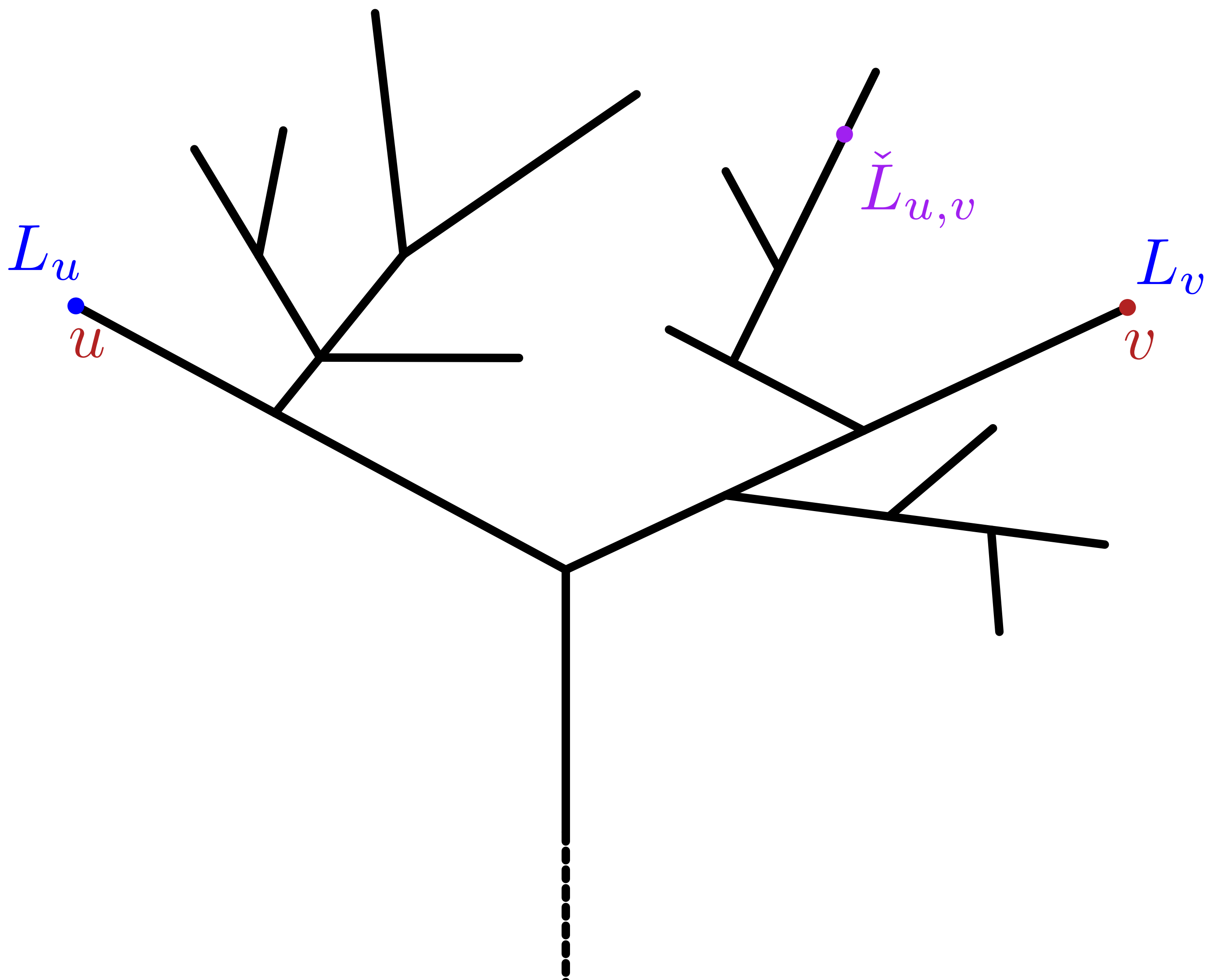

$$\check{L}_{u,v} = \min\{L_s, u \le s \le v\}$$

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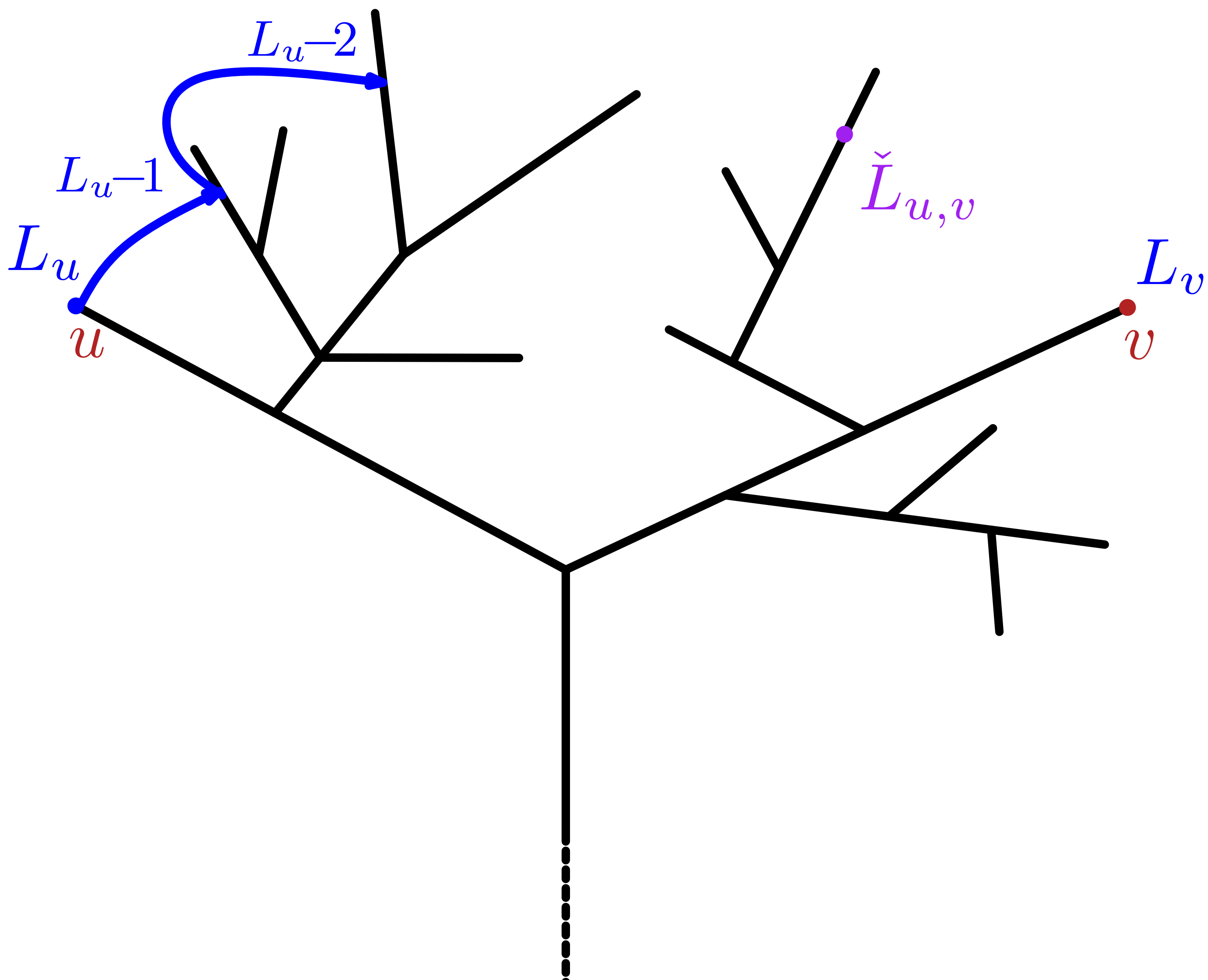

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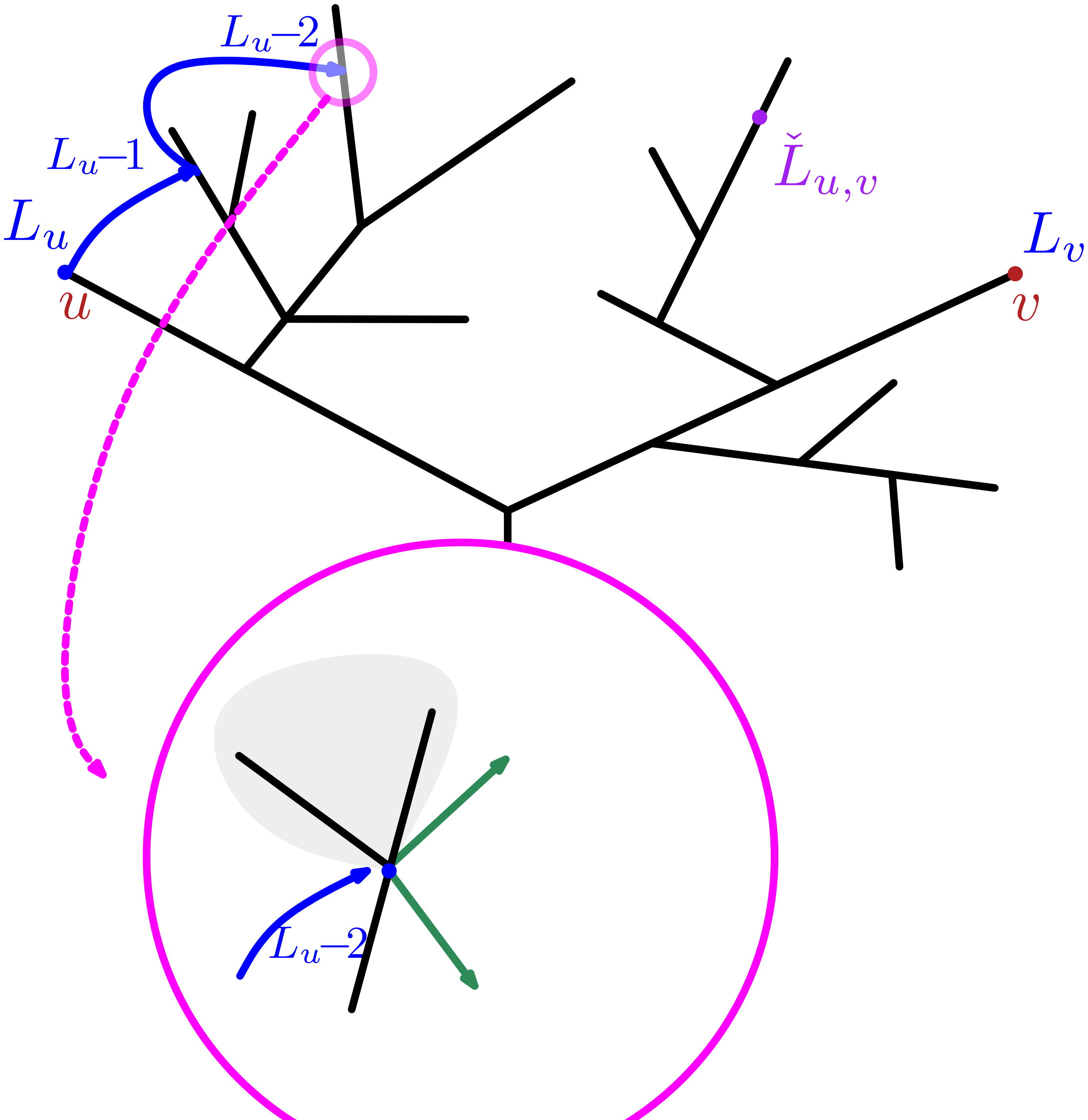

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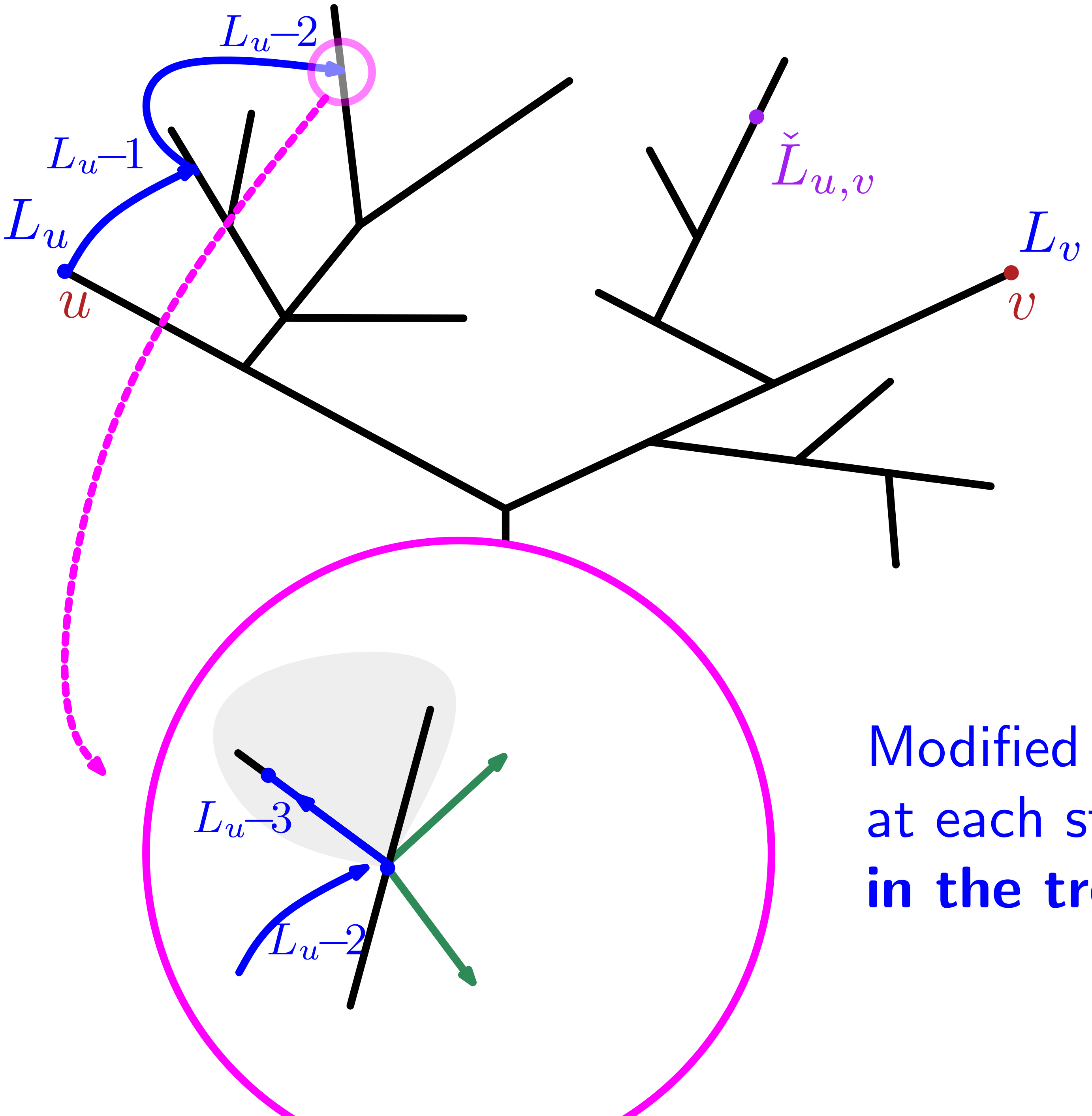

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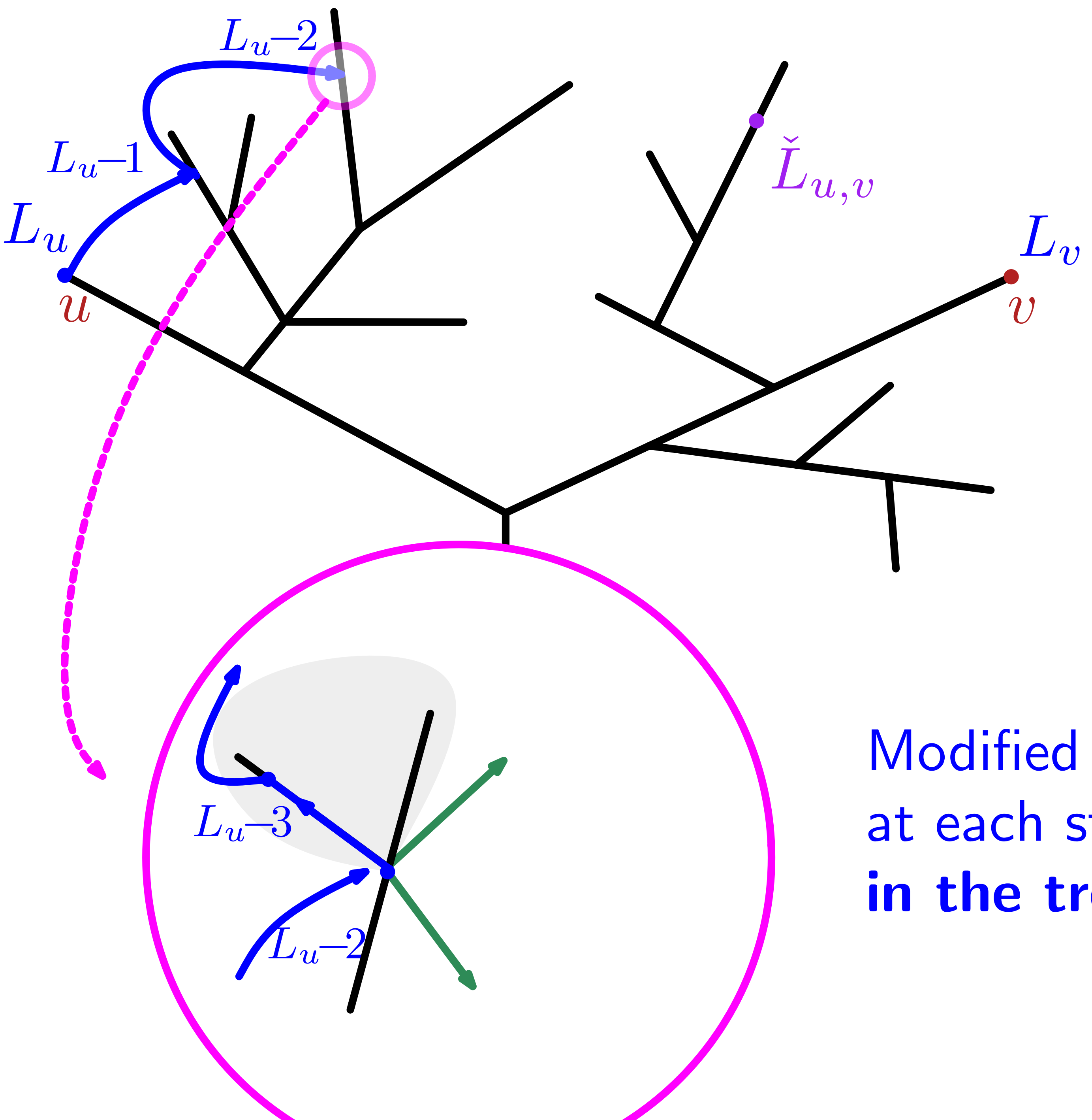
Modified LMP:
at each step, we take the first edge
in the tree

Distances are tight



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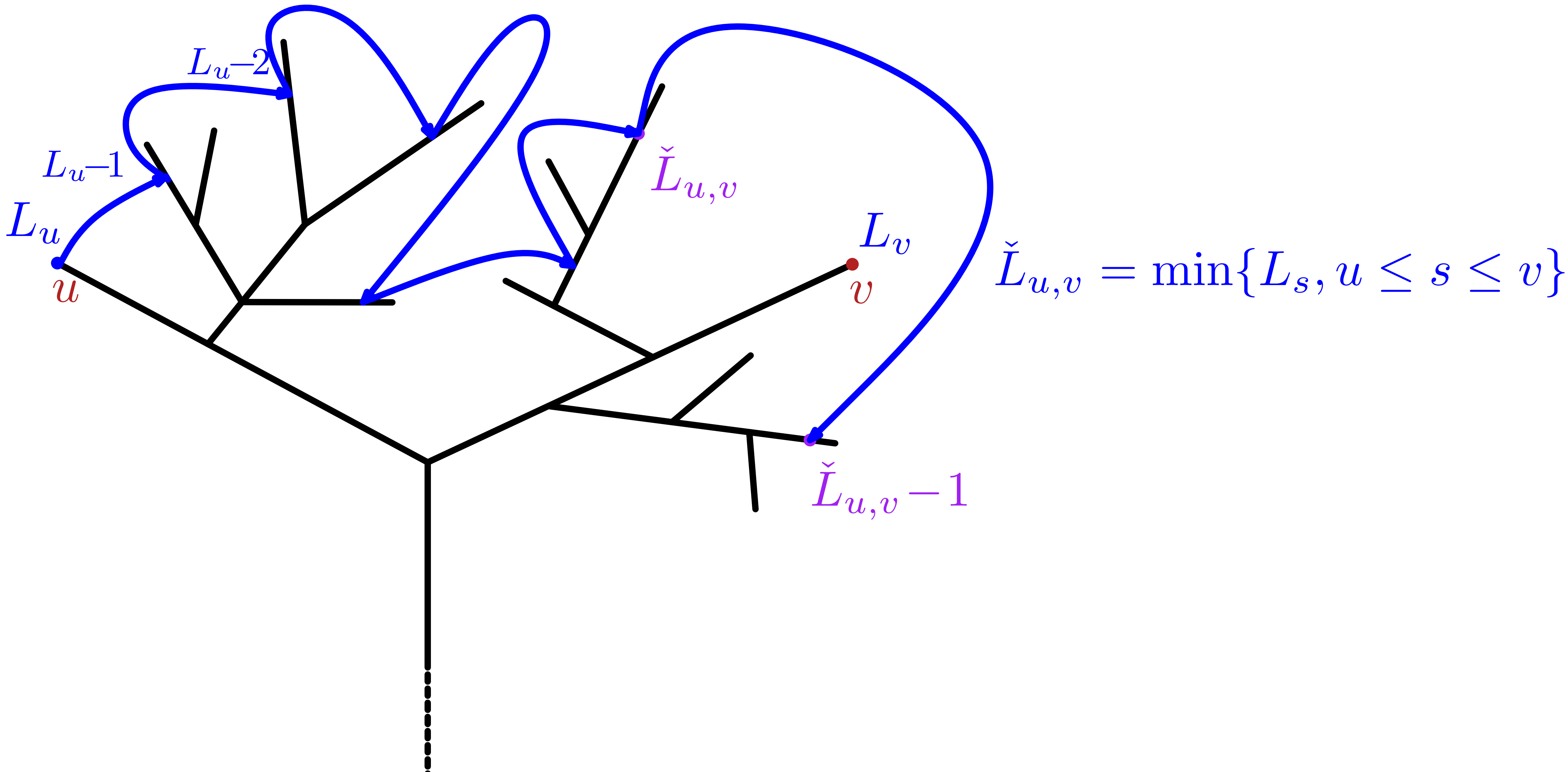

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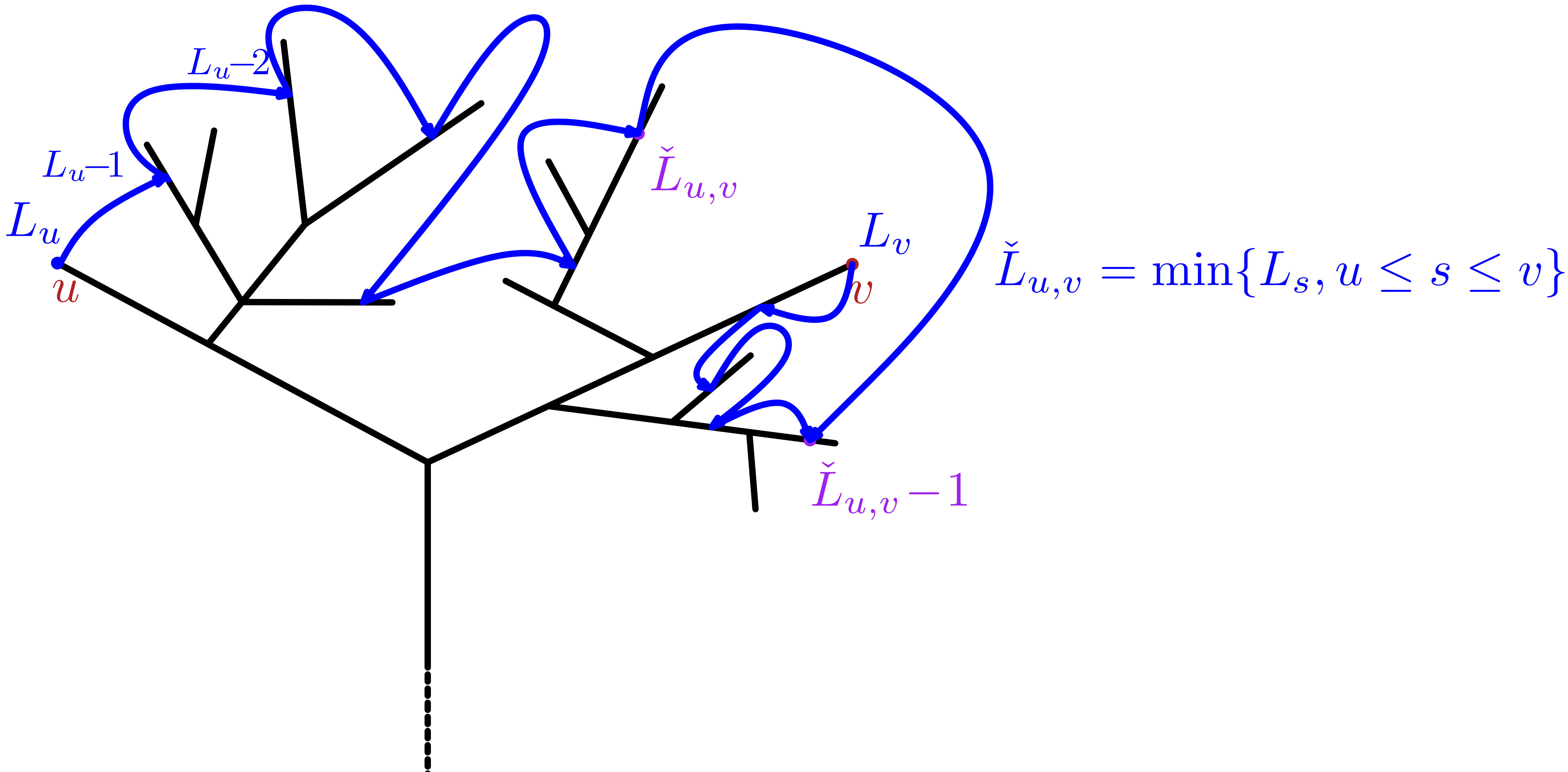

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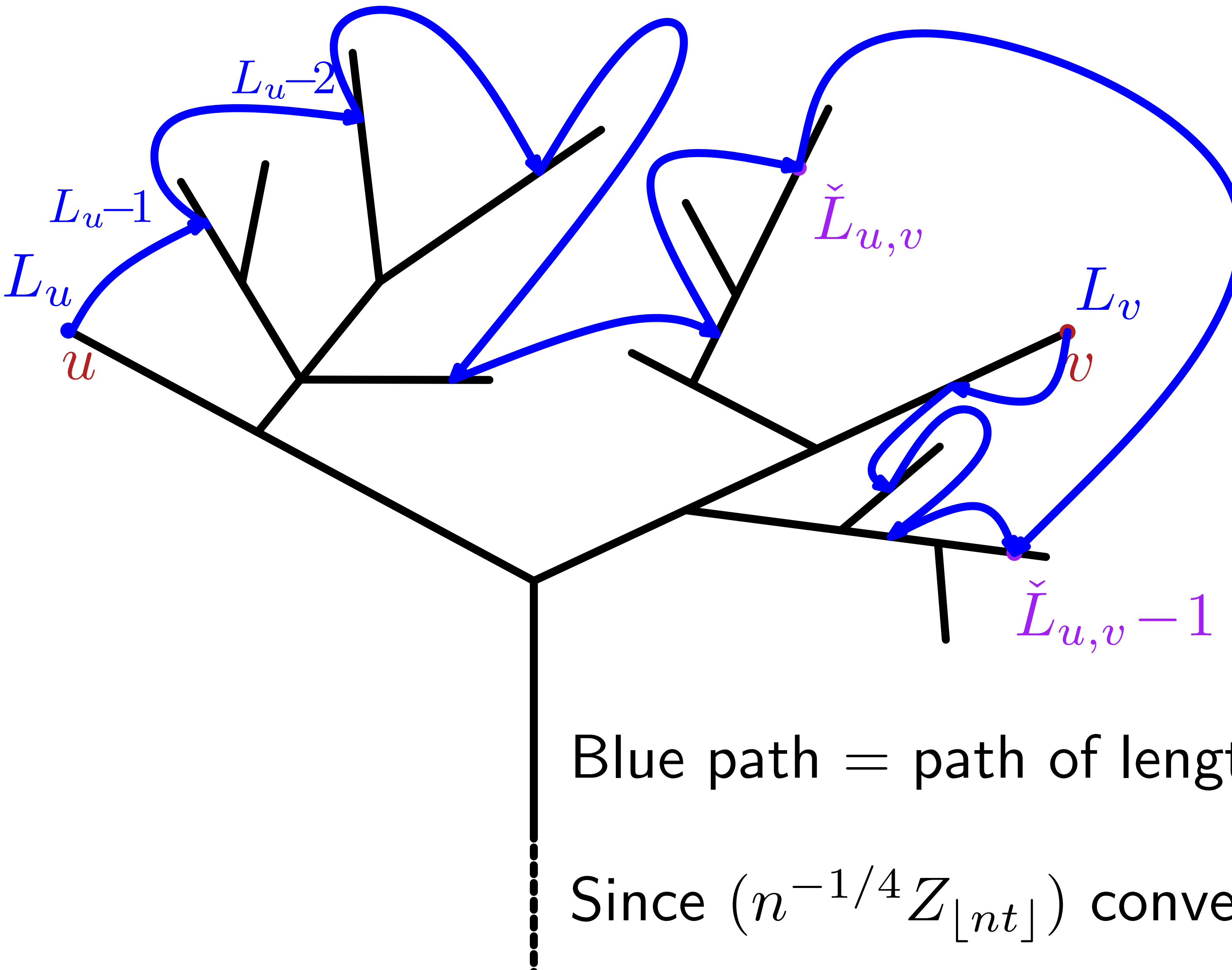


$\check{L}_{u,v} = \min\{L_s, u \le s \le v\}$

Blue path = path of length $L_u + L_v - 2\check{L}_{u,v} + 2$

Since $(n^{-1/4} Z_{\lfloor nt \rfloor})$ converges $\Rightarrow$ (d_n) tight

The result for the last time

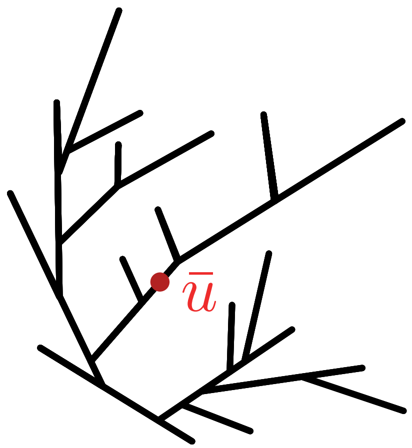
Theorem : [Addario-Berry, A.]
$(M_n) =$ sequence of random **simple** triangulations, then:

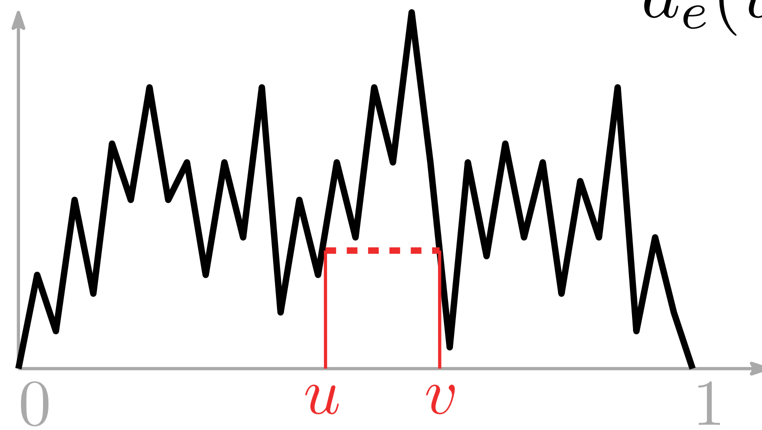
$$\left(M_n, \left(\frac{3}{4n}\right)^{1/4} d_{M_n} \right) \xrightarrow{(d)} (M, D^\star),$$

for the distance of Gromov-Hausdorff on the isometry classes of compact metric spaces.

The Brownian Map ??

The Brownian map





$d_e(u,v) = e_u + e_v - 2\min_{u \le s \le v} e_s$

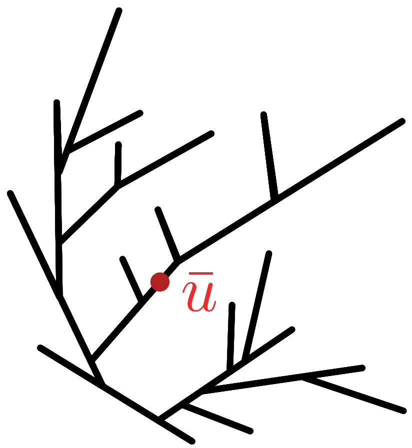
$\mathcal{T}_e = [0,1]/\sim_e$

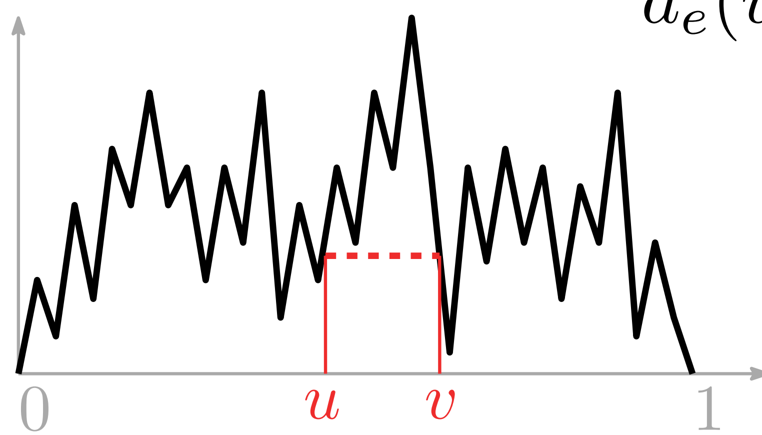
$u \sim_e v$ iff $d_e(u,v) = 0$

Conditional on $\mathcal{T}_e$, Z a centered Gaussian process with $Z_\rho = 0$ and $E[(Z_s - Z_t)^2] = d_e(s,t)$

$Z \sim$ **Brownian motion on the tree**

The Brownian map





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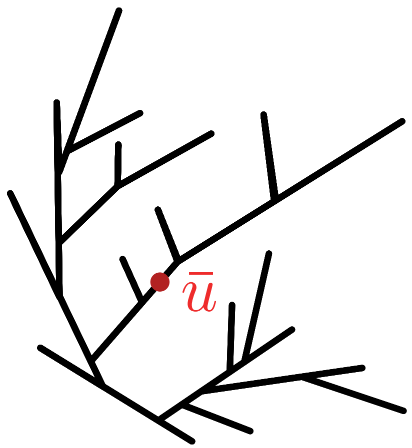

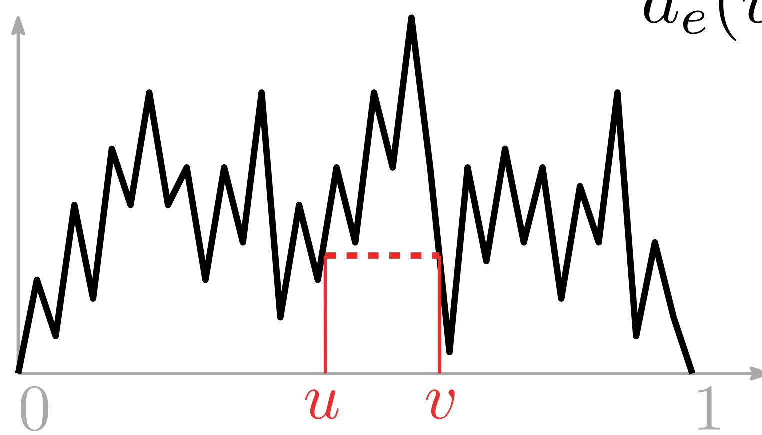

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$$D^*(a,b) = \inf\Big\{\sum_{i=1}^{k-1} D^\circ(a_i, a_{i+1}) : k \ge 1, a = a_1, a_2, \ldots, a_{k-1}, a_k = b\Big\},$$

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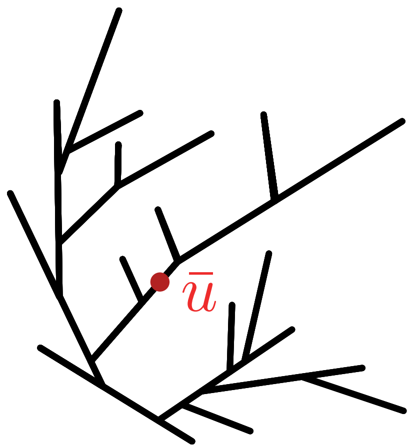

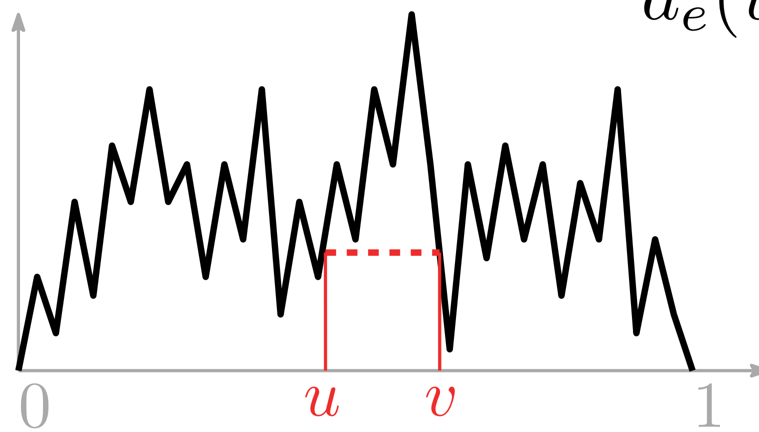

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Then $M = (\mathcal{T}_e/\sim_{D^\star}, D^*)$ is the **Brownian map**.

Perspectives

Same approach works also for simple quadrangulations.

Can it be generalized to other families of maps ?

- Generic bijection between blossoming trees and maps [Bernardi, Fusy] [A.,Poulalhon].
  Can we say something about distances ?
- Convergence of Hurwitz maps: bijection also with blossoming trees [Duchi, Poulalhon, Schaeffer].

Can we say something about the embedding of the Brownian map in the sphere via circle packing ?

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