Local limit of random discrete surfaces with (or without!) a statistical physics model.

Marie Albenque (CNRS, LIX, École Polytechnique)

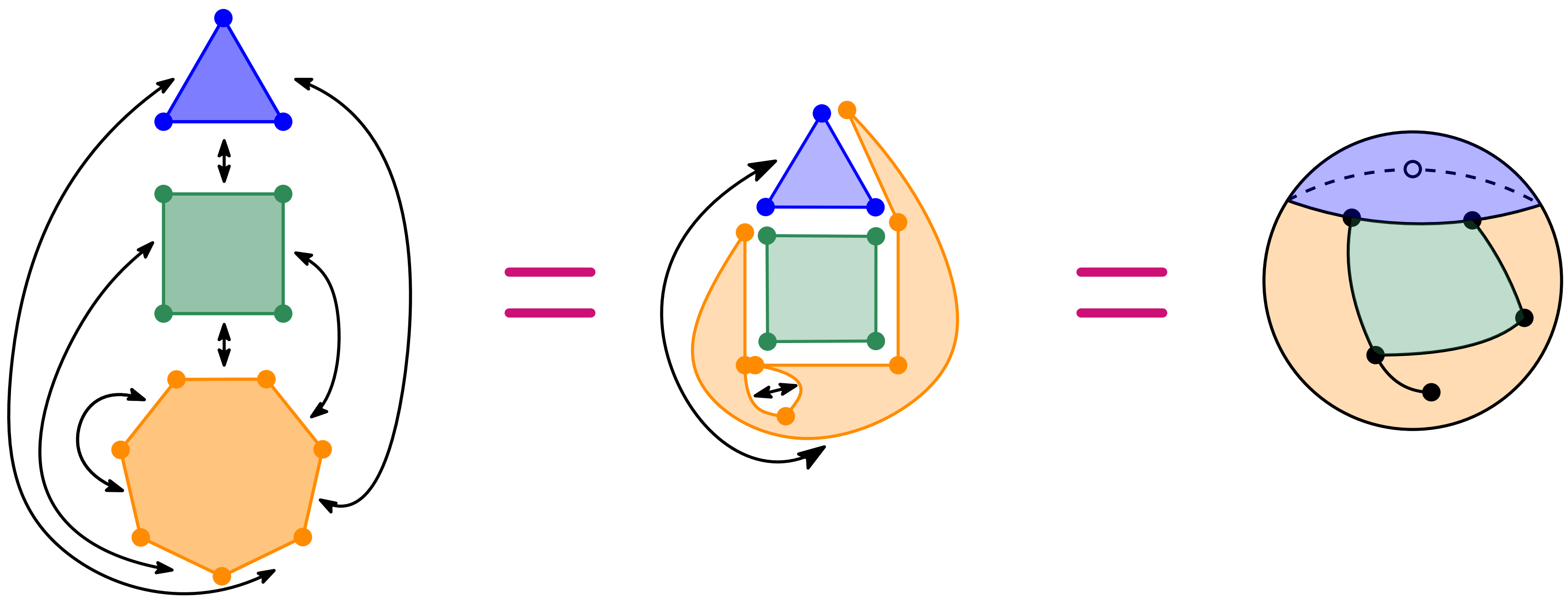
based on joint works with Laurent Ménard (Univ. Paris Nanterre – NYU Shanghai) and Gilles Schaeffer (CNRS, LIX, École Polytechnique)

STACS 22

I - Definition of planar maps

Maps – Definition(s)

A **map** is a collection of polygons glued along their sides (with some technical conditions).



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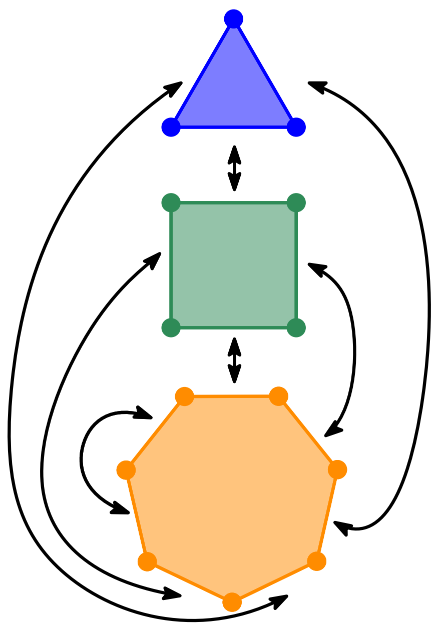

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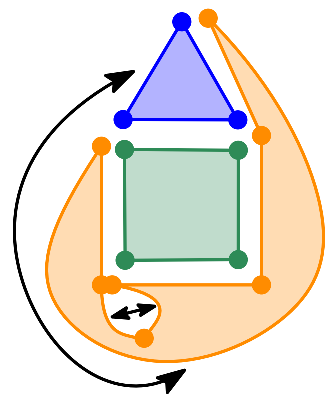

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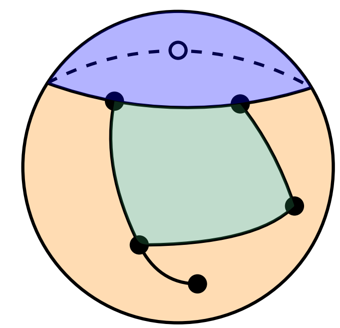

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this is a **planar map**.

If all the polygons have p sides, the resulting map is called a **p-angulation**.

3-angulation = **triangulation**, 4-angulation = **quadrangulation**

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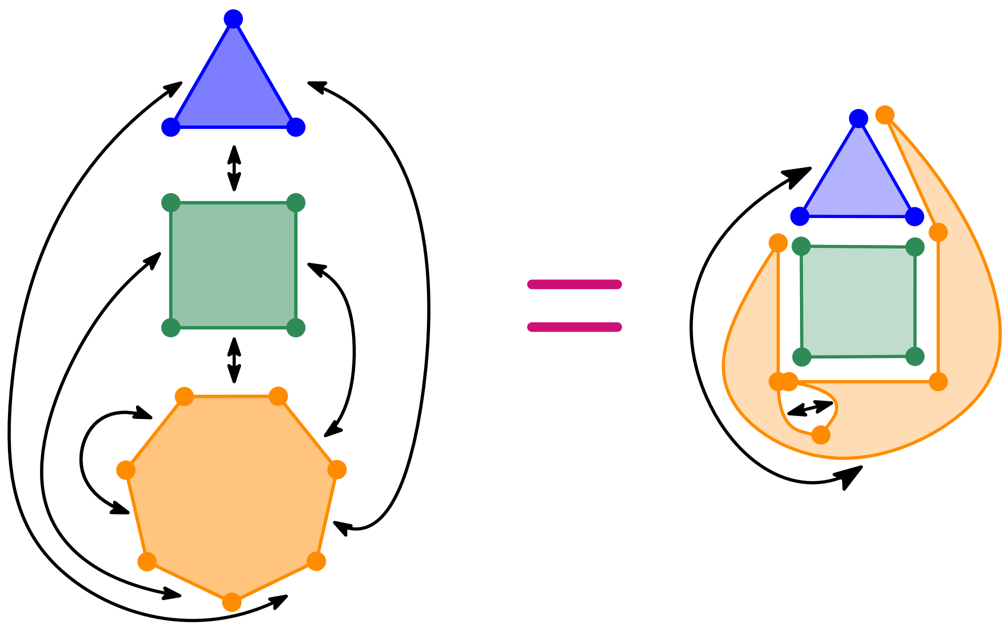

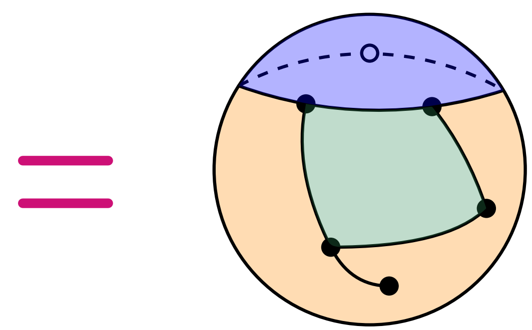

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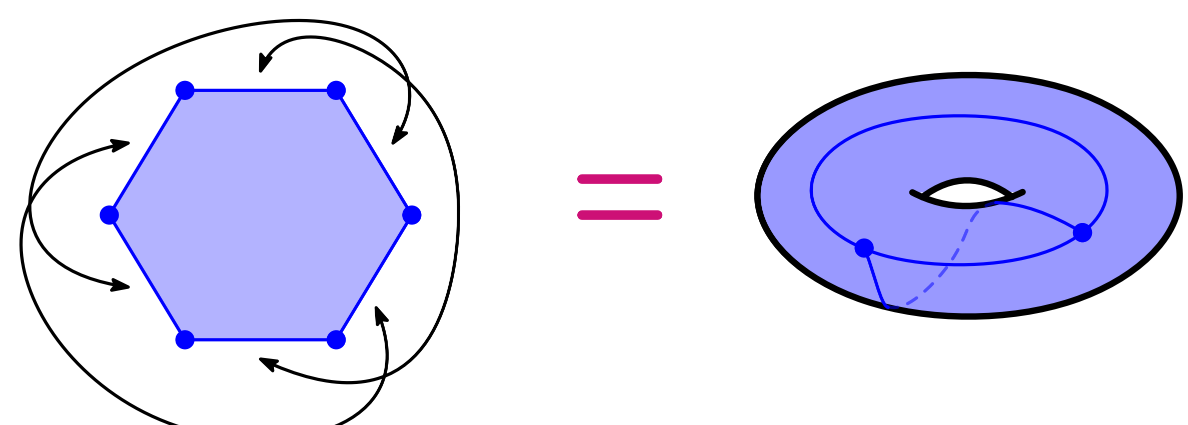
Side remark: we could also obtain
a surface different from the sphere
(and even not connected !)



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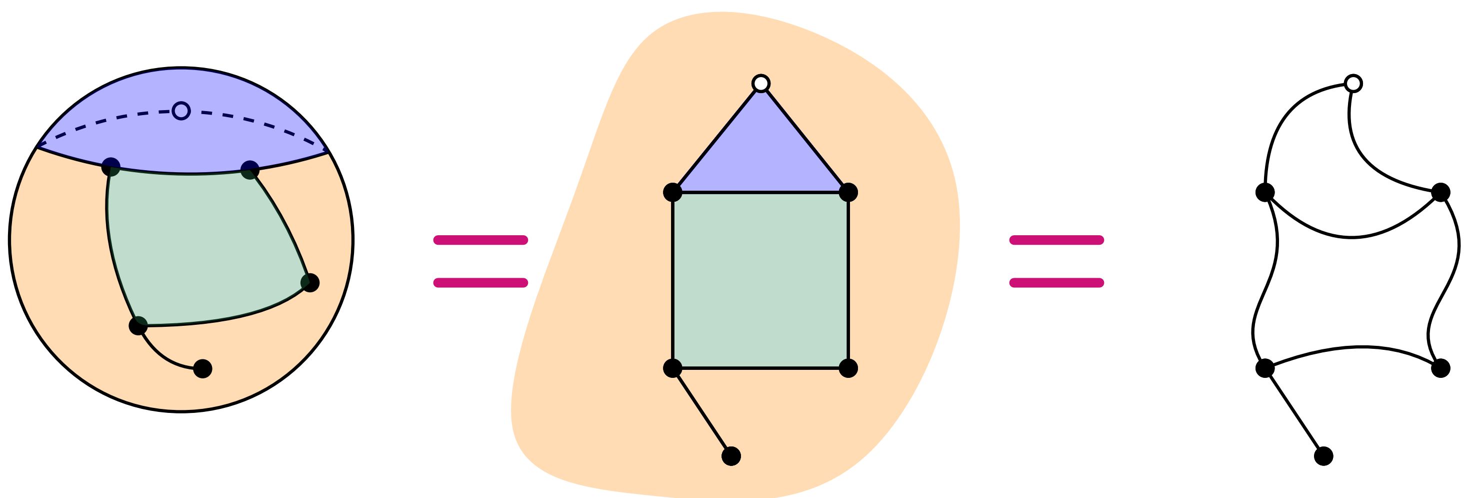
A **planar map** is a proper embedding of a planar connected graph in the 2-dimensional sphere (considered up to orientation-preserving homeomorphisms).



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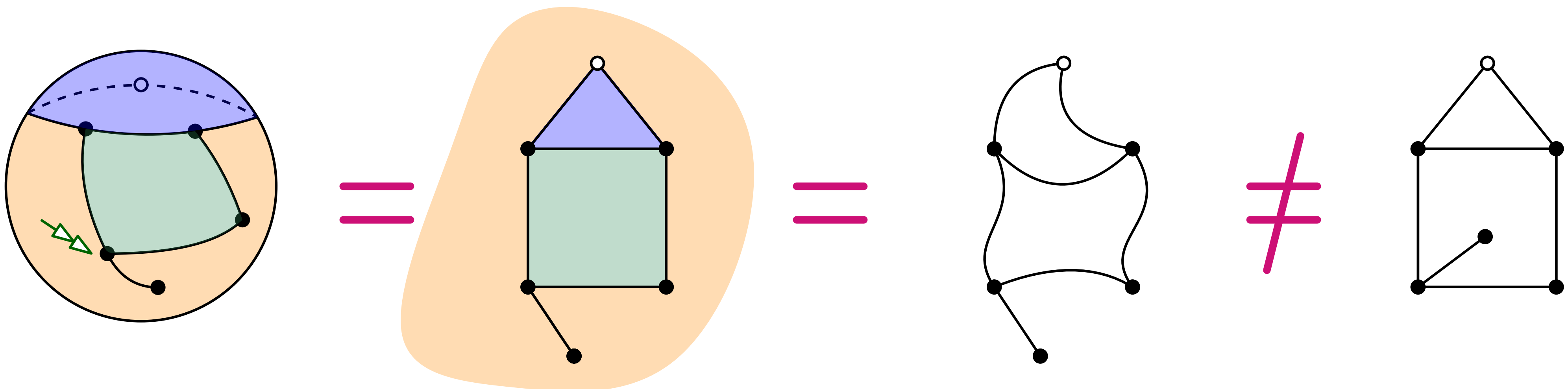

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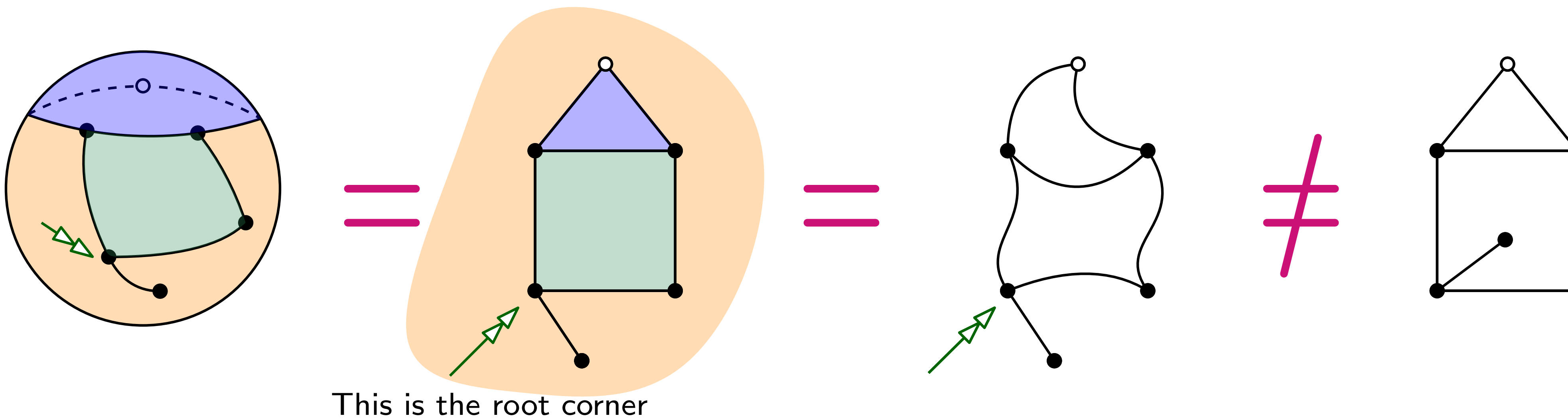

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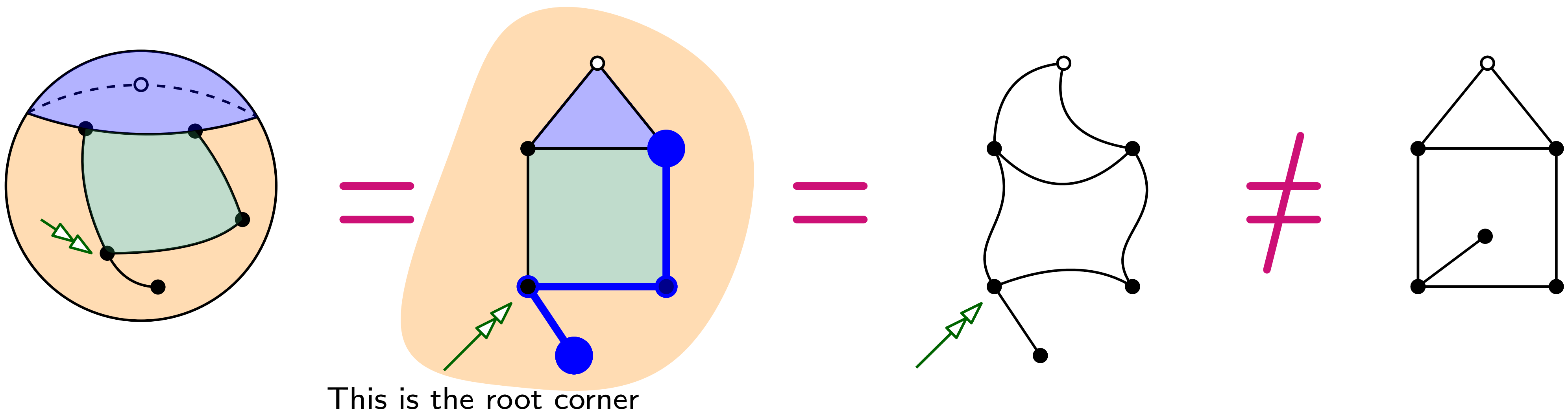

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A map M defines a discrete **metric space**:

- points: set of vertices of $M = V(M)$.
- distance: graph distance $= d_{gr}$.

Maps – Motivations

Maps appear in various fields of mathematics, computer science and statistical physics (connections with representation theory, KP-hierarchies, topological recurrence,...).

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Today, I focus on the study of **limits of random planar maps** and, more precisely on **local** limits of random planar **triangulations**.

Model: $\mathcal{T}_n = \{\text{Triangulations of size } n\}$
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Spoiler: In the second half of the talk, we will change the probability distribution.

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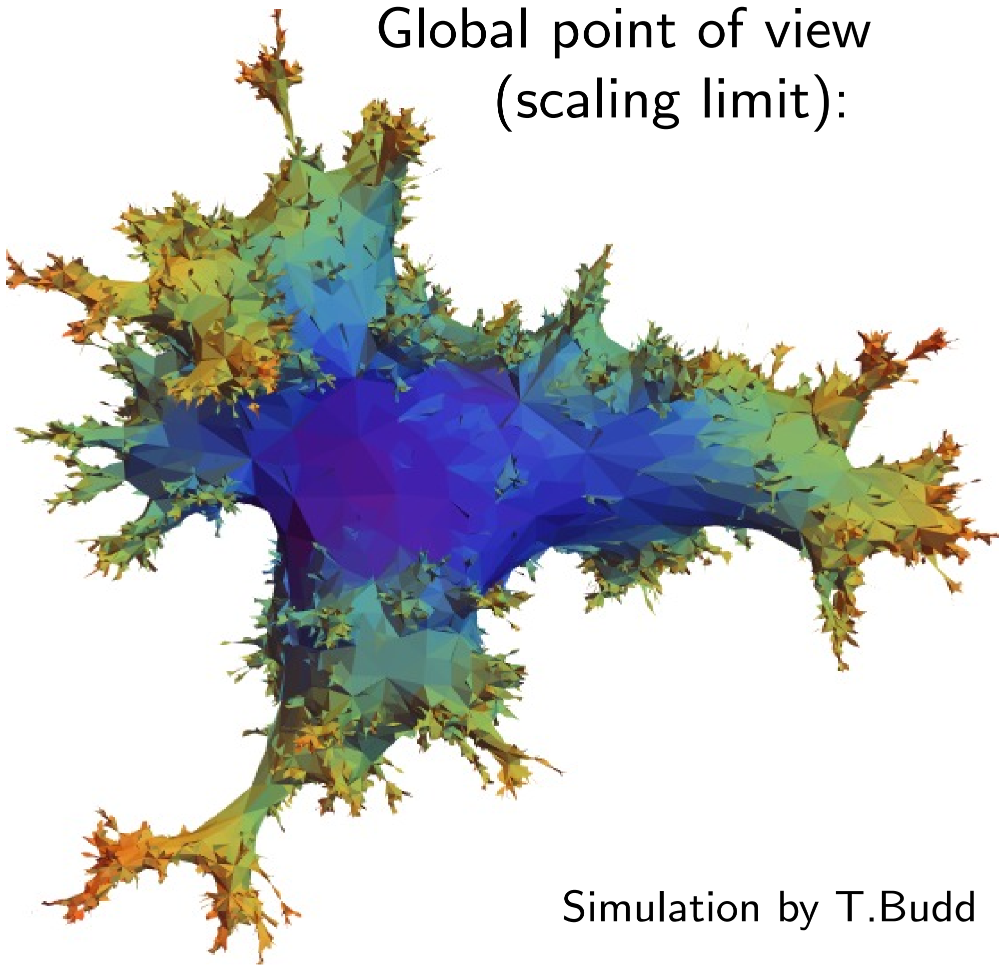
Two possible points of view:

Global point of view (scaling limit):



Simulation by T.Budd

Local point of view (Benjamini-Schramm topology):

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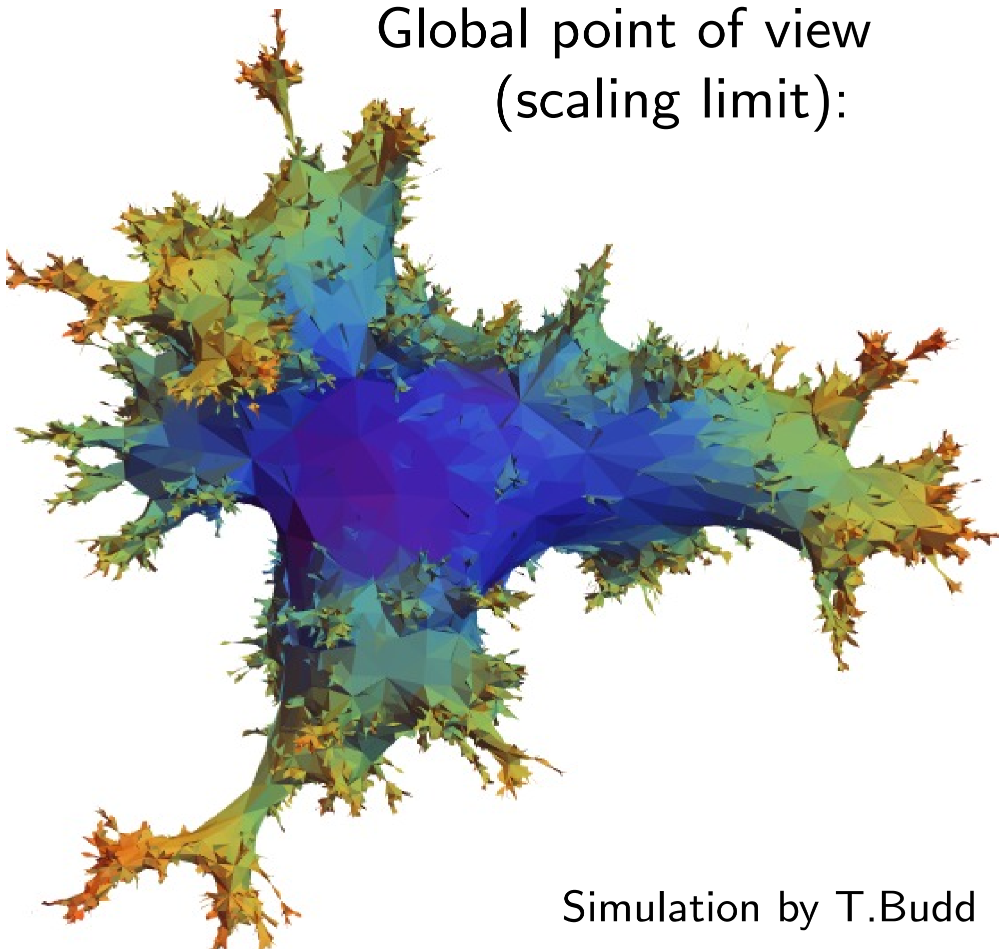

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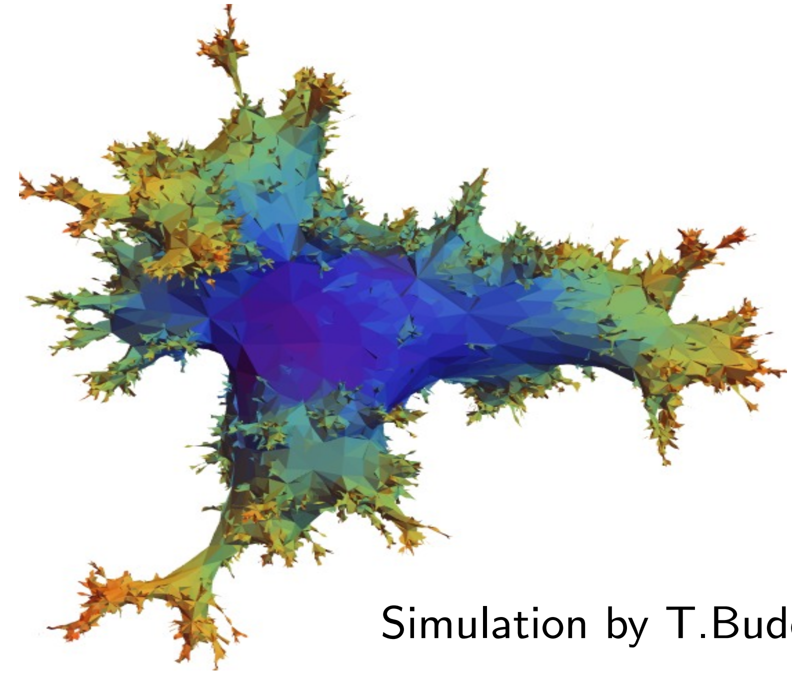

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When the size of the map goes to infinity, so does the typical distance between two vertices.

Idea: "scale" the map = length of edges decreases with the size of the map.
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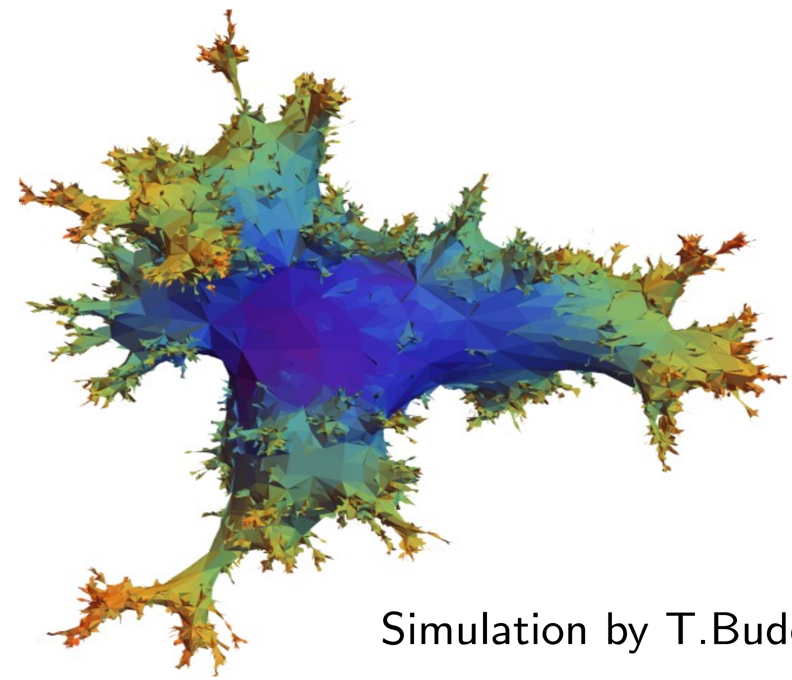

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Motivations + Results:

- Discretization of a continuous surface.
- Construction of a 2-dim. analogue of the Brownian motion: **The Brownian Map**, homeomorphic to the sphere, Hausdorff dimension = 4 [Miermont 13],[Le Gall 13].
- Link with Liouville Quantum Gravity, (will maybe be discussed at the end of the talk) [Duplantier, Sheffield 11], [Duplantier, Miller, Sheffield 14], [Miller, Sheffield 16,16,17]
- **Universality:** the scaling is "always" $n^{-1/4}$ + the limiting object does not depend on the precise combinatorics of the model (p-angulations, simple triangulations,...)

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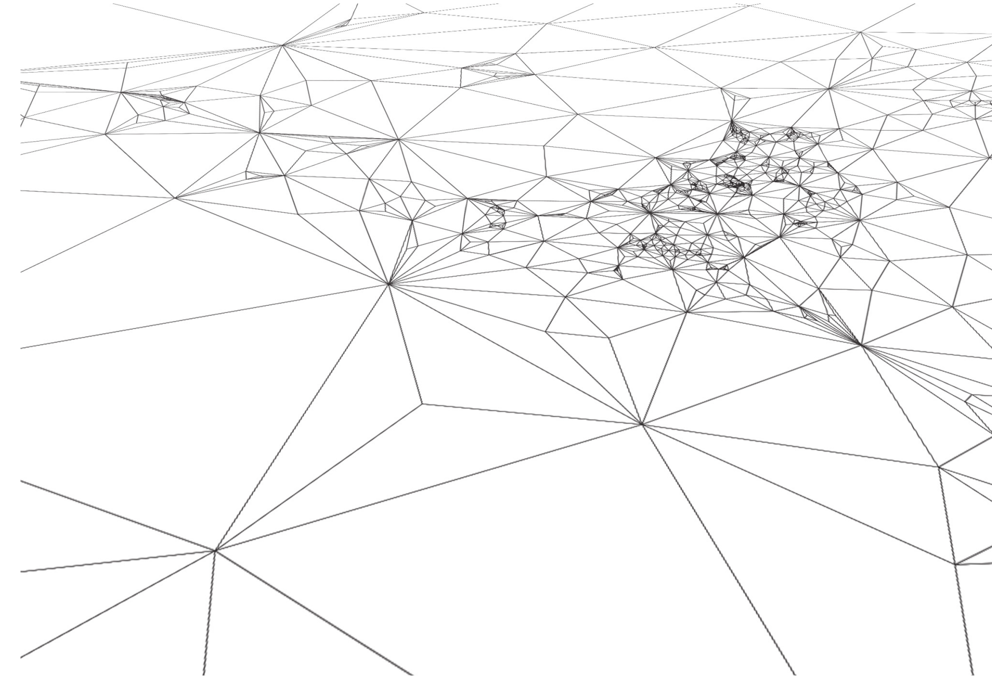

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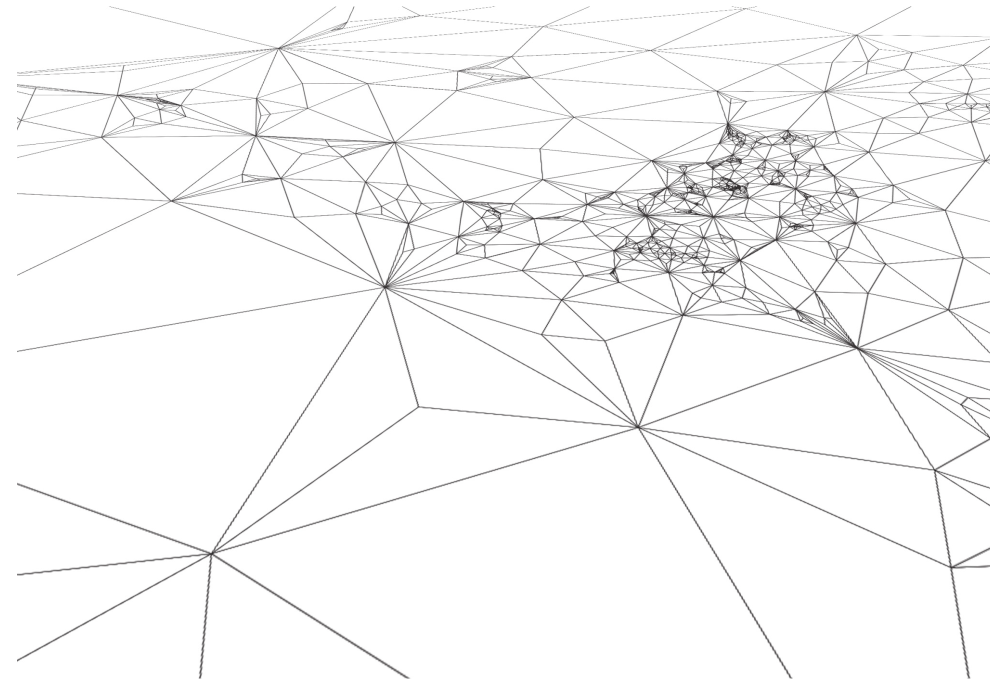
When the size of the map goes to infinity, so does the typical distance between two vertices.

Idea: **Do NOT scale** the distances

Look at **neighborhoods of the root**

Goal: obtain some (probability distribution on) infinite random maps.

Local limits of random maps



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Motivations + Results:

- Nice model of random discrete geometry.
- Construction of the Uniform Infinite Planar Triangulation (= **UIPT**). [Angel, Schramm]
- Connection with some models on Euclidean lattices via the KPZ formula (for Knizhnik, Polyakov and Zamolodchiko), [Duplantier, Sheffield 11]
- **Universality:** the number of vertices at distance R from the root is "always" of order R^4.

II - Local limits

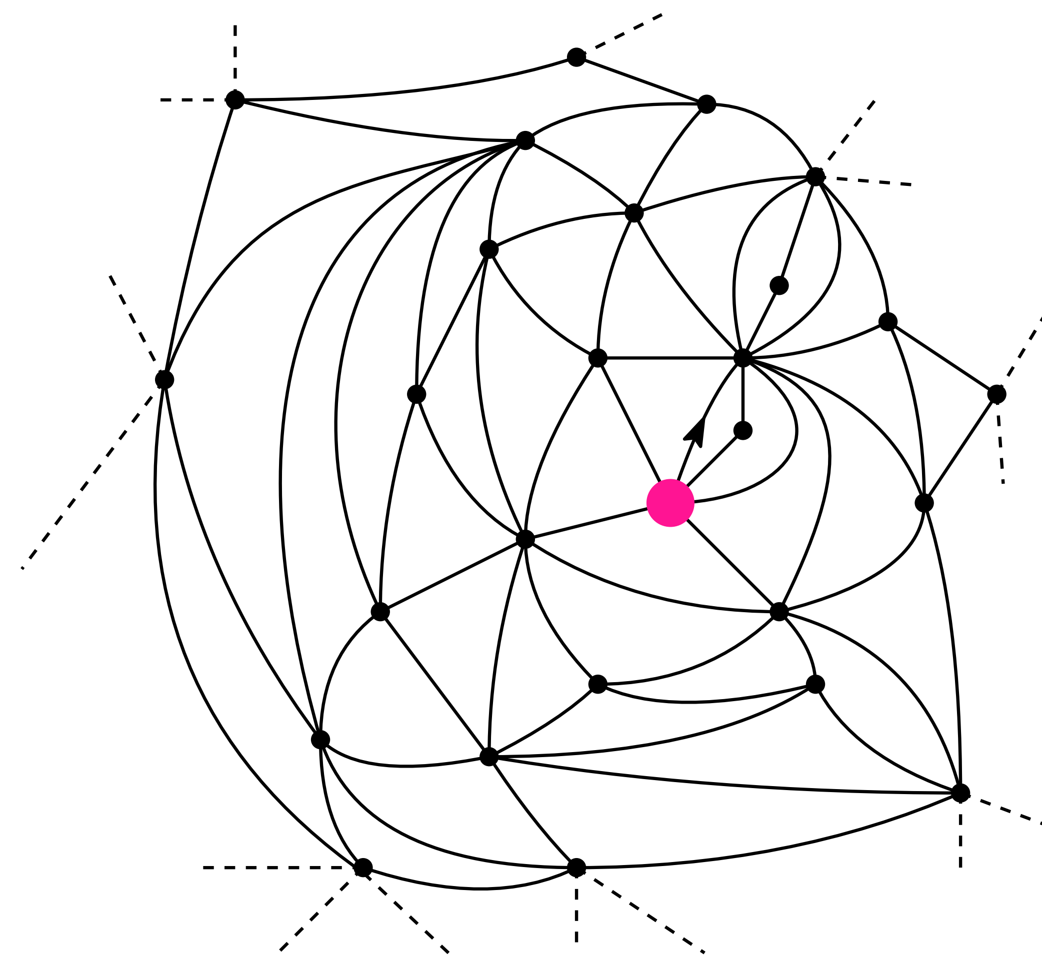
Definitions and first examples

Local topology ($\sim$ Benjamini–Schramm convergence)

$\mathcal{G}$:= family of (locally finite) rooted graphs

For $g \in \mathcal{G}$ and $R \in \mathbb{N}^\star$,

$$B_R(g) = \text{ball of radius } R \text{ around the root vertex of } g$$



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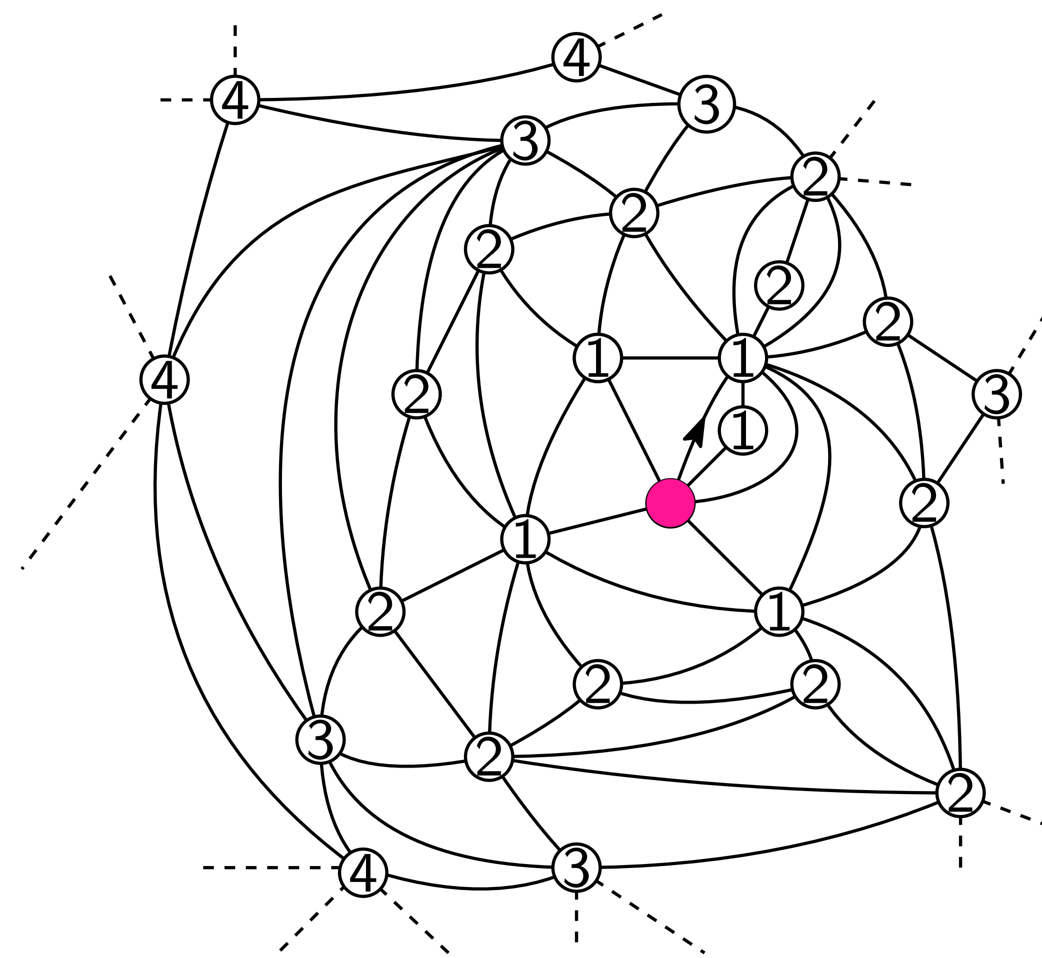

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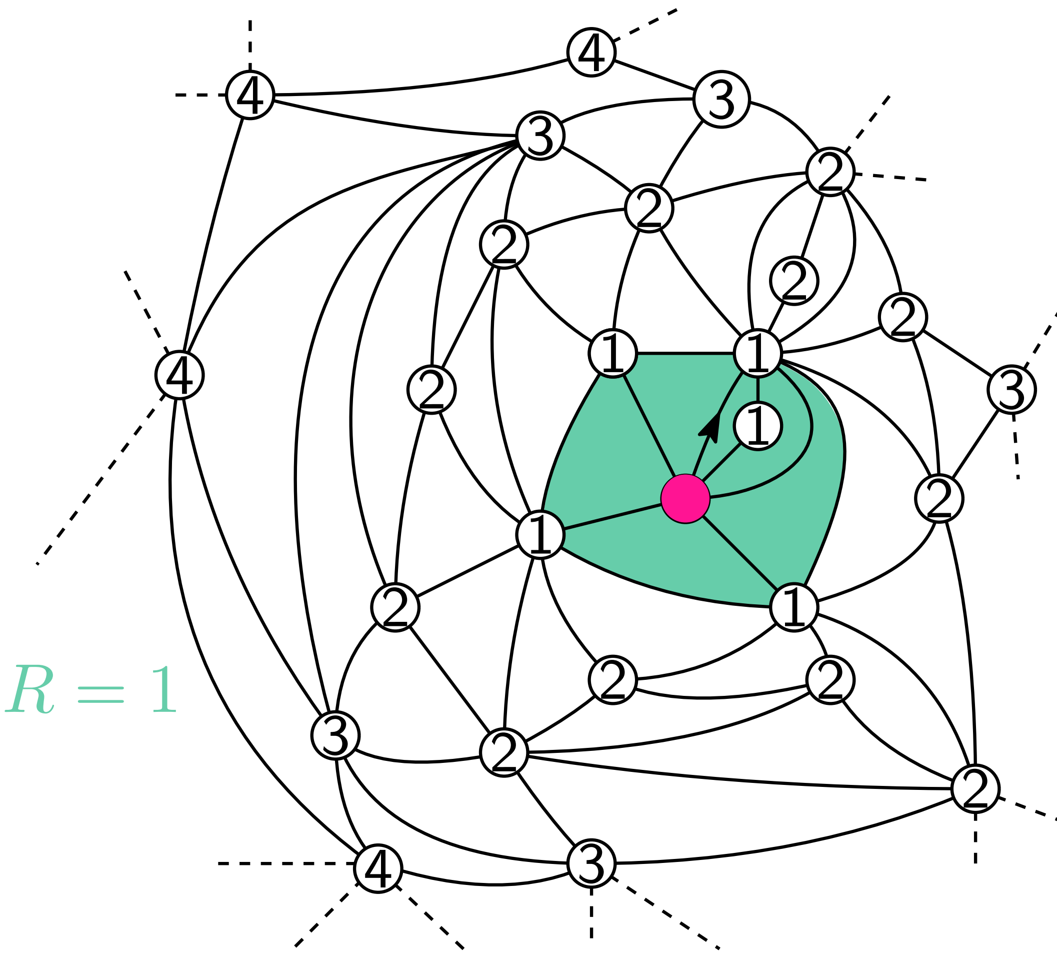

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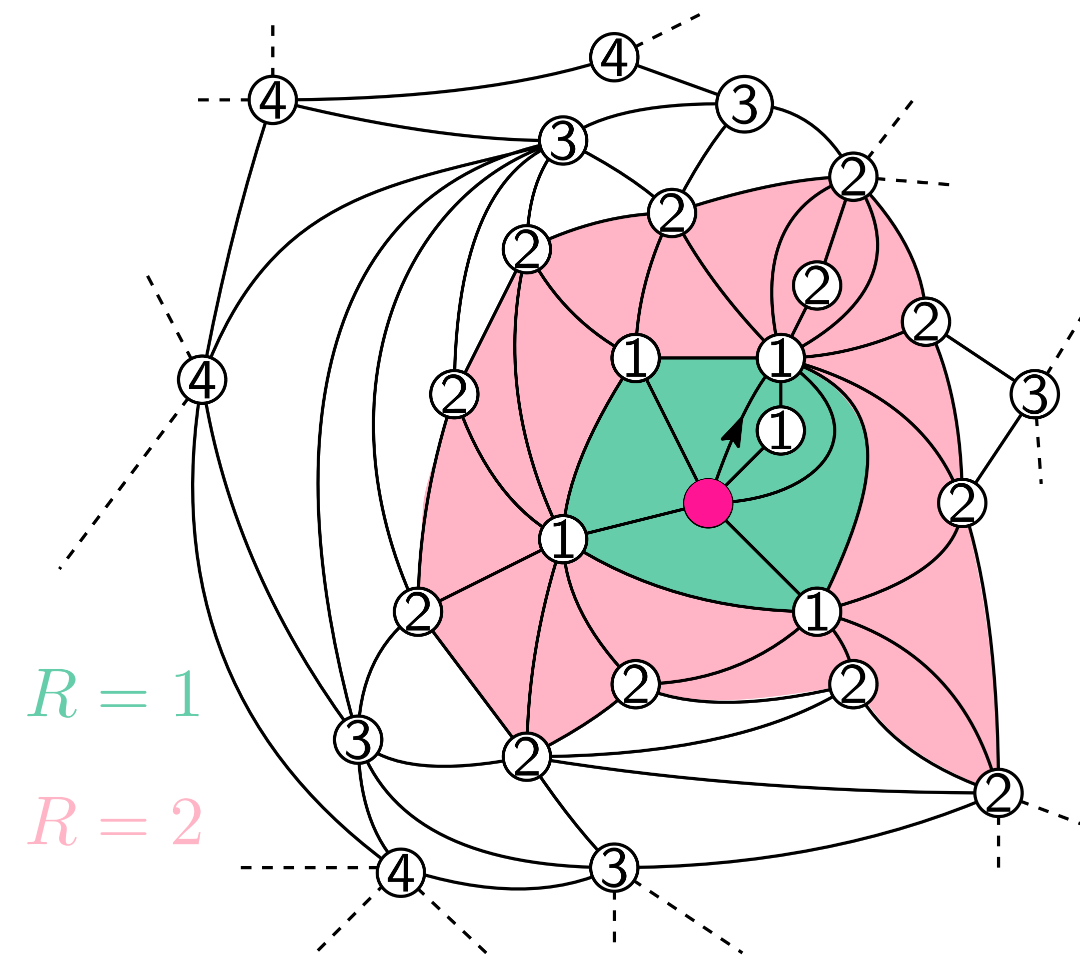

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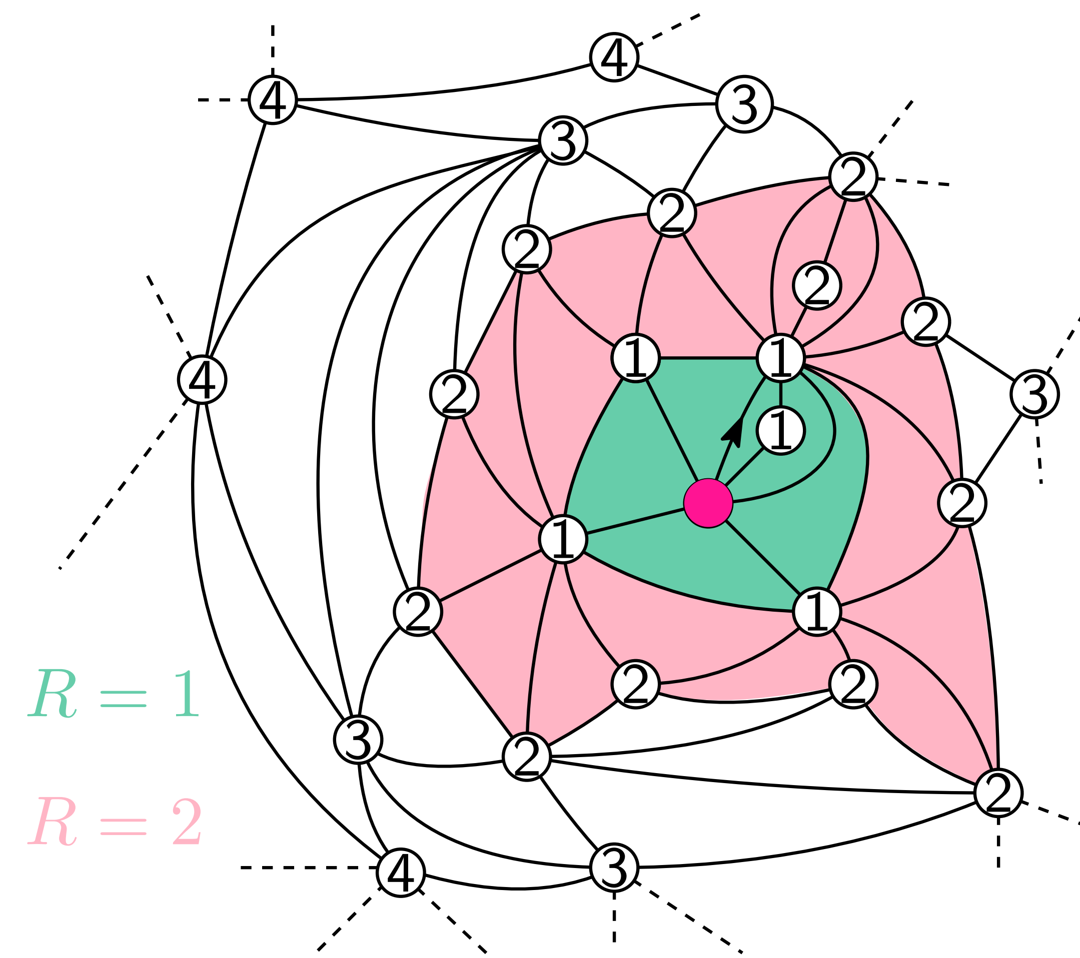
Definition:
The **local topology** on $\mathcal{G}$ is induced by the distance:

$$d_{loc}(g, g') := \frac{1}{1 + \max\{R \geq 0 : B_R(g) = B_R(g')\}}$$



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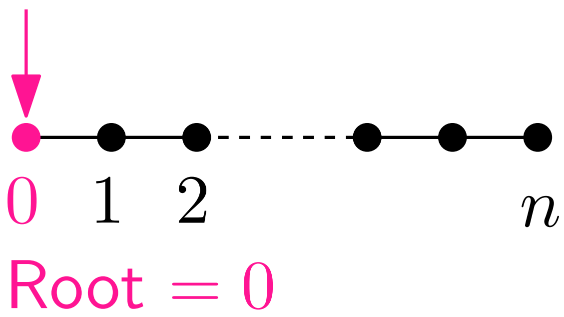
And for **random** graphs ?

(μ_n) = sequence of probability distributions on $\mathcal{G}$ (e.g. uniform distribution on $\mathcal{T}_n$)

if $\mu_n \xrightarrow{n \to \infty} \mu$ in distribution for the local topology,

we say that μ is the **local weak limit** of (μ_n).

Local convergence: simple examples



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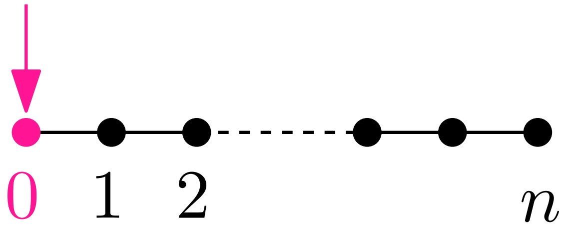

$\longrightarrow (\mathbb{Z}_+, 0)$

Root $= 0$

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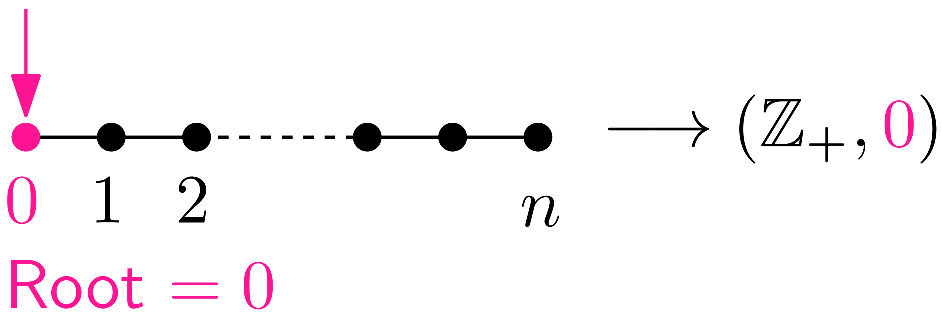

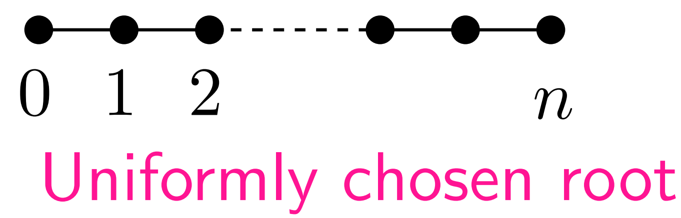

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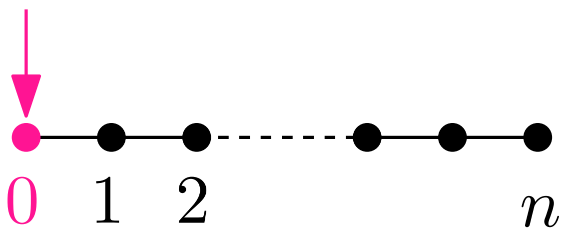

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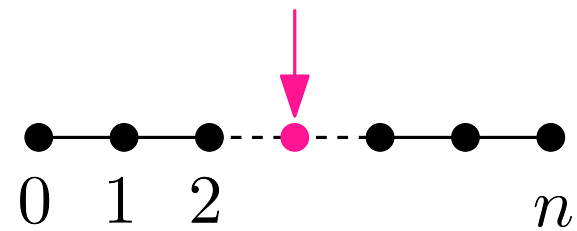

Uniformly chosen root

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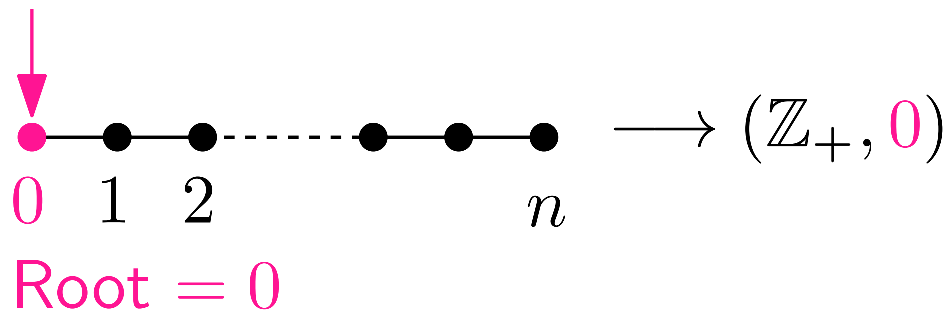

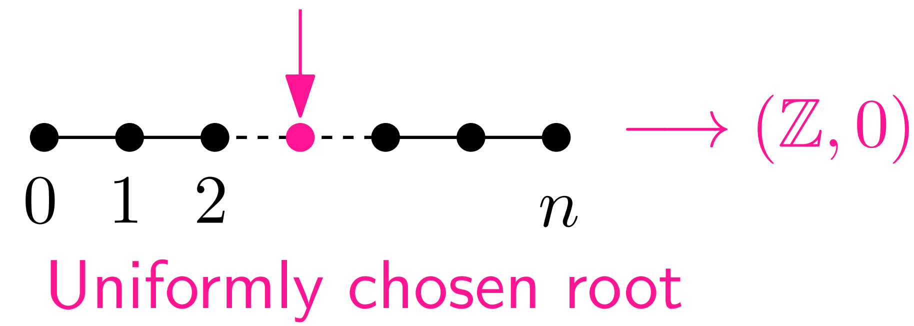

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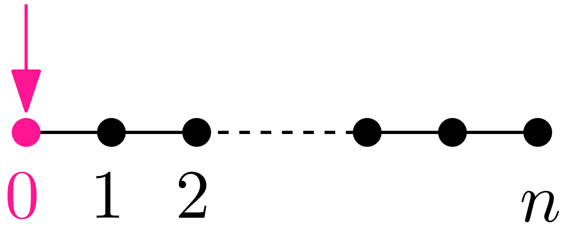

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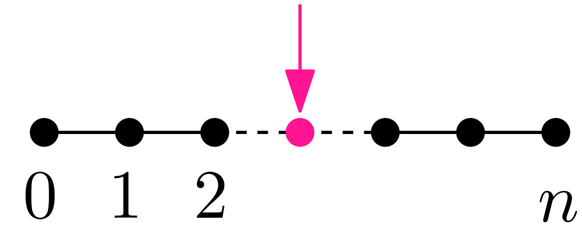

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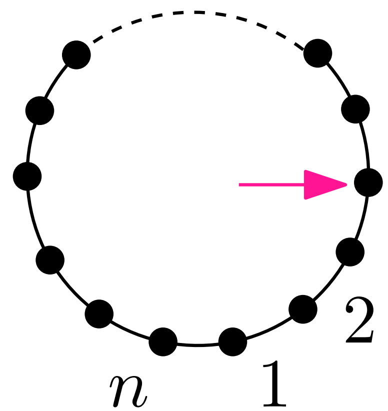

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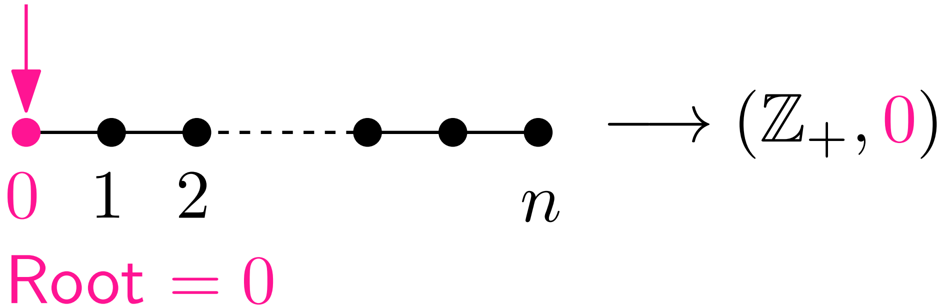
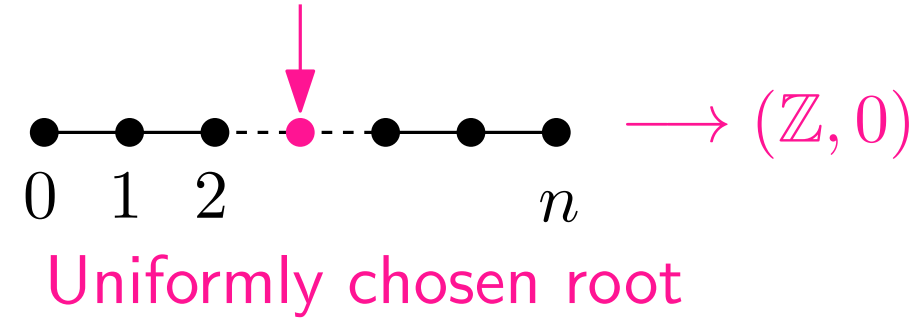
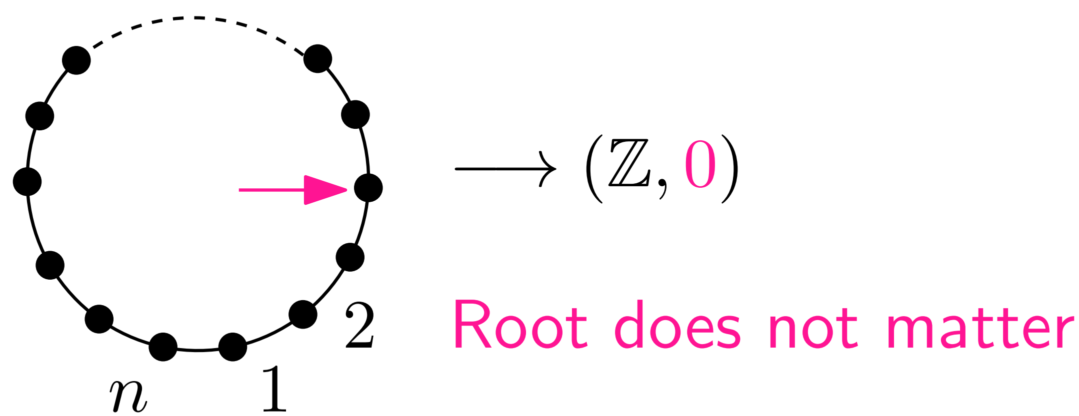
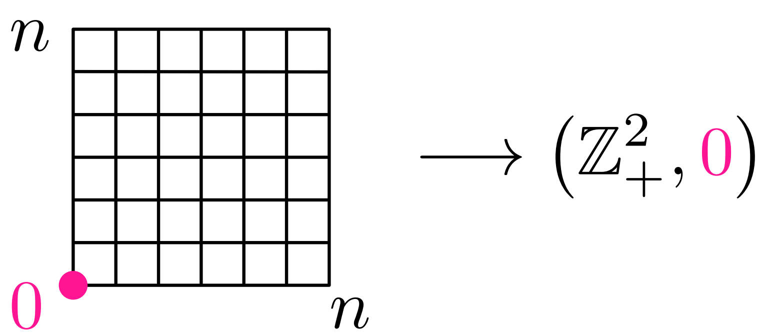
Root does not matter

Local convergence: simple examples









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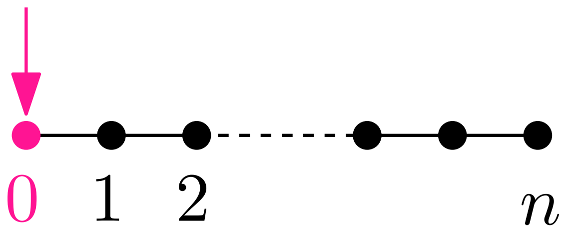

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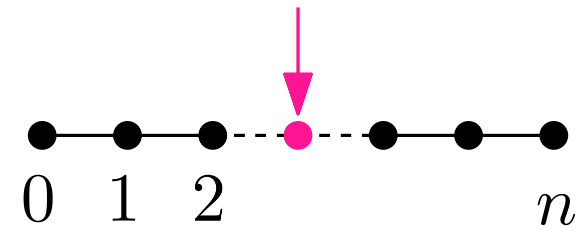

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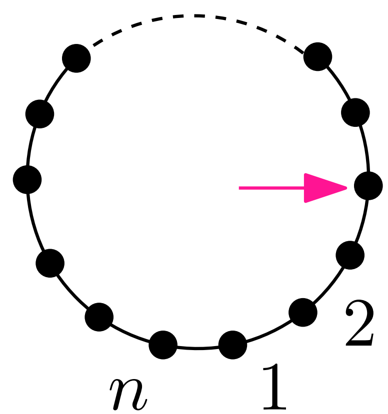
Uniformly chosen root



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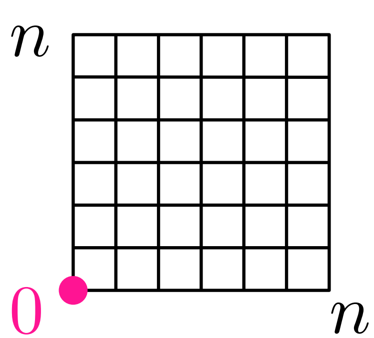
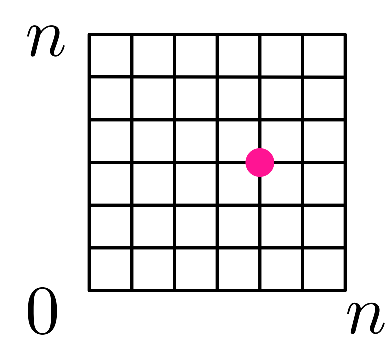


$\longrightarrow (\mathbb{Z}_+^2, 0)$



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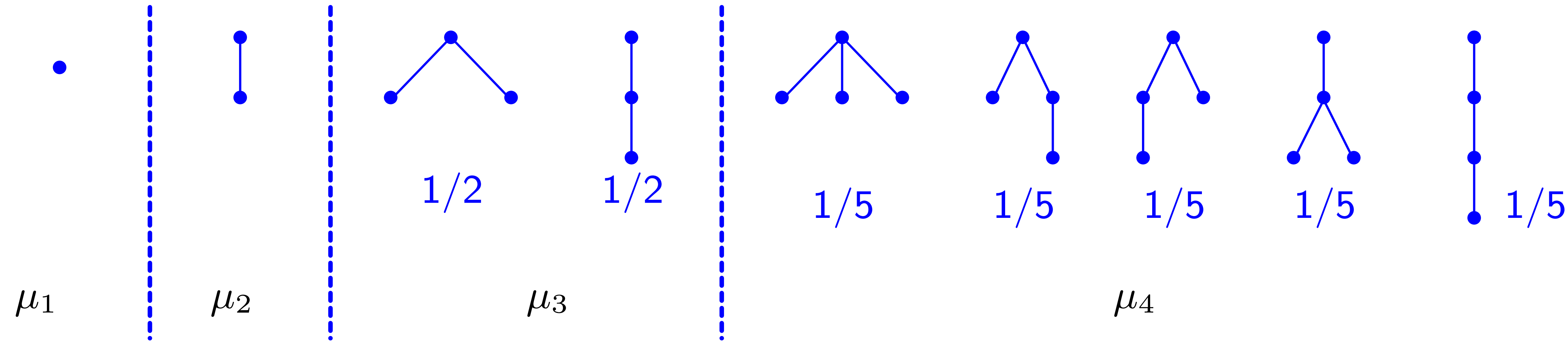
III - Local limits of random trees and maps

Local convergence: more complicated examples

μ_n = uniform measure on plane trees with n vertices:



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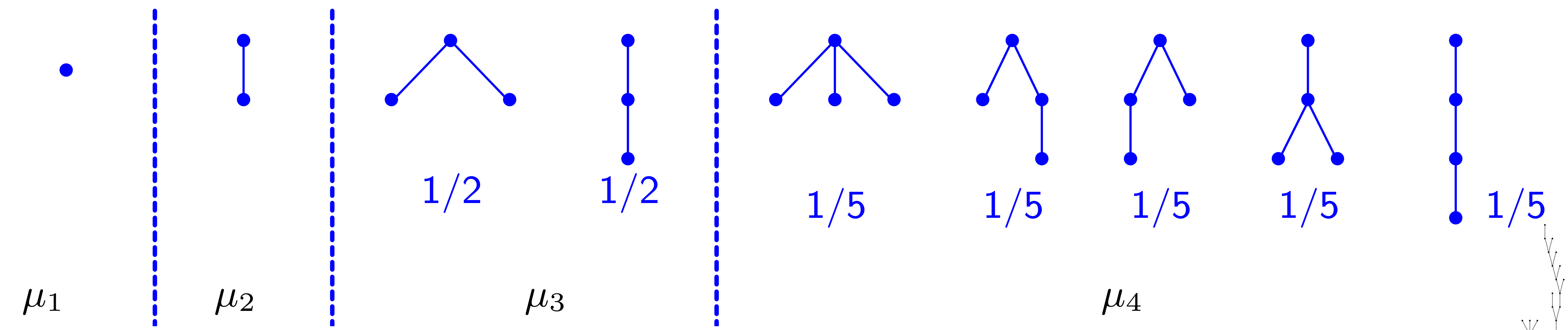
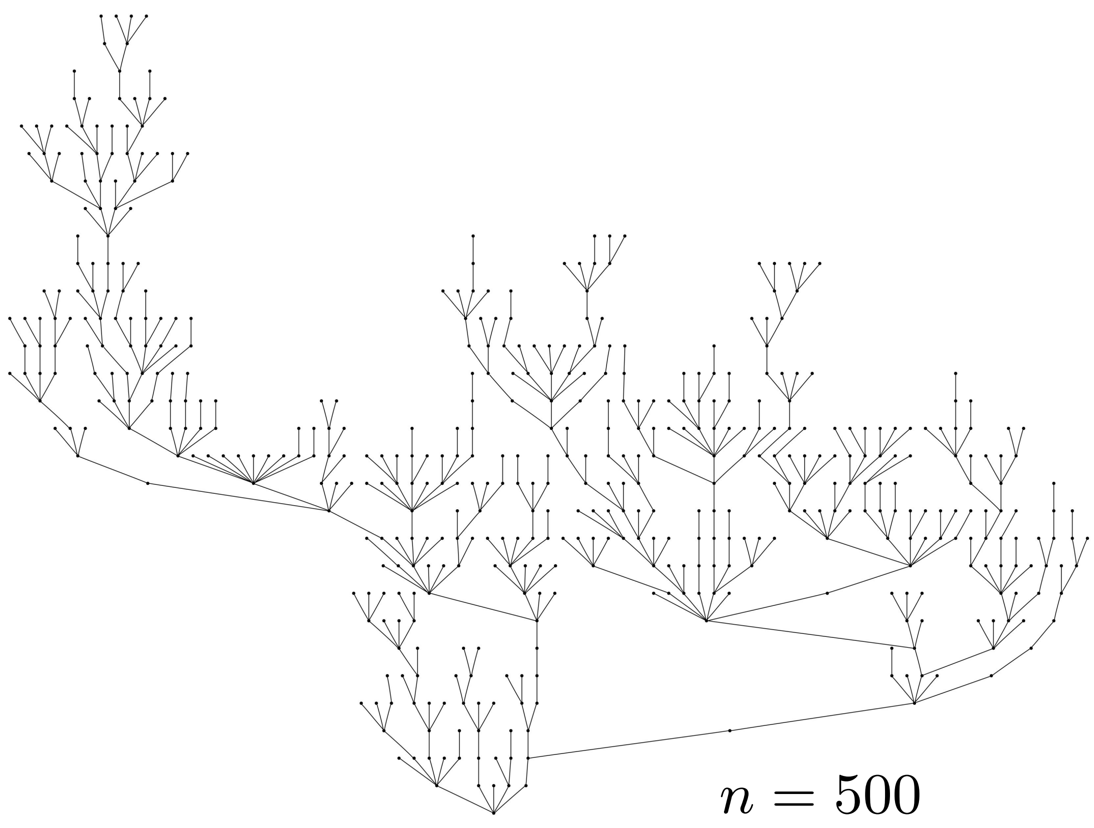
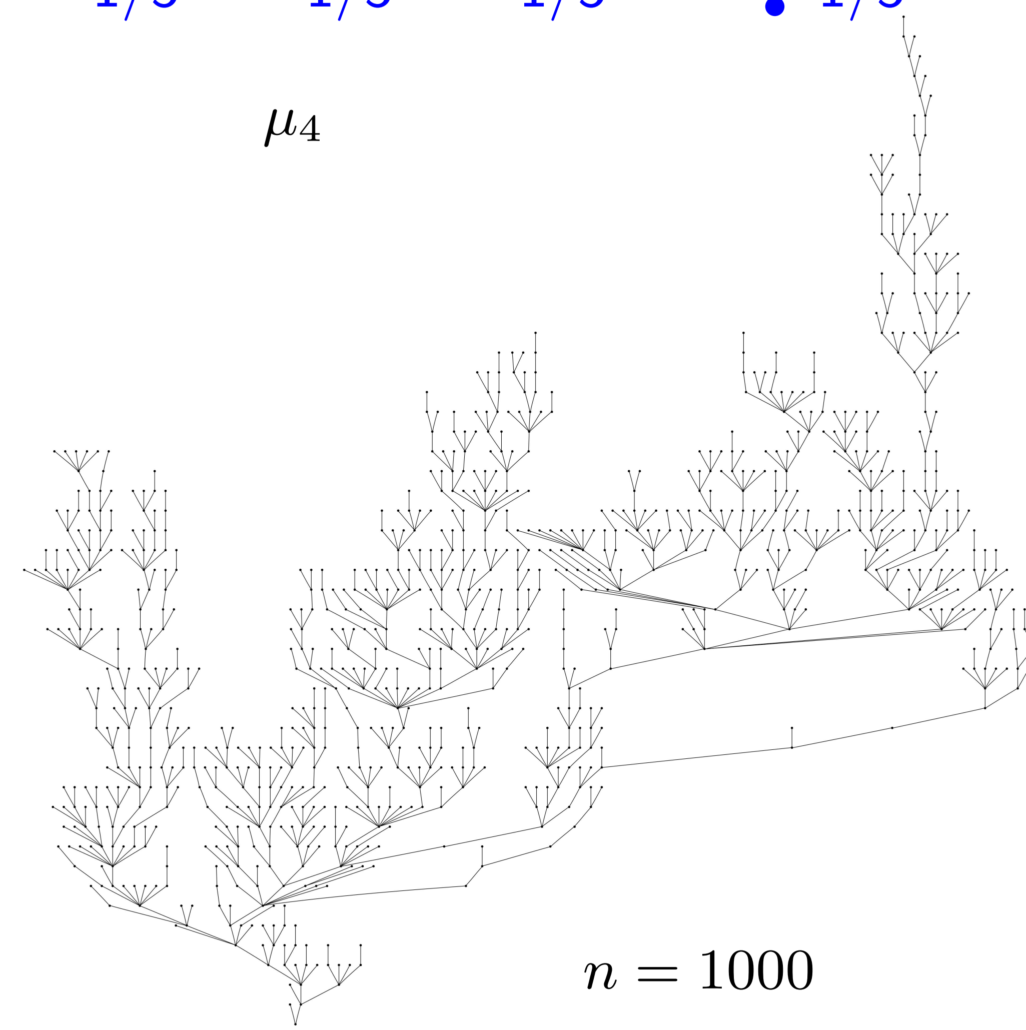




$n = 500$



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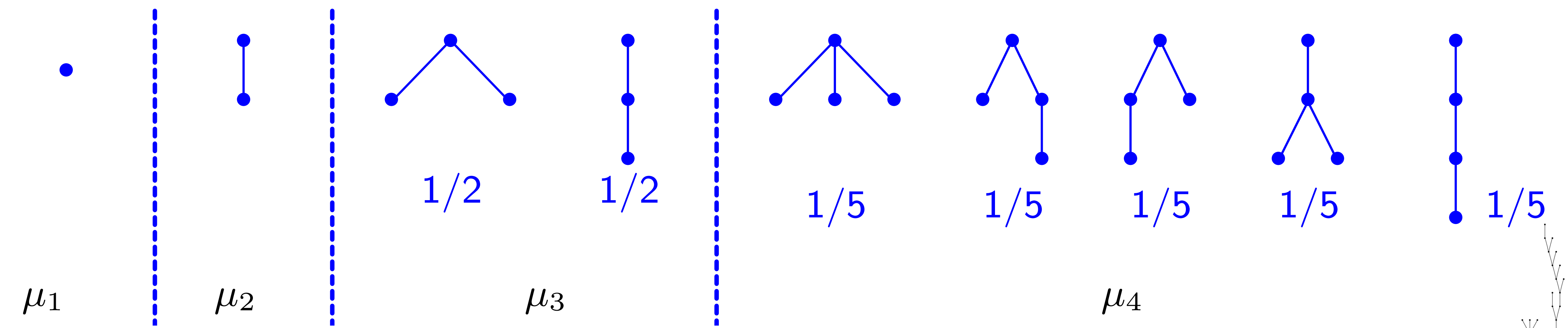
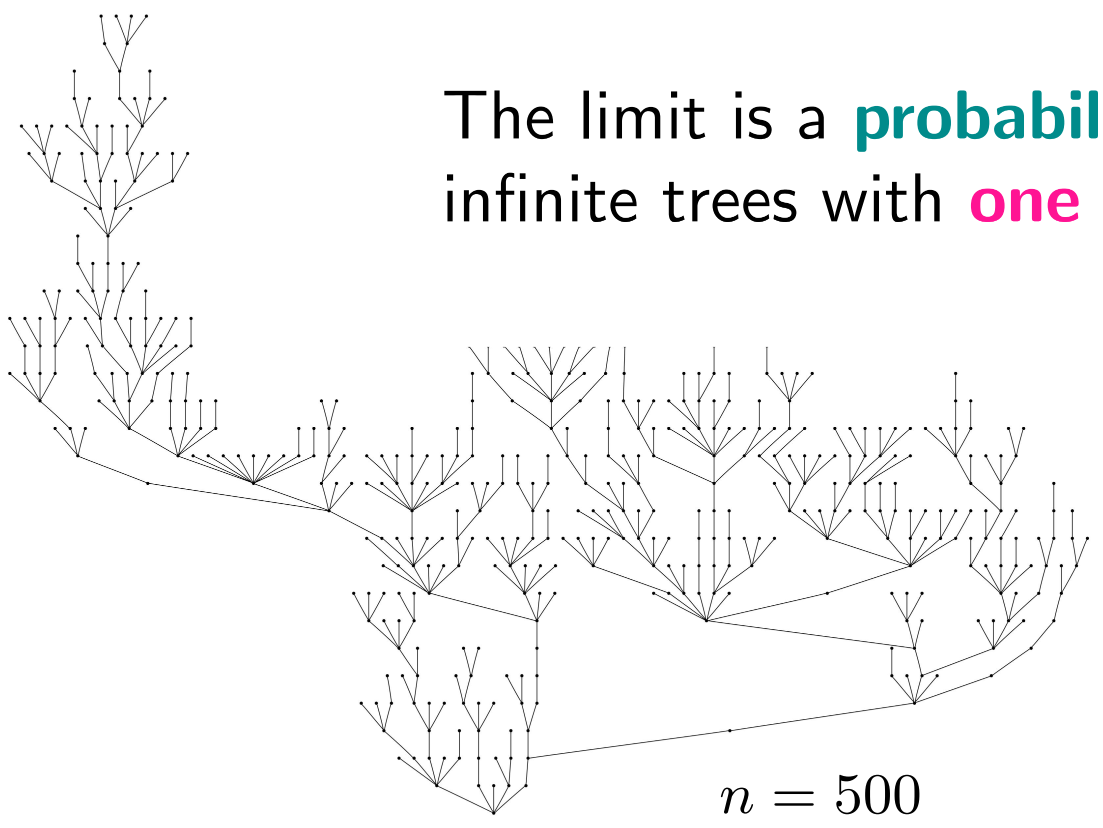
Local convergence: more complicated examples

μ_n = uniform measure on plane trees with n vertices:



The limit is a **probability distribution** on infinite trees with **one** infinite branch [Kesten].



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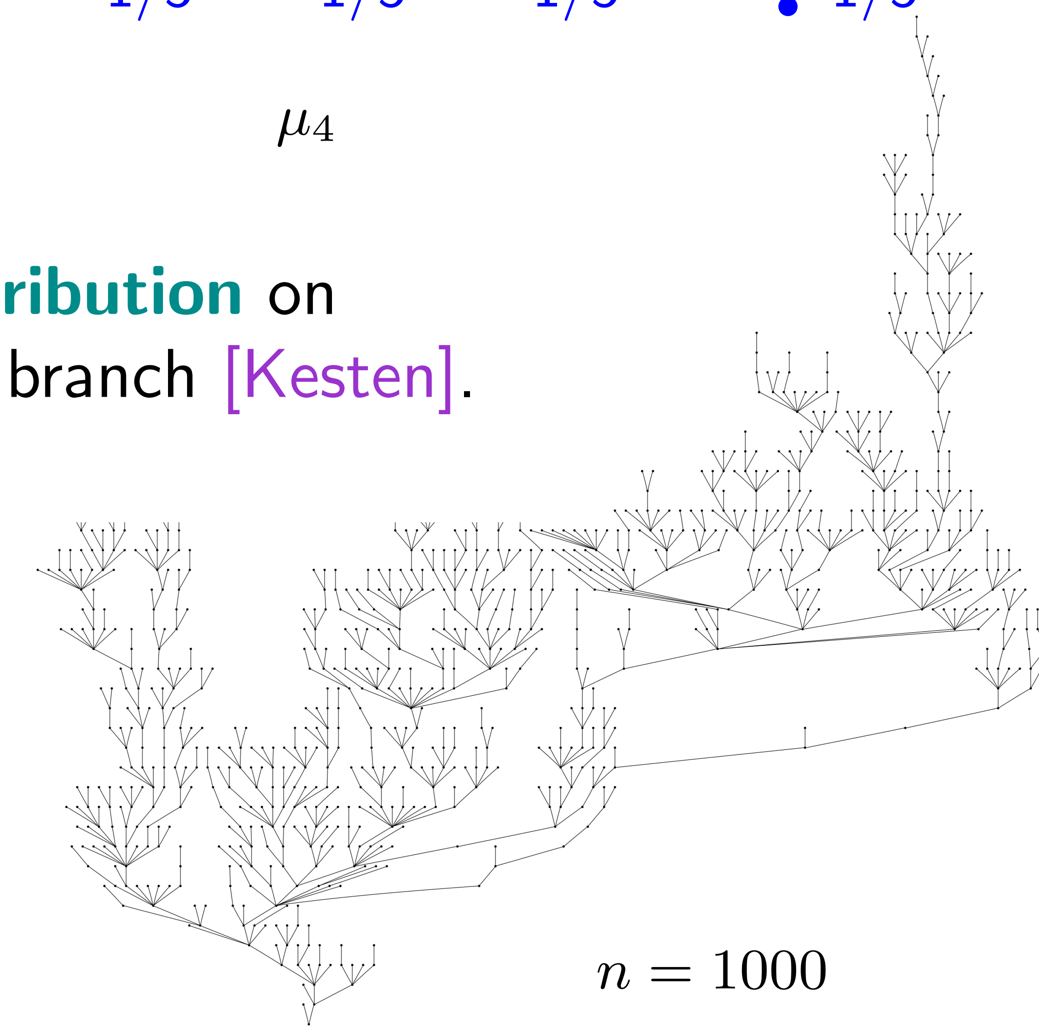

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Local limit of large uniformly random triangulations

Theorem [Angel – Schramm, '03]

Let $\mathbb{P}_n$ = uniform distribution on triangulations of size n.

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UIPT = Uniform Infinite Planar Triangulation
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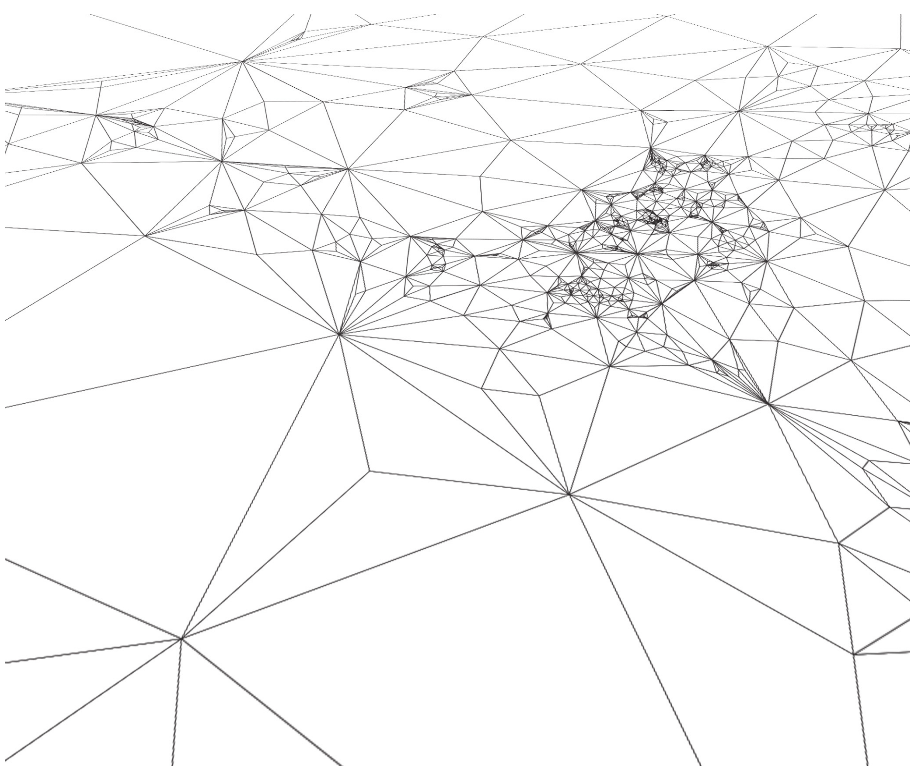
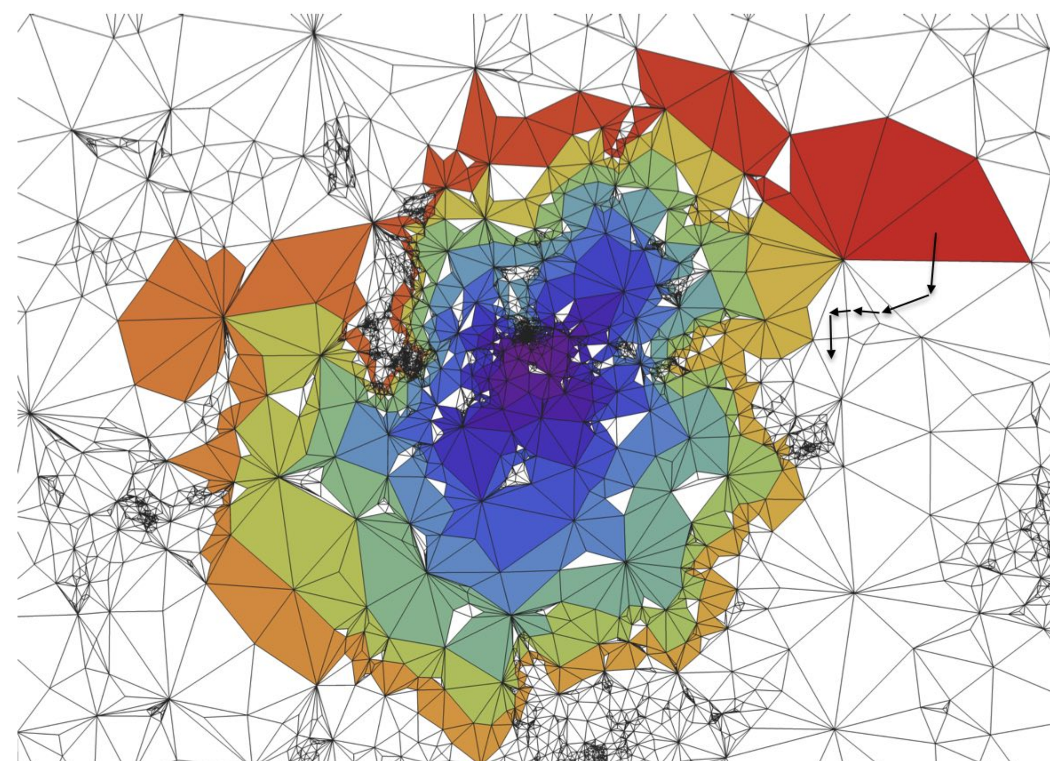


Simulation by I. Kortchemski



Simulation by T.Budd

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A very short idea of the proof:

Need to evaluate the probability that a given neighborhood Δ of the root appears:

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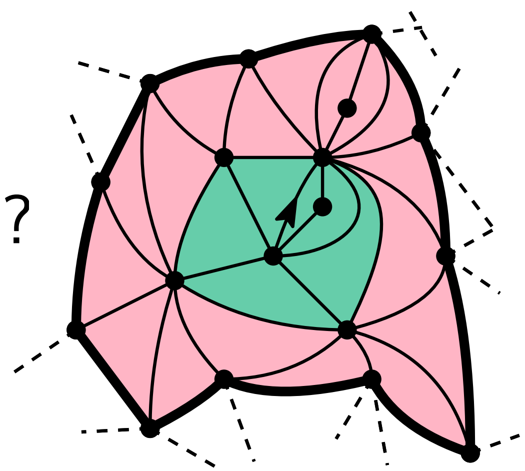

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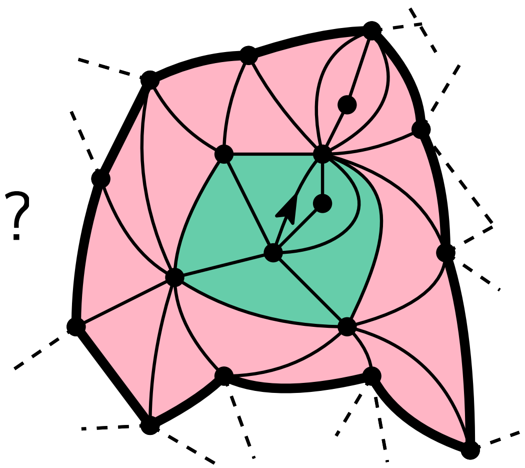

???$\Big) = \dfrac{\left|\mathcal{T}^{(k)}_{3n-e(\Delta)+\ell(\Delta)}\right|}{|\mathcal{T}_n|}$

$\ell(\Delta) = 11$

$$\begin{cases} e(\Delta) = & \#\{\text{edges of } \Delta\} \\ \ell(\Delta) = & \text{perimeter of } \Delta \end{cases}$$

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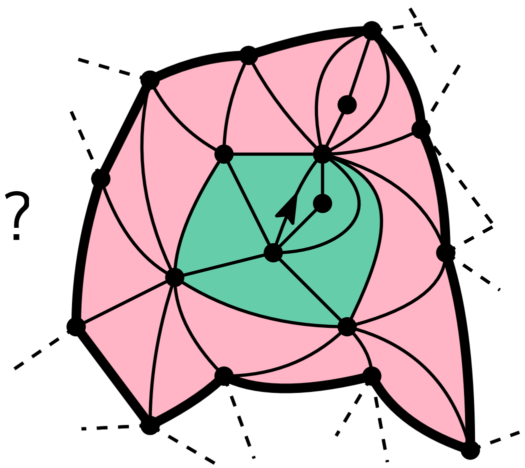
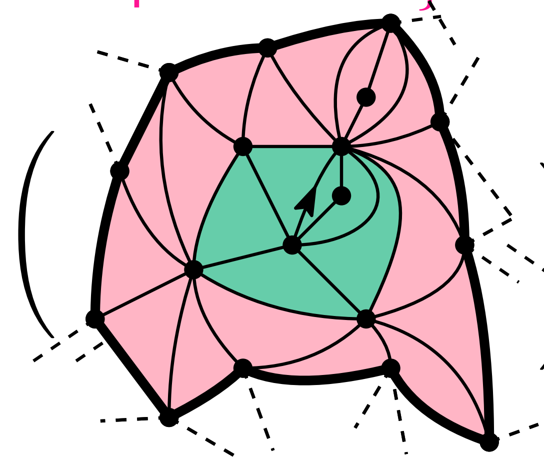
$\mathbb{P}_n\Big($???  ??? $\Big) = \dfrac{\left|\mathcal{T}^{(k)}_{3n-e(\Delta)+\ell(\Delta)}\right|}{|\mathcal{T}_n|} \xrightarrow{\ n \to \infty\ } \mathbb{P}_\infty\Big($  $\Big)$

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Some properties of the UIPT:

- The UIPT has almost surely one end [Angel – Schramm, 03]
- Volume (nb. of vertices) and perimeters of balls known to some extent.

$$\mathbb{E}\left[|B_R(\mathbf{T}_\infty)|\right] \sim \frac{2}{7}R^4 \qquad \text{[Angel 04, Curien – Le Gall 12]}$$

- Simple random Walk is recurrent [Gurel-Gurevich – Nachmias 13]

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Universality: we expect the **same behavior** for other **uniform** models of maps.
In particular, we expect the volume growth to be 4.
(proved for quadrangulations [Krikun 05], simple triangulations [Angel 04])

Intermezzo: why should we care about local limits ?

Suppose that a sequence of random graphs G_n admits a local weak limit G_∞,

$$\text{Then,} \quad f(G_n) \xrightarrow{proba} f(G_\infty) \quad \text{for any } f \text{ which is continuous for } d_{loc}.$$

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Main idea: The limiting object is often “nicer”.

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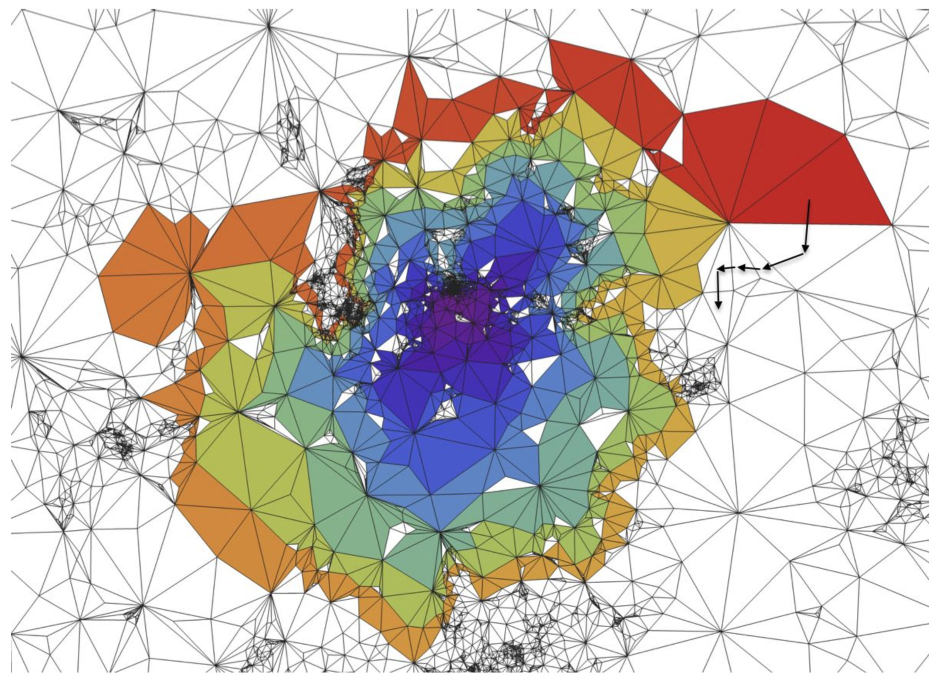
Two example for maps:

- one-endedness in the UIPT:

Allows to give an explicit description of what can happen when the map gets disconnected.

This is crucial to study a "peeling" exploration spiraling around the root, which gives the volume of the balls [Angel 03].



Simulation by T.Budd

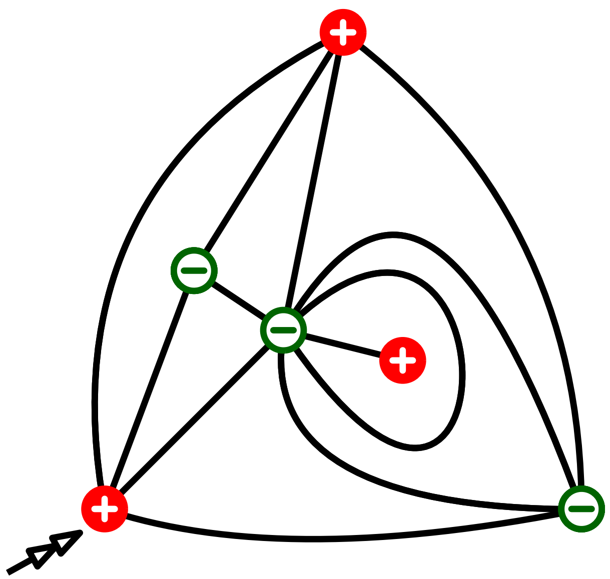
- spatial Markov property

Conditionally on their perimeter, the interior and exterior of a ball are independent

IV - Local limits of Ising-weighted triangulations

Escaping universality: adding matter

First, **Ising model** on a finite deterministic planar triangulation T:



Spin configuration on T:

$$\sigma : V(T) \to \{-1, +1\} = \{\ominus, \oplus\}.$$

Ising model on T: take a random spin configuration with probability:

$$P(\sigma) \propto e^{\beta J \sum_{v \sim v'} \mathbf{1}_{\{\sigma(v)=\sigma(v')\}}}$$

$\beta > 0$: inverse temperature.
$J = \pm 1$: coupling constant.
$h = 0$: no magnetic field.

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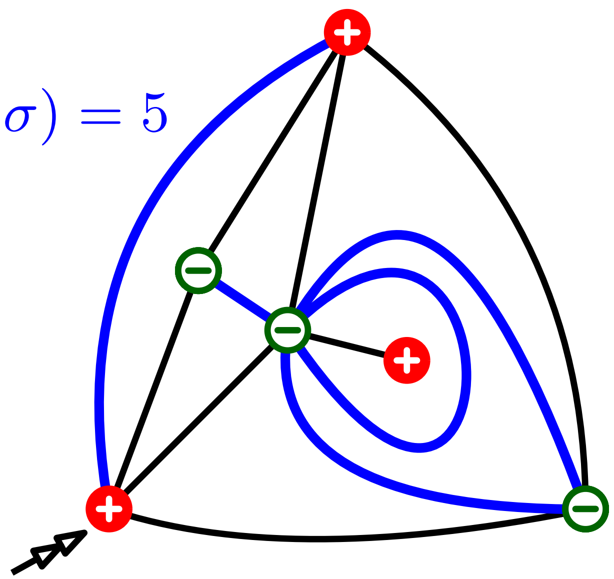

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Combinatorial formulation: $P(\sigma) \propto \nu^{m(\sigma)}$

with $m(\sigma)$ = number of monochromatic edges ($\nu = e^{\beta J}$).

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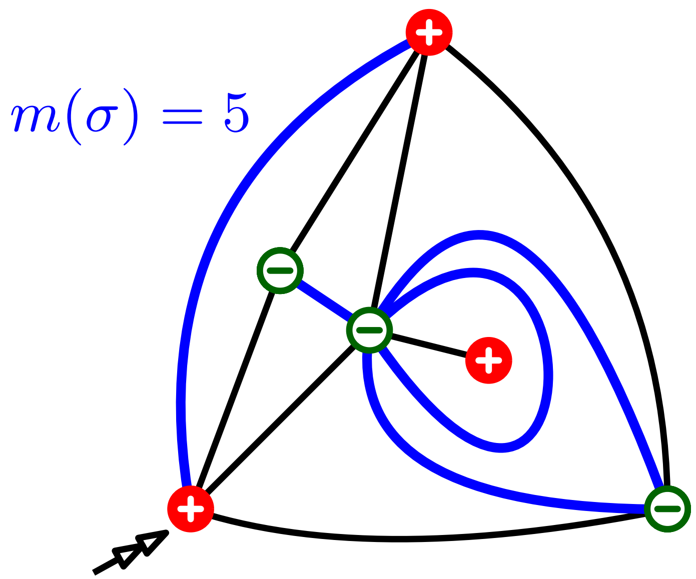

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Next step: Sample a triangulation of size n **together** with a spin configuration, with probability $\propto \nu^{m(T,\sigma)}$.

$$\mathbb{P}_n^{\nu}\Big(\{(T,\sigma)\}\Big) = \frac{\nu^{m(T,\sigma)}\delta_{|e(T)|=3n}}{\mathcal{Z}_n}.$$

$\mathcal{Z}_n =$ normalizing constant.

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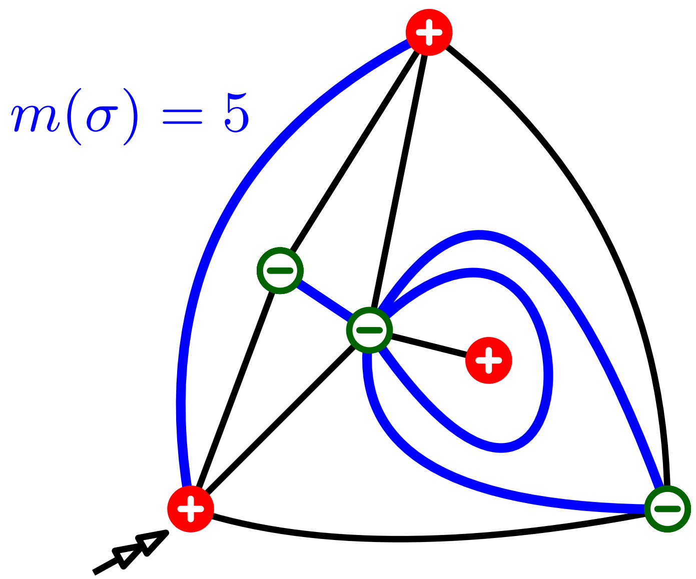

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$\mathcal{Z}_n =$ normalizing constant.

Remark: This is a probability distribution on triangulations **with** spins. But, forgetting the spins gives a probability a distribution on triangulations **without** spins **different from the uniform distribution**.

Escaping universality: new asymptotic behavior

Counting exponent for undecorated maps:

number of (undecorated) maps of size $n \sim \kappa \rho^{-n} n^{-\mathbf{5/2}}$

(e.g.: triangulations, quadrangulations, general maps, simple maps,...)

where κ and ρ depend on the combinatorics of the model.

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Generating series of Ising-weighted triangulations:

$$Q(\nu, t) = \sum_{T \in \mathcal{T}_f} \sum_{\sigma : V(T) \to \{-1,+1\}} \nu^{m(T,\sigma)} t^{e(T)}.$$

Theorem [Bernardi – Bousquet-Mélou 11]
For every $\nu > 0$, $Q(\nu, t)$ is algebraic and satisfies

$$[t^{3n}]Q(\nu,t) \underset{n\to\infty}{\sim} \begin{cases} \kappa\, \rho_{\nu_c}^{-n}\, n^{\mathbf{-7/3}} & \text{if } \nu = \nu_c = 1 + \frac{1}{\sqrt{7}}, \\ \kappa\, \rho_{\nu}^{-n}\, n^{\mathbf{-5/2}} & \text{if } \nu \neq \nu_c. \end{cases}$$

See also [Boulatov – Kazakov 1987], [Bousquet-Melou – Schaeffer 03] and [Bouttier – Di Francesco – Guitter 04].

This suggests a **different behavior** of the underlying maps for $\nu = \nu_c$.

Local convergence of triangulations with spins

Theorem [A. – Ménard – Schaeffer, 21]

Let $\mathbb{P}_n^\nu = \nu$–Ising weighted probability distribution for triangulations of size n:

$$\mathbb{P}_n^\nu \xrightarrow{(d)} \nu\text{-IIPT}, \qquad \text{for the local topology}$$

ν-IIPT $= \nu$-Ising Infinite Planar Triangulation

$=$ measure supported on infinite planar triangulations.

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Need to evaluate the probability that a given neighborhood of the root appears:

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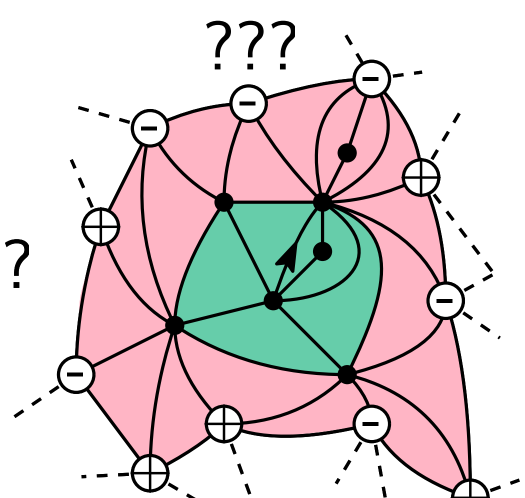

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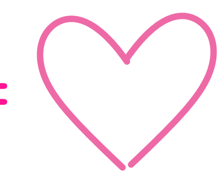
A very very short idea of the proof: Computations + Maple = 

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Generating series of triangulations with boundary conditions given by ω.

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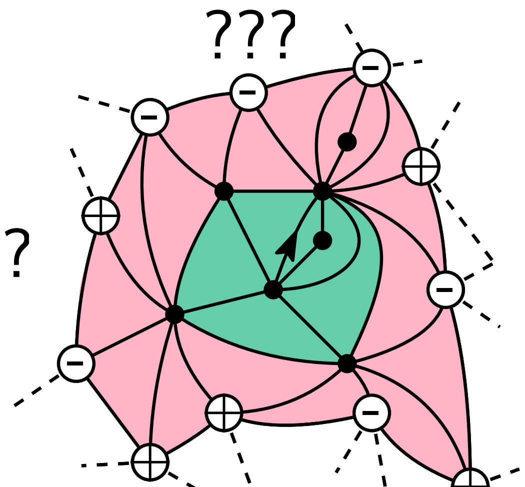
$\mathbb{P}_n^\nu\Big($???  ??? $\Big) = \dfrac{\nu^{m(\Delta)-m(\omega)}\,[t^{3n-e(\Delta)+|\omega|}]\mathbf{Z}_\omega(\nu,t)}{[t^{3n}]Q(\nu,t)}$

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But:

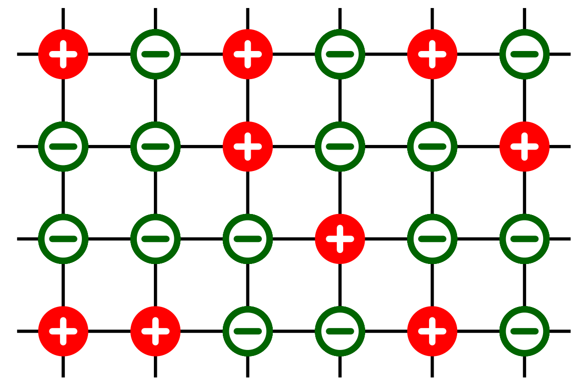
- Volume (nb. of vertices) and perimeters of balls is **unknown**.

Non-universality: we expect a **different** behavior for $\nu = \nu_c$
In particular, we expect the volume growth to be different from 4.

V - Connections and motivations from statistical physics

Motivations from statistical physics

Originally the Ising model was studied on **regular lattices** such as $\mathbb{Z}^2$ [Ising, Onsager].



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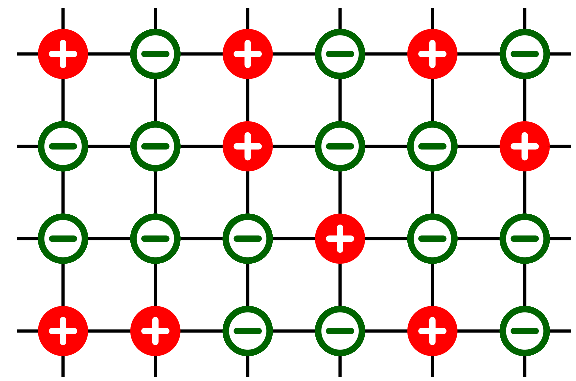

Why do we study it on **random** metric spaces ?

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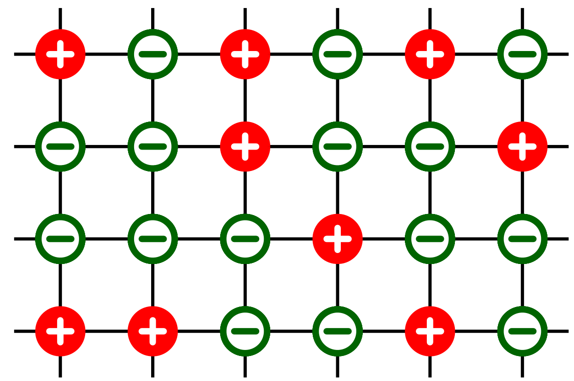

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In **general relativity**, the underlying space is not Euclidian anymore but is a Riemannian space, whose curvature describes the gravity.

One of the main challenge of modern physics is to make two theories consistent:

- quantum mechanics (which governs microscopic scales)
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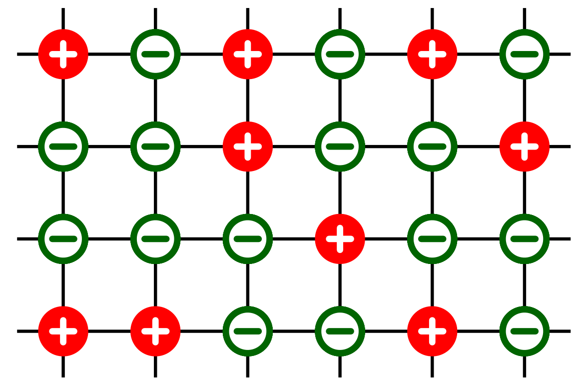

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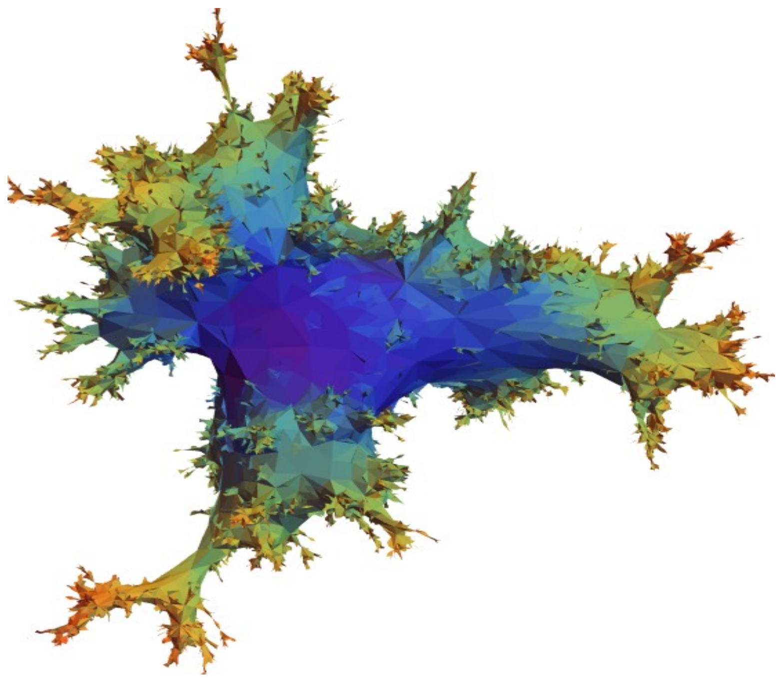
- quantum mechanics (which governs microscopic scales)
- general relativity (which governs macroscopic scales)

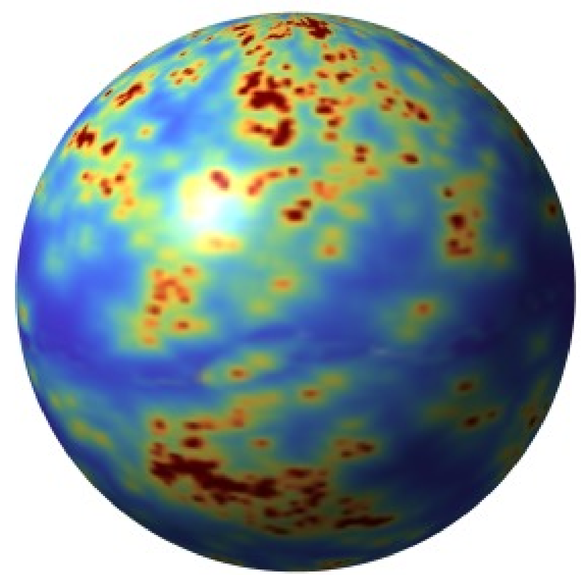
One attempt to reconcile these two theories, is the **Liouville Quantum gravity** which replaces the **deterministic** Riemannian space by a **random** metric space.

Link with Liouville Quantum Gravity

$\gamma \in (0,2)$, γ-Liouville Quantum Gravity = measure on a surface [Duplantier, Sheffield 11].



Simulation of the Brownian sphere by T.Budd



Simulation of $\sqrt{\frac{8}{3}}$-LQG by T.Budd

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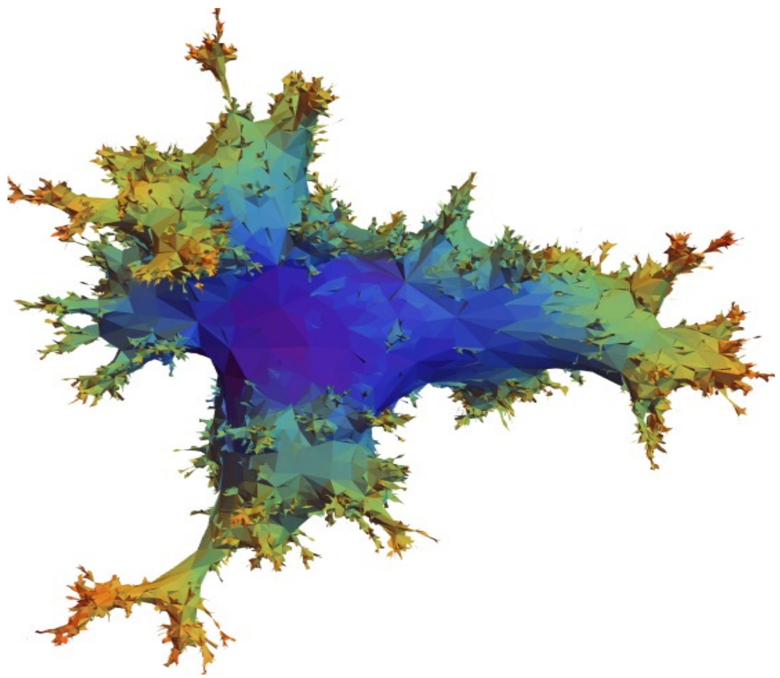

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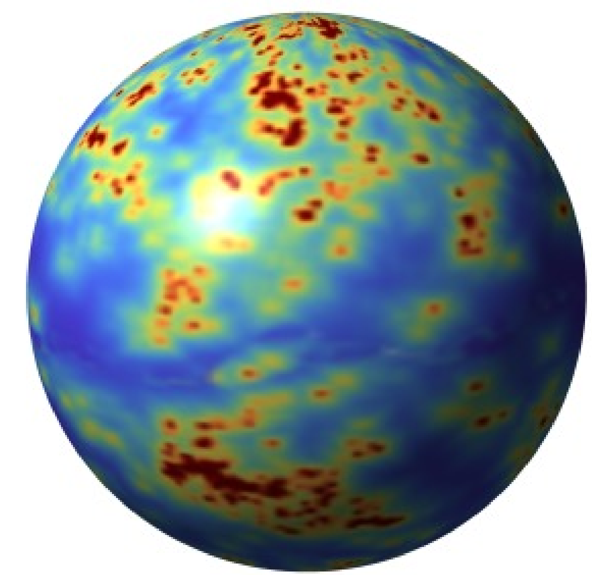
[Duplantier, Miller, Sheffield 14]
[Miller, Sheffield 16+16+17]
Construction in the continuum.

$\longleftrightarrow$



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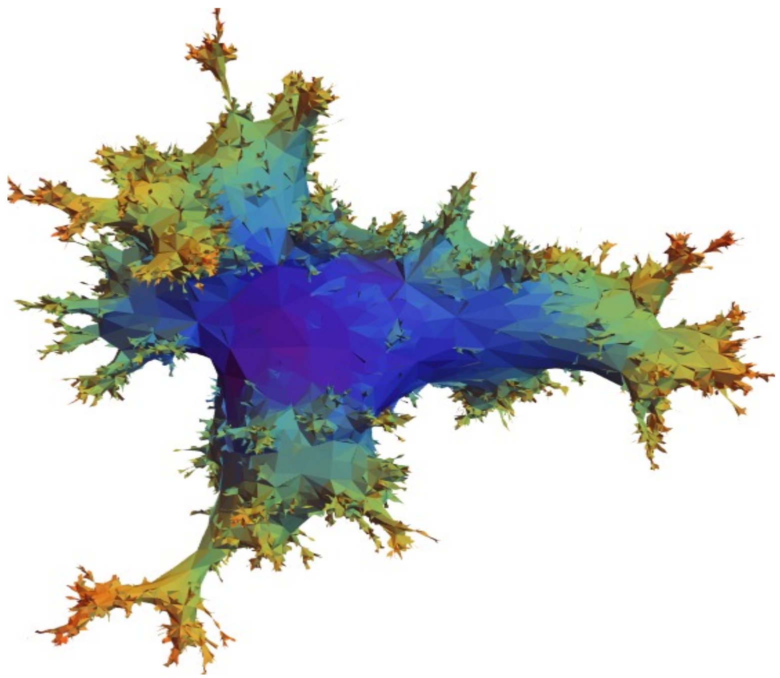

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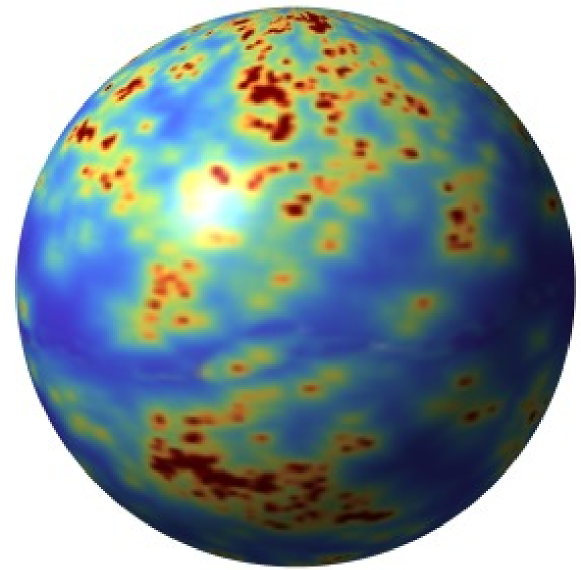
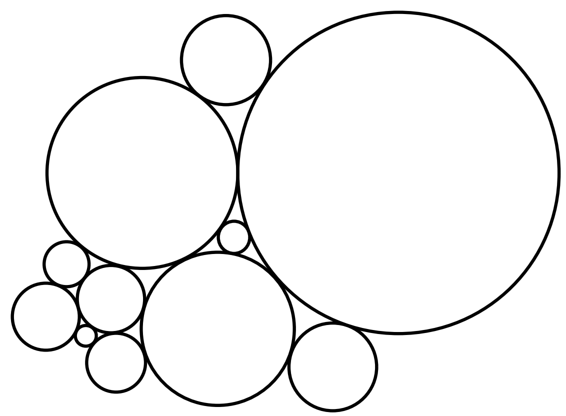
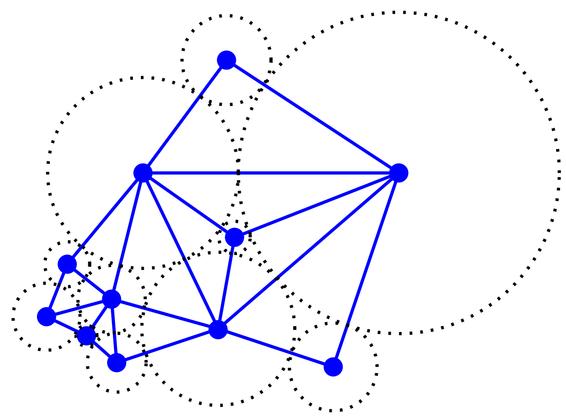


Simulation of $\sqrt{\frac{8}{3}}$-LQG by T.Budd

A priori , there is no canonical way to embed a planar map in the sphere.

But, for simple triangulations:
the **circle packing theorem**
gives a canonical embedding.
(Unique up to Möbius transformations.)





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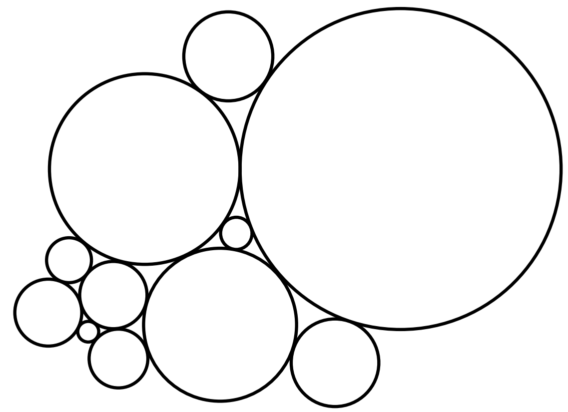
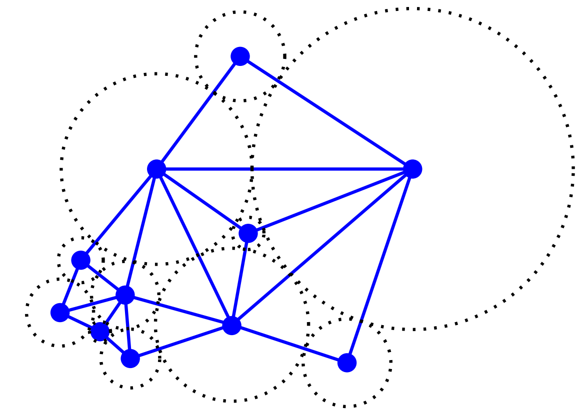
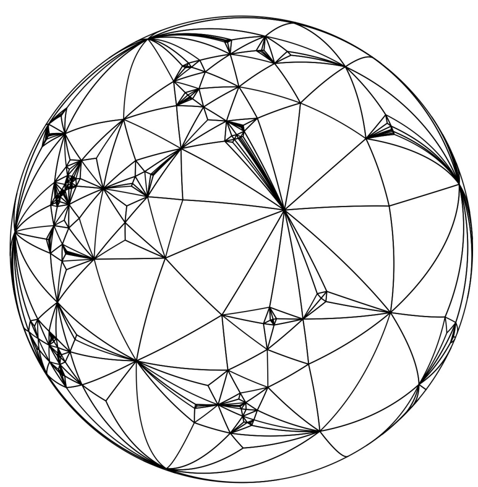
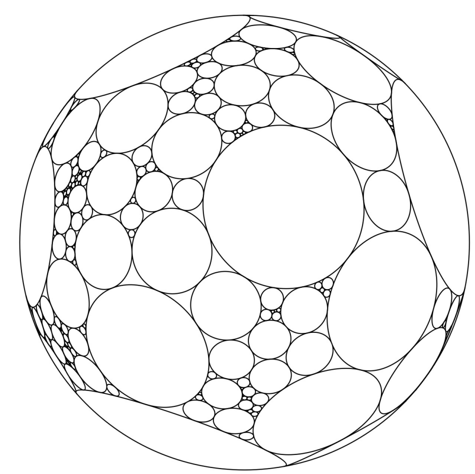


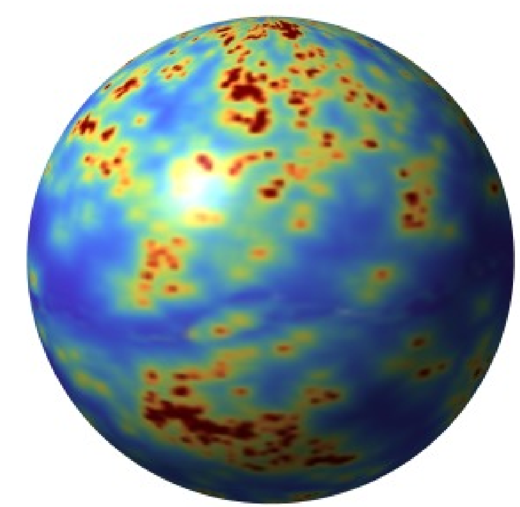






Simulation of a large simple triangulation
embedded in the sphere by circle packing.

Software CirclePack by K.Stephenson.



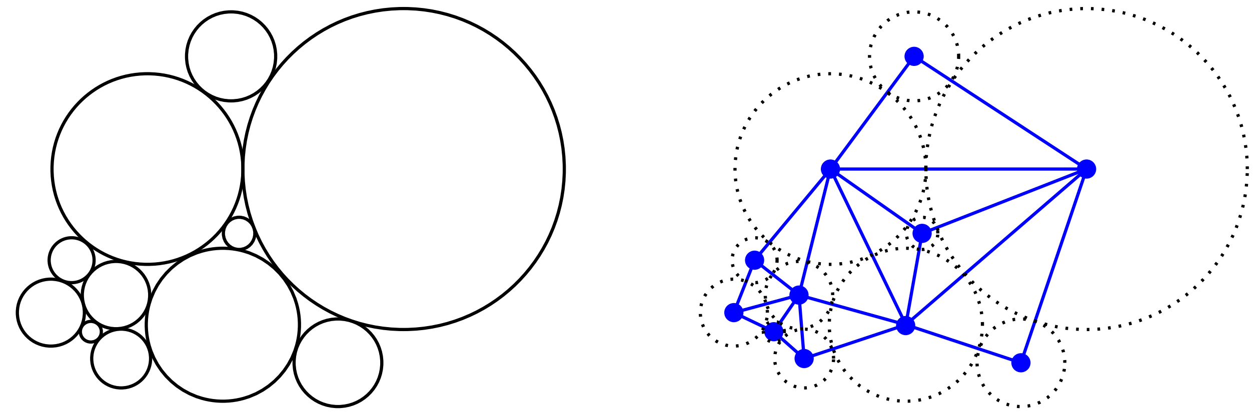
Simulation of $\sqrt{\frac{8}{3}}$-LQG by T.Budd

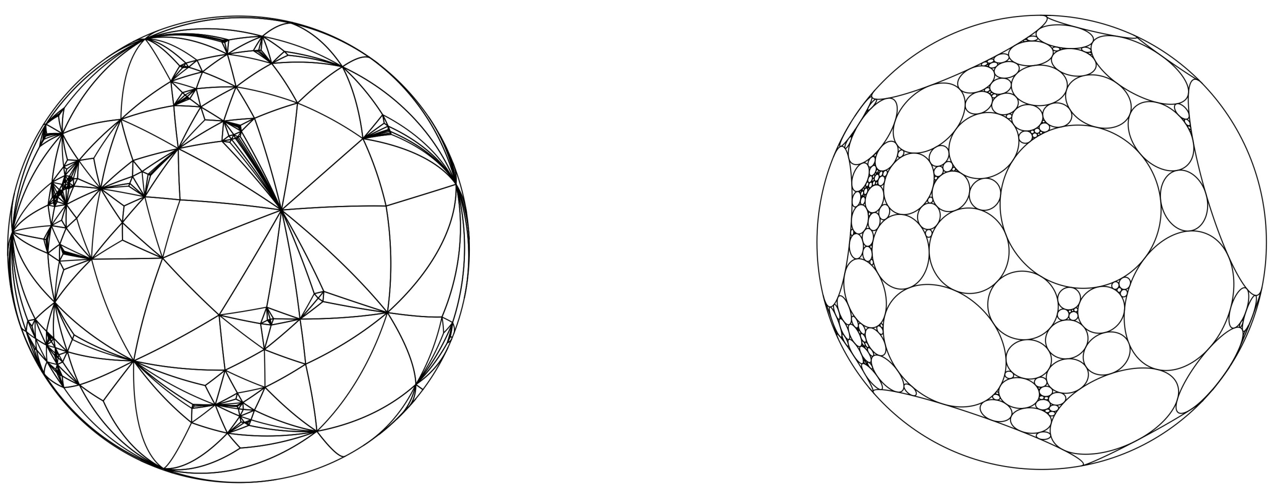
Unknown for embedding via circle packings, but fantastic result similar in spirit in [Holden – Sun 20]

Link with Liouville Quantum Gravity

But, for simple triangulations:
the **circle packing theorem**
gives a canonical embedding.

(Unique up to Möbius transformations.)





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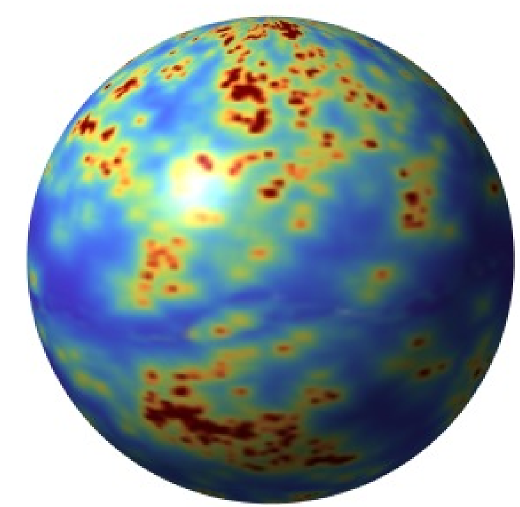

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The critical Ising model is *believed* to converge to $\sqrt{3}$-LQG.

Similar statements for other models of decorated maps
(with a spanning subtree ($\gamma = \sqrt{2}$), with a bipolar orientation ($\gamma = \sqrt{4/3}$),...).

For $\gamma \in (0, 2)$, there exists $d_\gamma =$ "fractal dimension of γ-LQG"

$d_\gamma =$ ball volume growth exponent for corresponding maps ??

YES, in some cases [Gwynne, Holden, Sun '17], [Ding, Gwynne '18]

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General bounds for d_γ [Ding, Gwynne '18], which give $4.18 \leq d_{\sqrt{3}} \leq 4.25$.

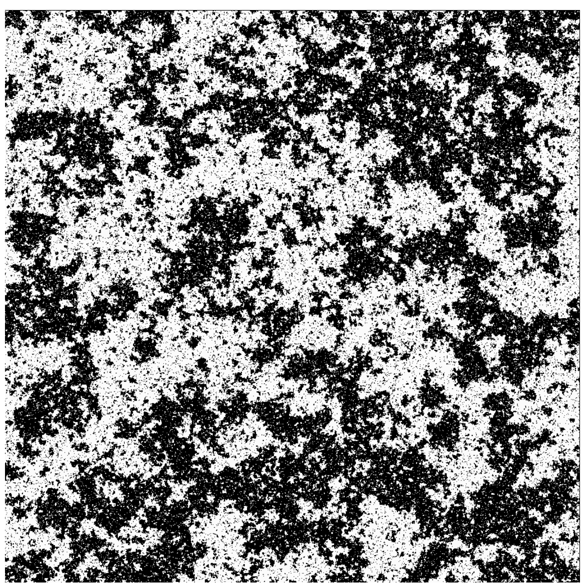
In particular $d_{\sqrt{3}} \neq 4$ and growth volume would then be different than the uniform models.

Perspectives and related works

- Compute the volume growth of the ν-IIPT
  or, at least, prove that it is different from 4 for $\nu = \nu_c$

- Study the connected components of the $+$ spins [A-Ménard, 22+]
  gives some insights about the Ising model on $\mathbb{Z}^2$ via the KPZ formula

$\nu < \nu_c$



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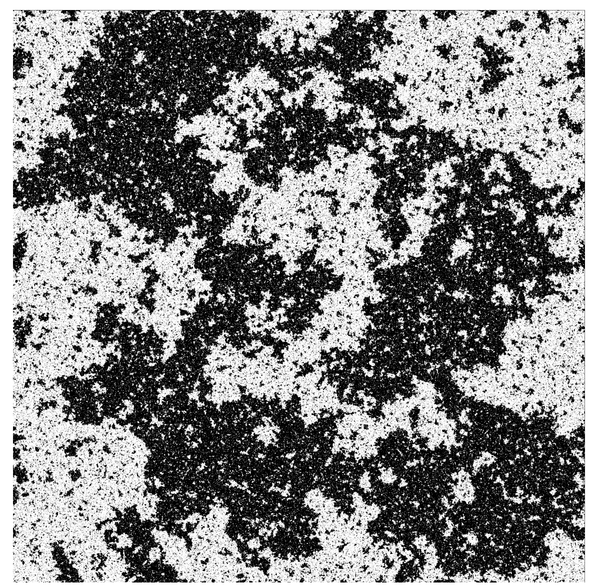

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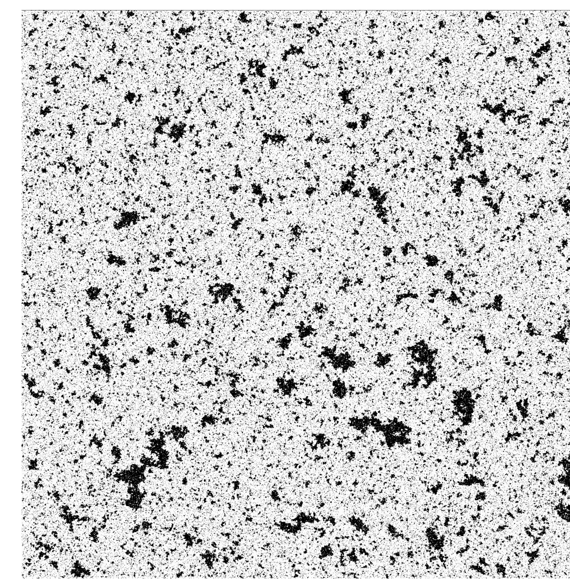

Simulations by R.Cerf.

- Investigate the different statistical physics models and their link with γ-LQG

Thank you !