Growth function for a class of monoids

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Formal Power Series and Algebraic Combinatorics

July, 24th 2009

First motivation = counting braids

braid diagram = a sequence of strand crossings.

$\sigma_{t,s} = \sigma_{s,t}$ $(s < t)$ = crossing of strands s and t, where strand s is above strand t

braid diagram = word on the alphabet $\{\sigma_{s,t}\}$

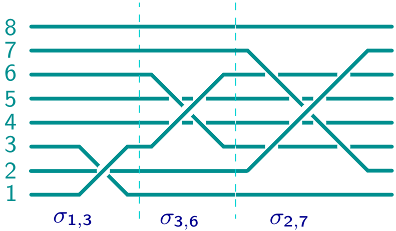
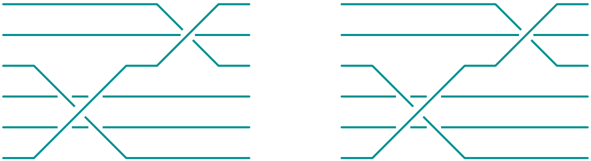


Figure: A braid diagram and the corresponding word

Equivalent diagrams



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Braid = equivalence class of diagrams.

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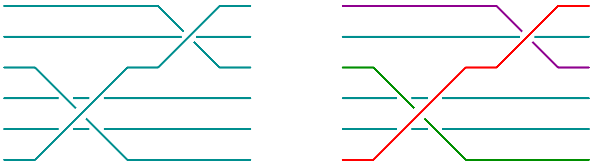

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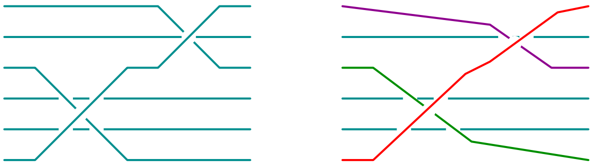

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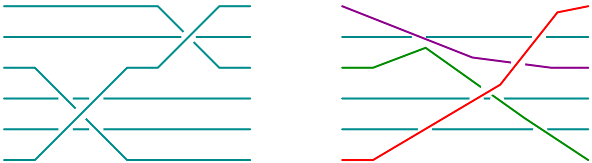

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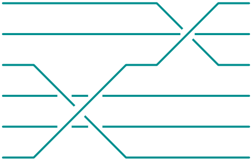
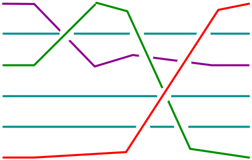

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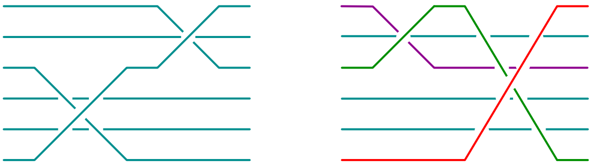

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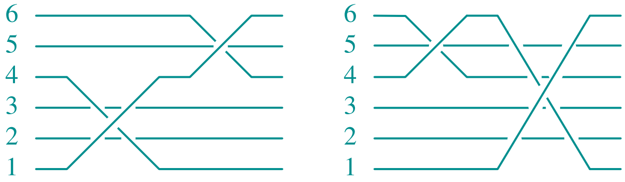

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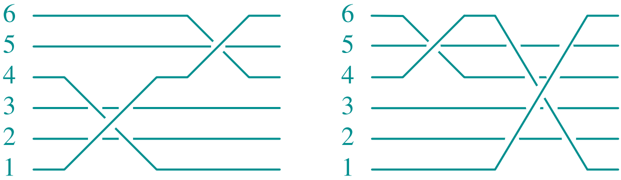

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Braid = equivalence class of diagrams.

Presentation of the dual braid monoid.

The set of generators of M is :

$$\mathcal{S} = \{\sigma_{s,t} = \sigma_{t,s} \text{ pour } 1 \leq s < t \leq n, \}$$

with the following equivalence relations :

$$\sigma_{s,t}\,\sigma_{u,v} = \sigma_{u,v}\,\sigma_{s,t} \text{ si } s <_s t <_s u <_s v,$$
$$\sigma_{s,t}\,\sigma_{t,u} = \sigma_{t,u}\,\sigma_{u,s} \text{ si } s <_s t <_s u.$$

where $<_s$ = cyclic order $\mathbb{Z}/n\mathbb{Z}$ defined by :

$$s <_s s+1 <_s s+2 <_s \ldots <_s s-1.$$

Length of a braid $= |m|_{\mathcal{S}}$

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Length of a braid $= |m|_{\mathcal{S}}$

How many braids ?

a_k = number of braids of length k

$$F_n(t) = \sum_{k \geq 0} a_k t^k = a_0 + a_1 t + a_2 t^2 \cdots$$

Theorem (A., Nadeau '08)

The growth function of the dual braid monoid on n strands is :

$$F_n(t) = \left[\sum_{k=0}^{n-1} \frac{(n-1+k)!(-t)^k}{(n-1-k)!k!(k+1)!} \right]^{-1}.$$

Steps of the proofs

Computation of the growth function of the monoid

Involution
ψ

Alternating generating series of lcm

A few definition about lcm

$\sigma \prec m \quad$ = there exists a diagram of m whose first letter is σ

Definition

$J \subset \mathcal{S}$ is a clique if it admits a common multiple.
The set of cliques is denoted $\mathcal{J}$

If $J \in \mathcal{J}$, then a least common multiple (lcm) exists, is unique and is denoted M_J.

We fix arbitrarily a linear ordering on $\mathcal{S}$, and denote a clique as

$$J = \sigma_1 \dots \sigma_n, \text{ with } \sigma_i < \sigma_{i+1}$$

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Theorem

$$\left(\sum_{J\in\mathcal{J}}(-1)^{|J|}M_J\right)\cdot\left(\sum_{m\in M}m\right)=1$$

Corollary (Bronfman '05, Kraamer '05)

The growth series of the monoid verifies then:

$$\left[\sum_{J\in\mathcal{J}}(-1)^{|J|}\,t^{|M_J|}\right]F(t)=1$$

A large class of monoids

Our approach works for every monoid M which admits a presentation with generators and relations and which is:

- atomic,
- left-cancellable : $a, u, v \in M$, $au = av \Rightarrow u = v$,
- if a subset of generators has a right common multiple then it has a least common multiple.

[Bronfman, 00], [Krammer, 04]

Trace monoids, Garside monoids, Artin-Tits monoids, . . .

To get the growth series from the involution, the relations must besides preserve the length.

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Proof of the inversion formula

$$\Big(\sum_{J\in\mathcal{J}}(-1)^{|J|}M_J\Big)\big(\sum_{\in M} m\big) = \sum_{(J,m)}(-1)^{|J|}M_J m = 1$$

Ψ is an involution with only $(1,1)$ as fixed point :

$$\Psi : \mathcal{J}\times M \rightarrow \mathcal{J}\times M$$
$$(J,m)\mapsto (J',m') \text{ with } M_J m = M_{J'}m' \text{ and } |J\Delta J'| = 1$$

$$\sigma_m = \max\{\sigma \text{ such that } \sigma \prec M_J m\}$$

$$\Psi(J,m) = \begin{cases}\big(J\cup\{\sigma_m\},\,(M_{J\cup\{\sigma_m\}})^{-1}\cdot m\big) & \text{if } \sigma_m\notin J\\ \big(J\backslash\{\sigma_m\},\,(M_{J\backslash\{\sigma_m\}})^{-1}M_J\cdot m\big) & \text{otherwise}\end{cases}$$

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Computation of the alternating generating series of lcm

$(M, \prec)$ = locally finite Poset

Möbius inversion formula : $(\sum \mu(m)m)(\sum m) = 1$

Our inversion formula is a generalization of Rota's cross-cut theorem.

Computation of the Möbius function :

- Use of NBB base with an appropriate order on $\mathcal{S}$ [Blass and Sagan, '96]
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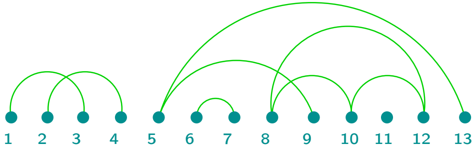
Common multiple of braids

Lcm of $\{\sigma_{1,3}, \sigma_{2,4}, \sigma_{5,13}, \sigma_{5,9}, \sigma_{6,7}, \sigma_{8,12}, \sigma_{8,10}, \sigma_{10,12}\}$?



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$|M_J|$ = number of vertices - number of parts = 13 - 4 = 9.

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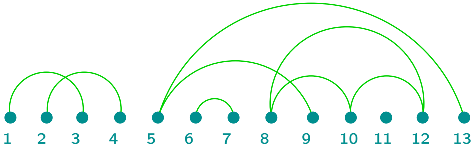

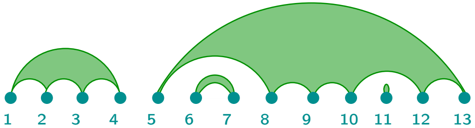

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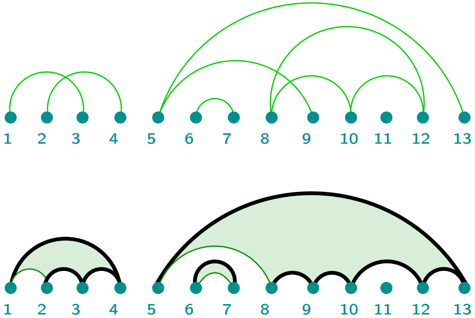

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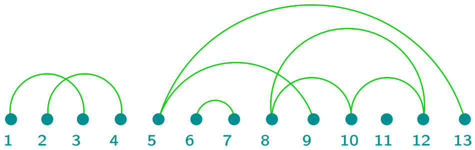
Involution on the edge configurations



$\Rightarrow$ Counting non-crossing alternating forests

Length of the lcm = number of edges of the forest

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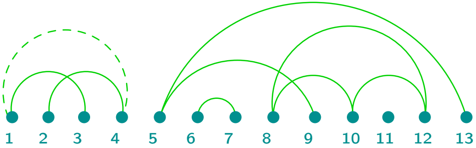

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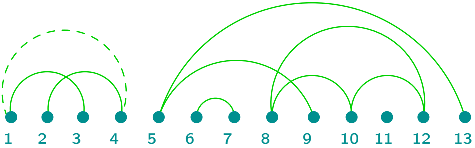

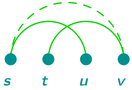

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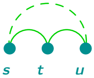

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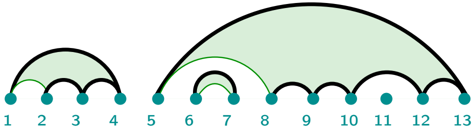

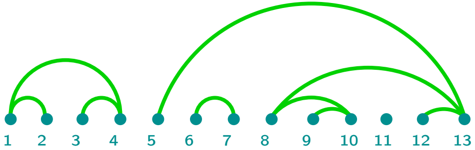

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Order compatible cliques

Definition

An order compatible (OC) clique is $\sigma_1 \dots \sigma_n$ such that :

$$\sigma_i = \max\{\sigma \prec M_{\sigma_1 \dots \sigma_i}\}$$

Theorem (Blass-Sagan, '96)

$$\mu(m) = \sum (-1)^{|J|}, \text{ where } J \text{ is an OC clique s.t. } M_J = m$$

$$\sigma_{i,j} < \sigma_{k,l} \iff [i,j] \subsetneq [k,l],$$

the OC-cliques are exactly the noncrossing alternating forests.

Steps of the proof

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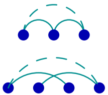
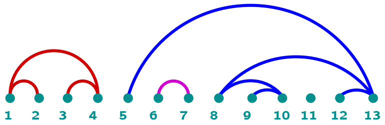


Noncrossing alternating forests

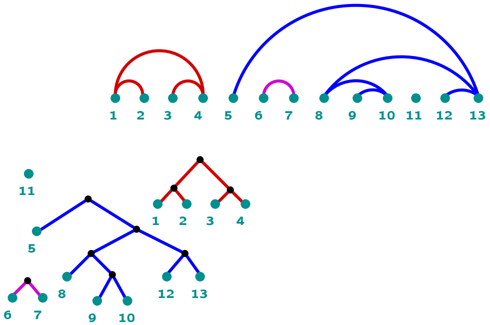
?

Noncrossing alternating forests and unary-binary trees



[Gelfand *et al.*, 97]
Bijection between the noncrossing alternating forests with n vertices and k edges and the unary binary trees with $n+k$ nodes, k of which being binary.

Noncrossing alternating forests and unary-binary trees



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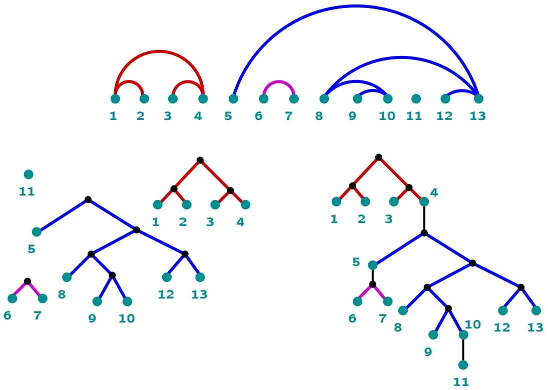

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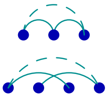

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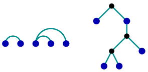

Unary-binary trees

Growth function of the dual braid monoid

Theorem (A., Nadeau '08)

The growth function of the dual braid monoid on n strands is :

$$F_n(t) = \sum \#\{\textit{braids of length } n\} t^n = \sum_{b \in B_n^{+\star}} t^{|b|_{\Sigma_n}}$$

$$F_n(t) = \left[\sum_{k=0}^{n-1} \frac{(n-1+k)!(-t)^k}{(n-1-k)!k!(k+1)!} \right]^{-1}.$$

Resolution of $\mathbb{Z}$

- $A := \mathbb{Z}M$: monoid algebra of M
- $B := \mathbb{Z}\mathcal{J}$: free module with basis $\mathcal{J}$
  $B_n := \mathbb{Z}\mathcal{J}_n$: submodule with basis $\mathcal{J}_n$ (cliques of size n)
- $C_n := B_n \otimes_{\mathbb{Z}} A$

Definition

$d_n : C_n \to C_{n-1}$ is a A-module homomorphism defined by:

$$d_n(\sigma_1 \ldots \sigma_n \otimes 1) = \sum_{i=1}^{n}(-1)^{n-i}\sigma_1 \ldots \hat{\sigma}_i \ldots \sigma_n \otimes \delta^{\sigma_i}_{\sigma_1 \ldots \hat{\sigma}_i \ldots \sigma_n},$$

where $M_{J_i}\delta^{\sigma_i}_{J_i} = M_{J_i \cup \{\sigma_i\}}$.

Theorem

$$0 \longrightarrow C_{|\mathcal{S}|} \xrightarrow{d_{|\mathcal{S}|}} C_{|\mathcal{S}|-1} \xrightarrow{d_{|\mathcal{S}|-1}} \cdots\cdots \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 = A \xrightarrow{\epsilon} \mathbb{Z}$$

is a resolution of $\mathbb{Z}$ as an A-module (i.e. $Im(d_n) = Ker(d_{n-1})$).

Koszul Algebras

- $\tilde{C}_n :=$ submodule of C_n with bases OC cliques of size n

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Artin-Tits monoids

$\mathcal{S}$ is a finite set, $\mathbb{M}$ a symmetric matrix, with $m_{s,t} \in \mathbb{N} \cup \{\infty\}$ and $m_{s,s} = 1$.

The Artin-Tits monoid associated to $\mathcal{S}$ and $\mathbb{M}$ is:

$$M = \langle s \in \mathcal{S} \mid \underbrace{sts\ldots}_{m_{s,t} \text{ terms}} = \underbrace{tst\ldots}_{m_{s,t} \text{ terms}} \text{ if } m_{s,t} \neq \infty \rangle$$

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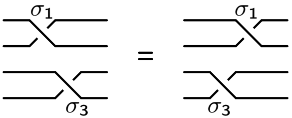
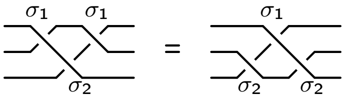
Braid monoids

From the classification of finite Coxeter groups, the classical braid monoids of type A, B and D are defined.

$$\mathcal{A}(A_N) = \left\langle \sigma_1, \ldots, \sigma_n \;\middle|\; \begin{array}{c} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \\ \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| \geq 2 \end{array} \right\rangle.$$





$$F_n(t) := \left[H_n(t)\right]^{-1}, \qquad H_n(t) = \sum_{k=1}^{n} (-1)^{k+1} t^{k(k-1)/2} H_{n-k}$$

Dual braid monoids:

W a Coxeter group :

T = New set of generators = { reflexions } = $\{wsw^{-1}, s \in \mathcal{S}, w \in W\}$

Definition of a dual structure [Birman,Ko,Lee, '98],[Bessis, '03], where the set of lcms is a lattice.

Lattice isomorphic to some lattice of non-crossing partitions.

- Type A
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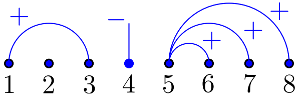
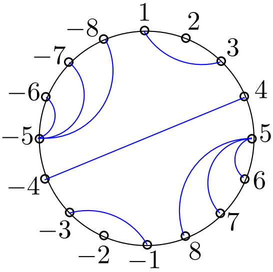
Dual braids of type B

Noncrossing partition of type B :

- Partition of the set $\{1, \ldots, n, -1, \ldots, -n\}$
- i, j in the same block $\Rightarrow -i, -j$ also.





Theorem

The monoid algebra of the dual braid monoid of type B is a Koszul algebra.

$$F_n^B(t) = \left(\sum_{k=0}^{n}(-1)^k \binom{n}{k}\binom{n+k-1}{k} t^k\right)^{-1}$$

Thank you !