Asymptotic behaviour of large random stack-triangulations

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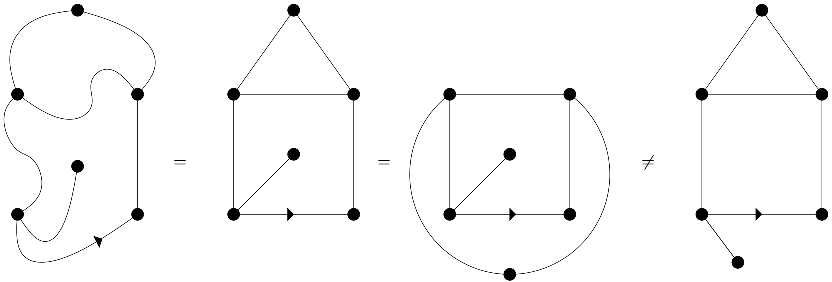
Outline

Definition of planar maps

- Planar map = planar connected graph embedded properly in the sphere up to a direct homomorphism of the sphere
- Rooted planar map = an oriented edge (e_0, e_1) is marked, e_0 = root vertex.



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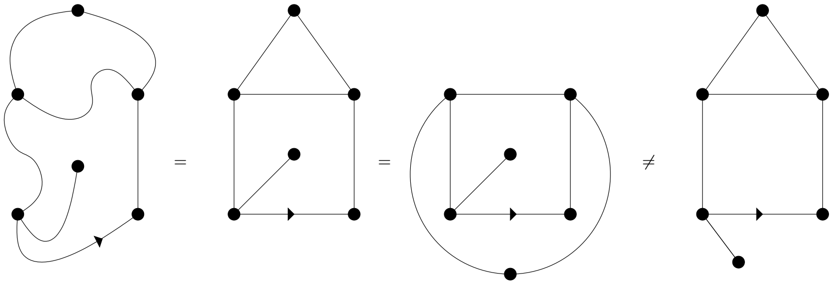

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Maps and faces

Faces = connected components of the sphere without the edges or the map.
Triangulation = map whose faces are all of degree 3.
Quadrangulation = map whose faces are all of degree 4.

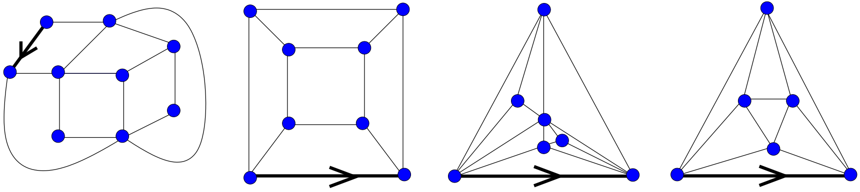
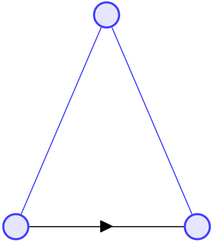


Figure: Two quadrangulations and two triangulations

Random Apollonian networks – Stack-triangulations

Stack-triangulations = triangulations obtained recursively:



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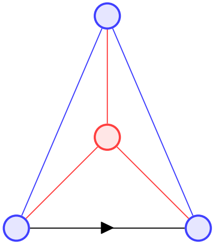

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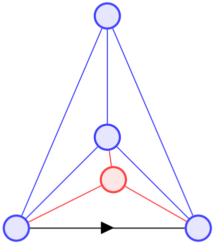

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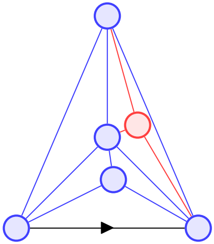

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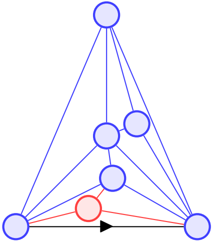

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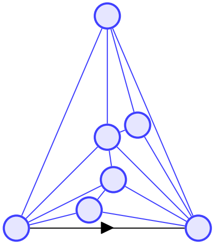

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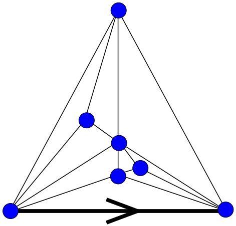
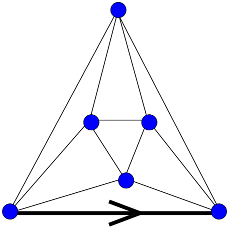
Stack-triangulations vs Triangulations

$$\{\text{Stack-triangulations}\} \subsetneq \{\text{Triangulations}\}$$





Convergence of large random planar maps

- Large ? Number of vertices grows to infinity.
- Random ? Which law ?
- Convergence ? Which notion of convergence ?

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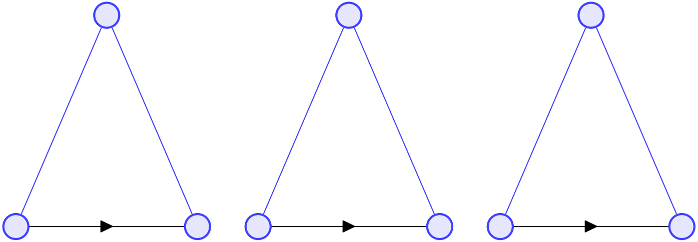
Two probability distributions

$\triangle_{2k}$ = set of stack-triangulations with $2k$ faces.
Two natural probability distributions on $\triangle_{2k}$:

- the uniform law, denoted $\mathbb{U}^{\triangle}_{2k}$,



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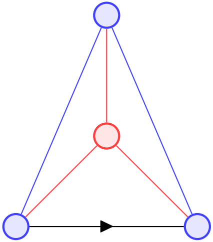 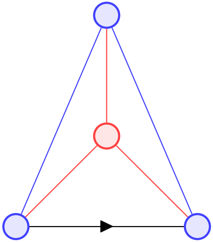 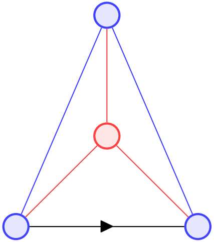

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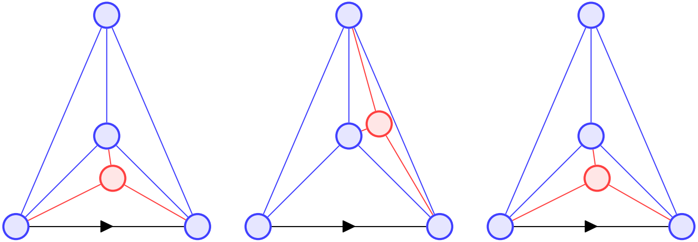

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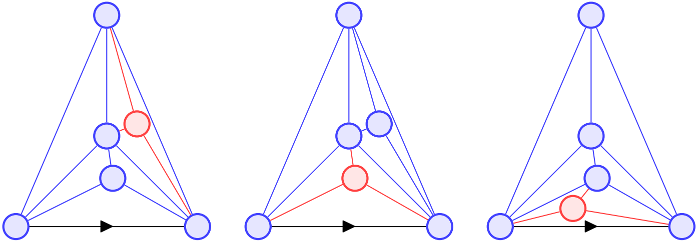

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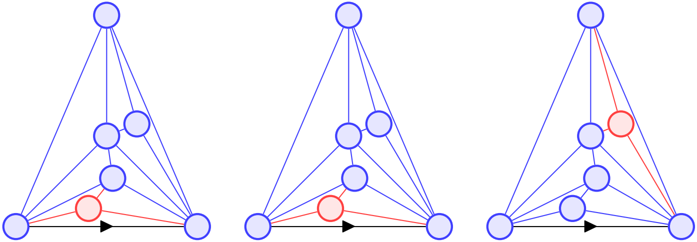

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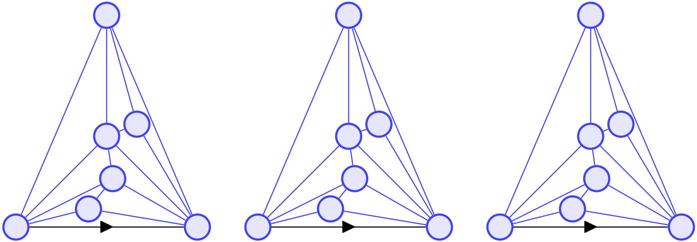

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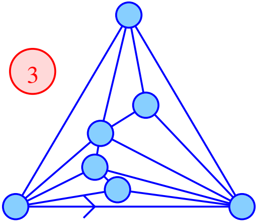

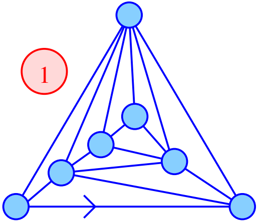

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Results on random stack-triangulations

According to $\mathbb{Q}_{2k}^{\triangle}$,

- Degree of a vertex and expected value of the distance between two vertices
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According to $\mathbb{U}_{2k}^{\triangle}$,

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Stack Triangulations		Quadrangulations uniform law
Uniform law	Historical law	

Which definition of convergence ?

Two notions of convergence : local convergence

$B_m(r)$ = ball of radius r centered at the root of m.

Definition

Let m and m' be two planar maps, the local distance between them is:

$$d_L(m, m') = \inf\{\frac{1}{1+r} \text{ where } B_m(r) \sim B_{m'}(r)\},$$

Local convergence = Convergence of the balls centered at the root.

	Stack-triangulations		Quadrangulations uniform law
	uniform law	historical law	
Local convergence			Angel and Schramm, 03 Chassaing and Durhuss, 06

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The Theorem

Theorem (A.,Marckert '08)

Under the uniform law on $\triangle_{2n}$,

$$\left(m_n, \frac{D_{m_n}}{(2/11)\sqrt{3n/2}}\right) \xrightarrow[n]{(d)} (\mathcal{T}_{2e}, d_{2e}),$$

for the Gromov-Hausdorff topology on the set of compact metric spaces.

- Gromov-Hausdorff ?
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Gromov-Hausdorff distance

Hausdorff distance between X and Y two compact sets of (E, d) :

$$d_H(X, Y) = \max\{\sup_{x\in X} \inf_{y\in Y} d(x, y), \sup_{y\in Y} \inf_{x\in X} d(x, y)\}$$

Gromov-Hausdorff distance between two compact metric spaces E and F:

$$d_{GH}(E, F) = \inf\ d_H(\phi(E), \psi(F))$$

Infimum taken on :

- all the metric spaces M
- all the isometric embeddings $\phi : E \to M$ et $\psi : F \to M$.

$\{$isometric classes of compact metric spaces$\}$
$=$ complete and separable ($=$ "polish") space.

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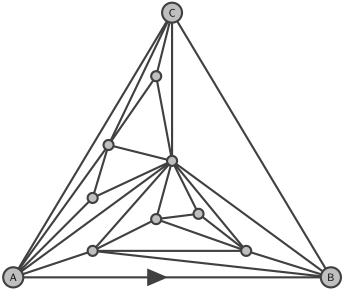
Triangulations and ternary trees



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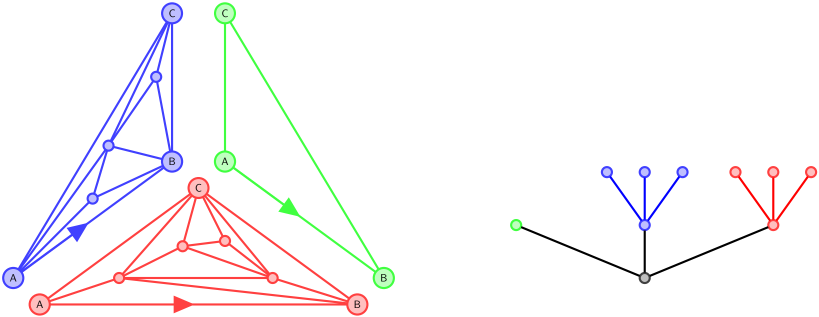

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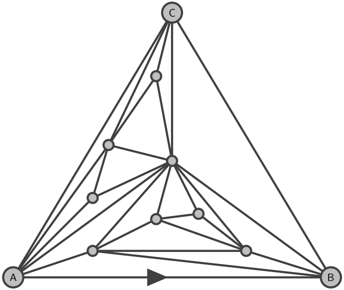
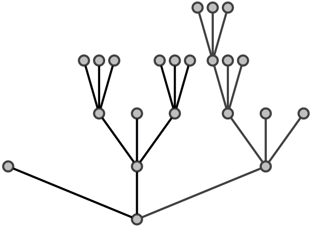
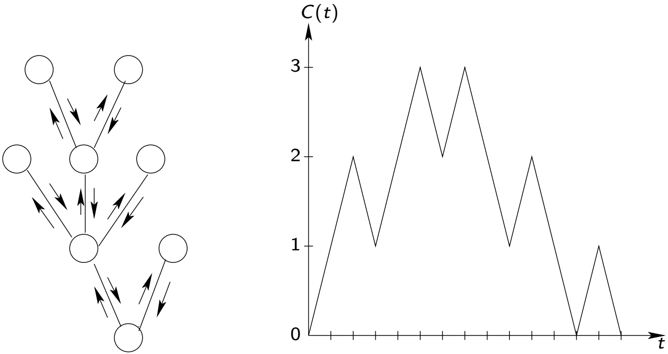


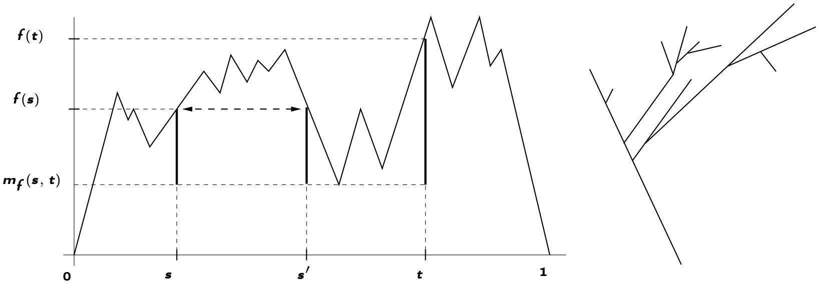


Harris walk of a tree



Continuum Tree

f = function from $[0,1]$ onto $\mathbb{R}^+$ such that $f(0) = f(1) = 0$.



- $s \sim s'$ if and only if $f(s) = f(s') = m_f(s, s')$
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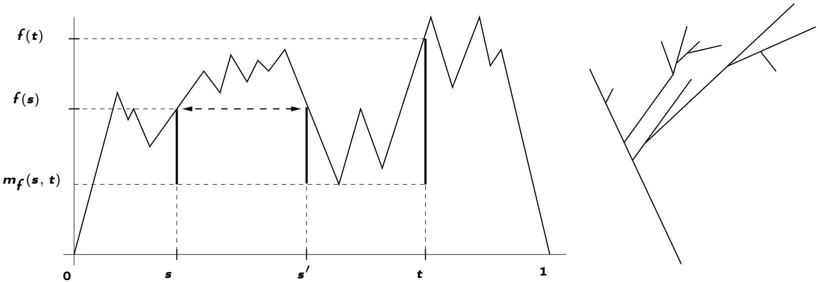

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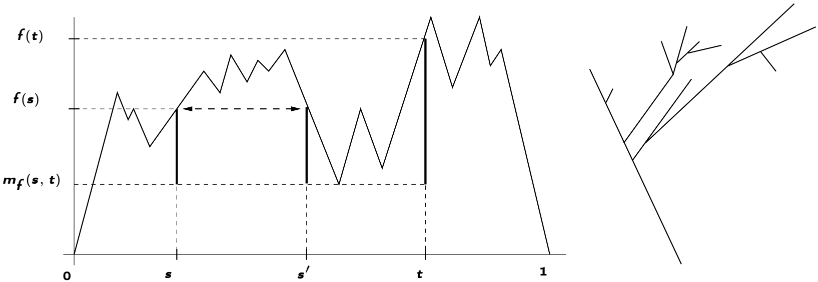

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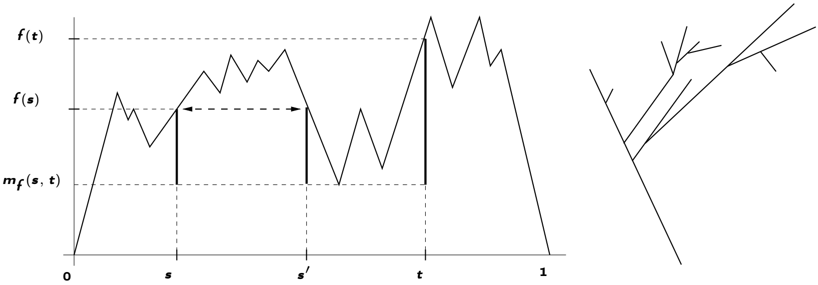

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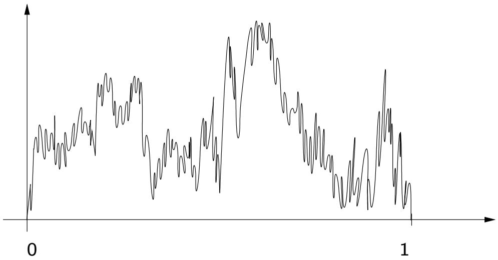
Continuum Random Tree – CRT

A normalized brownian excursion $\mathbf{e} = (\mathbf{e}_t)_{t\in[0,1]}$ is a brownian motion conditioned to satisfy $\mathcal{B}_0 = 0$, $\mathcal{B}_1 = 0$ and $\mathcal{B}(t) > 0$ for every $t \in]0,1[$.



CRT = Tree obtained from a normalized brownian excursion.
It is denoted $(\mathcal{T}_{2\mathbf{e}}, d_{2\mathbf{e}})$.

Convergence towards the CRT

Uniform law on stack-triangulations with $2n$ faces
$\Rightarrow$ uniform law $\mathbb{U}^{\text{ter}}_{3n-2}$ on the set of ternary trees with $3n-2$ nodes.

Proposition (Aldous)

Under $\mathbb{U}^{\text{ter}}_{3n+1}$, for the Gromov-Hausdorff topologogy :

$$\left(T, \frac{d_T}{\sqrt{3n/2}}\right) \xrightarrow[n]{(d)} (\mathcal{T}_{2e}, d_{2e}).$$

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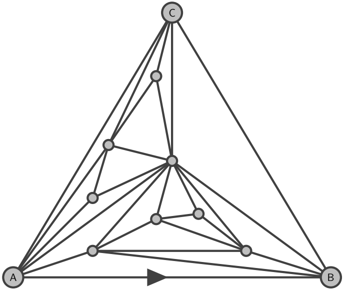

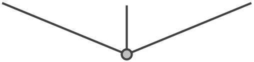

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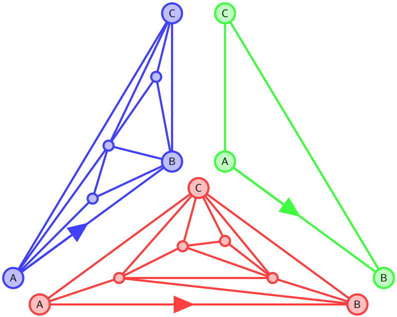

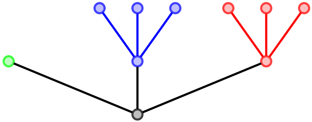

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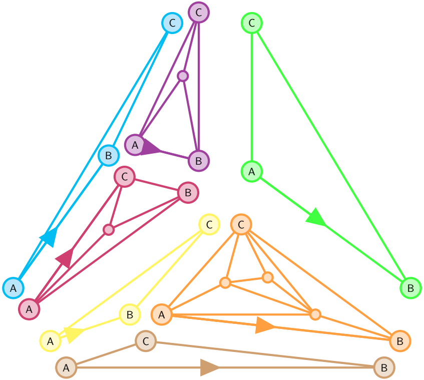

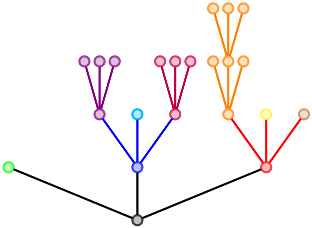

Bijection between trees and maps

Proposition

For any $K \geq 1$, there exists a bijection

$$\begin{array}{cccc} \Psi_K^{\triangle}: & \triangle_{2K} & \longrightarrow & \mathcal{T}_{3K-2}^{\text{ter}} \\ & m & \longmapsto & t := \Psi_K^{\triangle}(m) \end{array}$$

such that:
(i) (a) Every internal node u of m corresponds bijectively to an internal node v of t. u' denotes the image of u.
(b) Each leaf of t corresponds bijectively to a finite face of m.
(ii) For any internal node u of m, $|\Gamma(u') - d_m(root, u)| \leq 1$.
(ii′) For any pair on internal nodes u and v of m

$$|d_m(u,v) - \Gamma(u', v')| \leq 3.$$

Who is Γ ?

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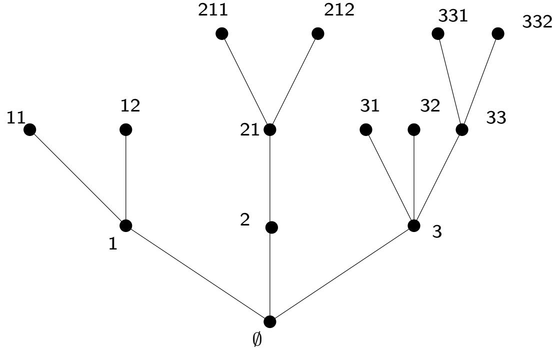
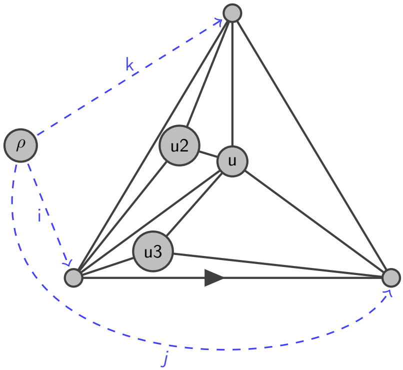
Who is Γ ?

Neveu formalism

- A ternary tree = set of words on the alphabet $\{1, 2, 3\}$.
- Vertex of the tree = a word



Type of faces and nodes



If $\text{type}(u) = (i, j, k)$,

$$\begin{cases} \text{type}(u1) = (& 1 + i \wedge j \wedge k, & j, & k &), \\ \text{type}(u2) = (& i, & 1 + i \wedge j \wedge k, & k &), \\ \text{type}(u3) = (& i, & j, & 1 + i \wedge j \wedge k &) \end{cases}$$

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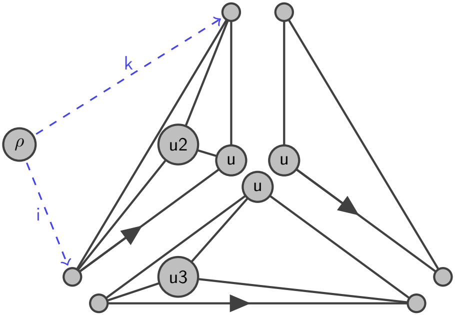

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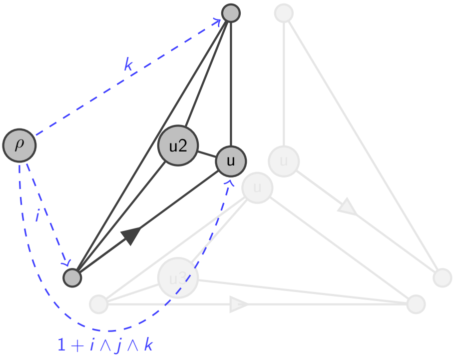

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A langage for distances

$\mathcal{L}_{1,2,3} = \{$ words of $\{1,2,3\}^\star$ with at least one occurence of 1, 2 and 3$\}$

Let $u \in \{1,2,3\}^\star$,

$$\Gamma(u) = \max\{k \text{ such that } u = u_1 \dots u_k,\ u_i \in \mathcal{L}_{1,2,3} \text{ for } i \in \{1,2,3\}\}$$

$$u = 12213213221223 2 \quad \Rightarrow \quad \Gamma(u) = 3.$$

Let $u = w \cdot u_1 \dots u_k$ et $v = w \cdot v_1 \dots v_l$ with $u_1 \neq v_1$, we denote :

$$\Gamma(u,v) = \Gamma(u_1 \dots u_k) + \Gamma(v_1 \dots v_l)$$

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Convergence of stack-triangulations

Lemma

Let $(X_i)_{i\geq 1}$ be a sequence of independant random variables uniformly distributed on $\{1,2,3\}$. Let W_n be the word $X_1 \ldots X_n$ then

$$\frac{\Gamma(W_n)}{n} \xrightarrow[n]{(a.s.)} \Gamma_\triangle, \text{ where } \Gamma_\triangle = 2/11$$

Distance in the map and in the tree:

$$|d_{m_n}(u,v) - \Gamma(u',v')| \leq 3$$

We show :

$$P\left(\sup\left|d_{m_n}(u,v) - \frac{2}{11} d_{T_n}(u',v')\right| \geq n^{1/3}\right) \xrightarrow[n\to\infty]{} 0$$

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Convergence of scaled stack-triangulations

Theorem

Under the uniform law on $\triangle_{2n}$,

$$\left(m_n, \frac{D_{m_n}}{\Gamma_\triangle \sqrt{3n/2}}\right) \xrightarrow[n]{(d)} (\mathcal{T}_{2e}, d_{2e}),$$

for Gromov-Hausdorff topology on the set of compact metric spaces.

	Stack-triangulations		Quadrangulations uniform law
	uniform law	historical law	
Local convergence			Angel-Schramm. 03 Chassaing-Durhuss, 06
Scaled convergence	cvg in law for Gromov-Hausdorff topology towards CRT normalization = $\sqrt{n}$		Chassaing-Schaeffer, 04 Marckert-Mokkadem, 06 Le Gall, 07 Le Gall-Paulin, 08

Convergence of stack-triangulations according to $\mathbb{Q}^{\triangle}$

Theorem (A.,Marckert '08)

Let M_n a stack-triangulation according to $\mathbb{Q}^{\triangle}_{2n}$. Let $k \in \mathbb{N}$ et $\mathbf{v}_1, \ldots, \mathbf{v}_k$, k nodes M_n chosen independently and uniformly amongst the internal nodes of M_n, then:

$$\left(\frac{D_{M_n}(\mathbf{v}_i, \mathbf{v}_j)}{3\Gamma_{\triangle} \log n}\right)_{(i,j)\in\{1,\ldots,k\}^2} \xrightarrow[n]{proba.} (1_{i\neq j})_{(i,j)\in\{1,\ldots,k\}^2}.$$

Study of the trees under the historical law = study of increasing trees ... [Broutin, Devroye, McLeish, de la Salle 08]

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Local convergence of stack-triangulations : Uniform law

Under $\mathbb{U}_{2n}^{\triangle}$:

Theorem (A.,Marckert '08)

The sequence $(\mathbb{U}_{2n}^{\triangle})$ weakly converges towards $P_{\infty}^{\triangle}$, for the topology of local convergence,
where the support of $P_{\infty}^{\triangle}$ is a set of infinite stack-triangulations.

Ingredients :

- Local convergence of Galton-Watson trees towards a tree with a unique infinite spine.
- Definition of an infinite planar map similar to the UIPT of Angel and Schramm.

Local convergence of stack-triangulations : Historical law

Degree of the root = number of white balls in an urn

- Initially : 2 white balls and 1 black ball
- matrix replacement : $\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$

[Flajolet, Dumas, Puyhaubert, 06]

$\Rightarrow$ The degree of the root grows to infinity.

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	Stack-triangulations		Quadrangulations uniform law
	uniform law	historical law	
Local convergence	cvg in law to a law supported by infinite triangulations	No convergence	Angel-Schramm. 03 Chassaing-Durhuss, 06
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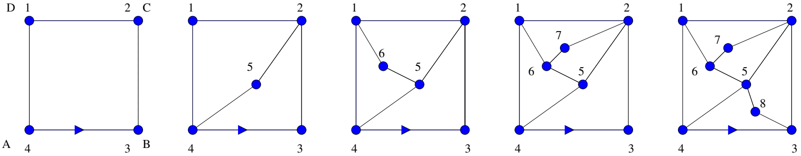
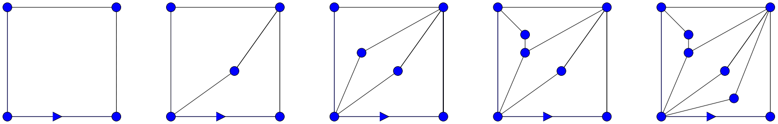
Stack-quadrangulations

We managed to deal with a special case of stack-quadrangulations



but more general models resist. . .



Brownian Map

Convergence of scaled quadrangulations under the uniform law ?

[Chassaing et Schaeffer, 04], [Marckert et Mokkadem, 06], [Marckert et Miermont, 07], [Le Gall, 07], [Le Gall et Paulin, 08]

- Universality principle ? Convergence of all the "reasonable" models to the same limit ?
- Which limit ? Brownian map...

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Thank you !