Lambda calculs et catégories

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Summary of the lecture

1 – Adjunctions

2 – Monads
Adjunctions

A notion of duality between functors
An \textbf{adjunction} is a triple \((L, R, \phi)\) where \(L\) and \(R\) are two functors
\[
L : \mathcal{A} \to \mathcal{B} \quad \quad \quad R : \mathcal{B} \to \mathcal{A}
\]
and \(\phi\) is a family of bijections, for all objects \(A\) in \(\mathcal{A}\) and \(B\) in \(\mathcal{B}\),
\[
\phi_{A,B} : \mathcal{B}(LA, B) \cong \mathcal{A}(A, RB)
\]
natural in \(A\) et \(B\). One also writes
\[
\begin{array}{ccc}
LA & \longrightarrow & B \\
\phi_{A,B} & \longrightarrow & \mathcal{B}(LA, B) \\
A & \longrightarrow & \mathcal{A}(A, RB) \\
\end{array}
\]
One says that \(L\) is \textbf{left adjoint to} \(R\), noted \(L \dashv R\).

The 2-dimensional version of isomorphism
The naturality of the bijection $\phi$

Natural in $A$ and $B$ means that the family of bijections

$$\phi_{A,B} : \mathcal{B}(LA, B) \cong \mathcal{A}(A, RB)$$

transforms every commutative diagram

\[ \begin{array}{ccc}
LA & \overset{g}{\longrightarrow} & B \\
\downarrow{Lh_A} & & \downarrow{h_B} \\
LA' & \underset{f}{\longrightarrow} & B'
\end{array} \]

into a commutative diagram

\[ \begin{array}{ccc}
A & \overset{\phi_{A,B}(g)}{\longrightarrow} & RB \\
\downarrow{h_A} & & \downarrow{Rh_B} \\
A' & \underset{\phi_{A',B'}(f)}{\longrightarrow} & RB'
\end{array} \]
Example: the free vector space

\[ \text{Set} \xrightarrow{L} \text{Vect} \xleftarrow{R} \]

where

- \( \mathcal{A} = \text{Set} \): the category of sets and functions
- \( \mathcal{B} = \text{Vect} \): the category of vector spaces on a field \( k \)
- \( R \): the « forgetful » functor \( V \mapsto U(V) \)
- \( L \): the « free vector space » functor \( X \mapsto kX \)

\[ kX := \left\{ \sum_{x \in X} \lambda_x x \mid \lambda_x \in k \text{ null almost everywhere.} \right\} \]
Illustration: the tensor algebra

\[ \text{Vec} \perp \text{Alg} \]

where

\[ A = \text{Vec} \quad : \quad \text{the category of vector spaces} \]
\[ B = \text{Alg} \quad : \quad \text{the category of algebras and homomorphisms}, \]

\[ R \quad : \quad \text{the « forgetful » functor } A \mapsto U(A) . \]
\[ L \quad : \quad \text{the « free algebra » functor } V \mapsto TV . \]

\[ TV := \bigoplus_{n \in \mathbb{N}} V^\otimes n \]
Definition of a Lie algebra

Vector space $g$ equipped with a Lie bracket

Anti-symmetry:

$$[x, y] = -[y, x]$$

Jacobi identity:

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

Example: the vector space of vector fields on a smooth manifold.
Illustration: the enveloping algebra of a Lie algebra

\[ \begin{array}{ccc}
\text{Lie} & \perp & \text{Alg} \\
\downarrow & L & \downarrow \\
\end{array} \]

where

- \( \mathcal{A} = \text{Lie} \): the category of Lie algebras,
- \( \mathcal{B} = \text{Alg} \): the category of algebras,
- \( R \): equips \( \mathcal{A} \) with the canonical Lie bracket \([a, b] = ab - ba\),
- \( L \): « enveloping algebra » functor \( g \mapsto U(g) \).

\[ U(g) := Tg / I(g) \]

where \( I(g) \) is the ideal generated by \( ab - ba - [a, b] \).
Illustration: the free category

where

\[ A = \text{Graph} : \text{the category of graphs}, \]
\[ B = \text{Cat} : \text{the category of categories and functors}, \]
\[ R : \text{the « forgetful » functor} \]
\[ L : \text{the « free category » functor} \]
Illustration: the terminal object

where

\[ A = C \quad : \quad \text{any category equipped with a terminal object} \quad 1 \]
\[ B = 1 \quad : \quad \text{the singleton category} \]
\[ L \quad : \quad \text{the canonical (and unique) functor} \]
\[ R \quad : \quad \text{the functor whose image is the terminal object} \quad 1 \]
Illustration: cartesian categories

\[
\begin{array}{c}
\text{C} \\
\downarrow \quad \perp \\
\text{C} \times \text{C}
\end{array}
\]

where

\[ A = \text{C} \quad : \quad \text{any cartesian category} \]
\[ B = \text{C} \times \text{C} \quad : \quad \text{the product category} \]

\[ L \quad : \quad \text{the diagonal functor } A \mapsto (A, A) \]
\[ R \quad : \quad \text{the functor } (A, B) \mapsto A \times B \]
Illustration: cartesian closed categories

where

\[ A = B = C \] : any cartesian closed category \( C \)

\[ L \] : the functor \( B \mapsto A \times B \)

\[ R \] : the functor \( B \mapsto A \Rightarrow B \)

for a given object \( A \) of the cartesian closed category \( C \).
Illustration: negation

where

\[ A = C \quad : \quad \text{any cartesian closed category } C \]
\[ B = C^{\text{op}} \quad : \quad \text{the opposite category } C^{\text{op}} \]
\[ L \quad : \quad \text{the negation functor } A \mapsto A \Rightarrow \bot \]
\[ R \quad : \quad \text{the negation functor } A \mapsto A \Rightarrow \bot \]

for a given object \( \bot \) of the cartesian closed category \( C \).
Adjunction in the 2-category $\text{Cat}$

A bijection $\phi$ between the natural transformations

Here, a morphism $X \to Y$ in the category $\mathcal{C}$ is seen as a natural transformation $[X] \to [Y]$. 
Adjunction in the 2-category $\mathbf{Cat}$

A bijection $\phi$ between the natural transformations

Here, a morphism $X \to Y$ in the category $\mathcal{C}$ seen as a natural transformation $[X] \to [Y]$. 
A 2-dimensional naturality condition

One reformulates the naturality condition in that way:

The bijection $\phi$ is natural with respect to the natural transformations $\alpha$ and $\beta$. 
**Adjunction in the 2-category $\text{Cat}$**

This point of view leads to a more satisfactory definition of adjunction:

A bijection $\phi$ between the natural transformations

\[
A \xrightarrow{\phi_{A,B}} B 
\]
Adjunction in the 2-category $\text{Cat}$

One reformulates the naturality condition as follows:

The bijection $\phi$ is natural with respect to the natural transformations $\alpha$ et $\beta$. 
Algebraic presentation of the adjunction

An **adjunction** is a quadruple \((L, R, \eta, \varepsilon)\) where \(L\) and \(R\) are functors

\[
L : \mathcal{A} \rightarrow \mathcal{B} \quad \quad \quad R : \mathcal{B} \rightarrow \mathcal{A}
\]

and \(\eta\) and \(\varepsilon\) are natural transformations:

\[
\eta : \text{Id}_\mathcal{A} \Rightarrow RL \quad \quad \quad \varepsilon : LR \Rightarrow \text{Id}_\mathcal{B}
\]

such that the composite

\[
\begin{align*}
R \xrightarrow{\eta_R} RLR \xrightarrow{R\varepsilon} R \\
L \xrightarrow{L\eta} LRL \xrightarrow{\varepsilon F} L
\end{align*}
\]

depicted as

\[
\begin{array}{c}
\begin{tikzpicture}
  \node (a) at (0,0) {$\mathcal{A}$};
  \node (b) at (0,-3) {$\mathcal{B}$};
  \node (c) at (3,0) {$\mathcal{A}$};
  \node (d) at (3,-3) {$\mathcal{B}$};
  \draw[->] (a) to node[auto] {$\text{Id}_\mathcal{A}$} (c);
  \draw[->] (b) to node[auto] {$\text{Id}_\mathcal{B}$} (d);
  \draw[->] (a) to node[auto,swap] {$R$} (b);
  \draw[->] (a) to node[auto] {$L$} (c);
  \draw[->] (b) to node[auto] {$\varepsilon$} (d);
  \draw[->] (c) to node[auto] {$\varepsilon$} (d);
\end{tikzpicture}
\end{array}
\]

\[
\begin{array}{c}
\begin{tikzpicture}
  \node (a) at (0,0) {$\mathcal{A}$};
  \node (b) at (0,-3) {$\mathcal{B}$};
  \node (c) at (3,0) {$\mathcal{A}$};
  \node (d) at (3,-3) {$\mathcal{B}$};
  \draw[->] (a) to node[auto] {$\text{Id}_\mathcal{A}$} (c);
  \draw[->] (b) to node[auto] {$\text{Id}_\mathcal{B}$} (d);
  \draw[->] (a) to node[auto,swap] {$R$} (b);
  \draw[->] (a) to node[auto] {$L$} (c);
  \draw[->] (b) to node[auto] {$\varepsilon$} (d);
  \draw[->] (c) to node[auto] {$\varepsilon$} (d);
\end{tikzpicture}
\end{array}
\]

are the natural identities

\[
\text{Id}_R : R \Rightarrow R \quad \quad \quad \text{Id}_L : L \Rightarrow L
\]

of the functors \(R\) and \(L\).
Dual definition (but equivalent) of adjunction

By duality, an adjunction is given by a family of bijections $\psi$ between the sets of 2-cells

$$
\begin{array}{c}
A \\ L \downarrow \theta \downarrow C \\
B \\
\end{array}
\quad \psi_{A,B} 
\quad
\begin{array}{c}
A \\ R \downarrow \zeta \downarrow C \\
B \\
\end{array}
$$

natural in $A$ and $B$. 

The 2-dimensional topology of adjunctions

The **unit** and **counit** of the adjunction $L \dashv R$ are depicted as

\[
\eta : \text{Id} \Rightarrow R \circ L
\]

\[
\varepsilon : L \circ R \Rightarrow \text{Id}
\]
A typical 2-cell generated by an adjunction
A purely diagrammatic composition
The 2-dimensional dynamics of adjunctions

\[
\begin{align*}
\eta &\quad \varepsilon \\
\mathcal{A} &\quad \mathcal{B}
\end{align*}
\]

\[
\begin{align*}
\eta &\quad \varepsilon \\
\mathcal{A} &\quad \mathcal{B}
\end{align*}
\]

\[
\begin{align*}
\eta &\quad \varepsilon \\
\mathcal{B} &\quad \mathcal{A}
\end{align*}
\]

\[
\begin{align*}
\eta &\quad \varepsilon \\
\mathcal{B} &\quad \mathcal{A}
\end{align*}
\]
String diagrams

The $\lambda$-term

$$\varphi : \neg \neg A, \psi : \neg \neg B \vdash \lambda k. \varphi (\lambda a. \psi (\lambda b. k(a,b))) : \neg \neg (A \otimes B)$$

has the following control flow diagram
Illustration: the 2-category of sets and relations

Show that a relation

$$f : A \longrightarrow B$$

is left adjoint if and only if it is functional:

$$\forall a \in A. \ \exists! b \in B. \ a[f]b$$

Show that its right adjoint $g$ is the relation defined as

$$\forall a \in A. \ \forall b \in B. \ a[f]b \iff b[g]a.$$
Monads

Kleisli category, Eilenberg-Moore category
Monads

Suppose given a 0-cell $C$ in a 2-category $W$.

A monad $T$ on a 0-cell $C$ is a 1-cell

$$T : C \rightarrow C$$

equipped with a multiplication

$$\mu : T \circ T \Rightarrow T : C \rightarrow C$$

and with a unit

$$\eta : Id_C \Rightarrow T : C \rightarrow C$$

satisfying the expected associativity and unit laws.
Monads

- **Associativity law:**

\[
\begin{array}{c}
T \circ T \circ T \\
\mu \circ T \\
\end{array}
\xrightarrow{T \circ \mu}
\begin{array}{c}
T \circ T \\
\mu \\
\end{array}
\]

- **Left and right unit laws:**

\[
\begin{array}{c}
T \circ T \\
\eta \circ T \\
T \\
\end{array}
\xrightarrow{id}
\begin{array}{c}
T \\
\mu \\
T \\
\end{array}
\]

\[
\begin{array}{c}
T \circ T \\
T \circ \eta \\
T \\
\end{array}
\xrightarrow{id}
\begin{array}{c}
T \\
\mu \\
T \\
\end{array}
\]
Every adjunction defines a monad

(with a graphical proof)
Illustration: the state monad

Every set $S$ induces a monad

$$X \mapsto S \Rightarrow (S \times X) : \text{Set} \rightarrow \text{Set}$$

called the state monad. This monad is induced by the adjunction

\[
\begin{array}{cccccc}
\text{Set} & & \perp & & \text{Set} \\
\downarrow & & & & \uparrow \\
\text{Set} & & \downarrow & & \text{Set} \\
& & L & & R
\end{array}
\]

where

$$L : X \mapsto S \times X$$
$$R : X \mapsto S \Rightarrow X.$$
Suppose given a monad \( T \) on a category \( \mathcal{C} \).

An algebra of the monad \((T, \mu, \eta)\) is a pair \((A, h)\) consisting of

- an object \( A \) of the category \( \mathcal{C} \)
- a morphism
  \[
  h : TA \to A
  \]

making the diagrams commute.
Algebra homomorphism

An algebra homomorphism

$$f : (A, h_A) \longrightarrow (B, h_B)$$

is a morphism

$$f : A \longrightarrow B$$

making the diagram

$$
\begin{array}{ccc}
TA & \xrightarrow{Tf} & TB \\
h_A & & h_B \\
\downarrow & & \downarrow \\
A & \xrightarrow{f} & B
\end{array}
$$

commute in the category $\mathcal{C}$. 
Kleisli category

The Kleisli category $C_T$ of a monad $(T, \mu, \eta)$ is the category $C$

- with the same objects as the category $C$,

- with the morphisms

$$A \rightarrow TB$$

in the category $C$ as morphisms

$$A \rightarrow B$$

in the Kleisli category.
Kleisli category

The identities

\[ \text{id}_A : A \to A \]

are given by the morphisms

\[ \eta_A : A \to TA. \]

The two morphisms

\[ f : A \to B \quad g : B \to C \]

are composed as follows

\[ g \circ_K f := \]

\[ g \circ_K f := \]

\[ \begin{array}{ccc}
A & \xrightarrow{f} & TB \\
& & \\
B & \xrightarrow{g} & TC \\
\end{array} \]

\[ \begin{array}{ccc}
TTC & \xrightarrow{Tg} & TC \\
& \xrightarrow{\mu_C} & \\
\end{array} \]
Exercise

Show that:

- that the identities of the Kleisli category are identities
- that its composition is associative.

Remark: checking associativity requires to consider the diagram

and to show that the two maps from $A$ to $TD$ coincide.
Short bibliography of the course

On categorical semantics of linear logic and 2-categories:

Categorical semantics of linear logic.
Survey published in
« Interactive models of computation and program behaviour ».

On string diagrams:

Christian Kassel
Quantum groups
Graduate Texts in Mathematics 155
Springer Verlag 1995.

Peter Selinger
A survey of graphical languages for monoidal categories.
New Structures for Physics

Functorial boxes in string diagrams
Lecture Notes in Computer Science 4207.