1 Distributivity laws between functors and monads

§1. Suppose given two categories $\mathcal{A}$ and $\mathcal{B}$, each of them equipped with a monad

$$(S, \mu_S, \eta_S) : \mathcal{A} \longrightarrow \mathcal{A} \quad (T, \mu_T, \eta_T) : \mathcal{B} \longrightarrow \mathcal{B}$$

A homomorphism

$$(F, \lambda) : (\mathcal{A}, S) \longrightarrow (\mathcal{B}, T) \quad (1)$$

is defined as a functor $F: \mathcal{A} \rightarrow \mathcal{B}$ equipped with distributivity law

$$\lambda : T \circ F \Rightarrow F \circ S$$

making the diagrams of natural transformations below commute:

§1. Formulate the two commutative diagrams $(a)$ and $(b)$ as families of commutative diagrams between maps living in the category $\mathcal{B}$.

§2. Depict the commutative diagrams $(a)$ and $(b)$ in the language of string diagrams.

§3. Show that every homomorphism $(F, \lambda)$ as in (1) induces a functor

$$\tilde{F} : \text{Alg}(S) \longrightarrow \text{Alg}(T)$$

making the diagram below commute:

$$\text{Alg}(S) \xrightarrow{\tilde{F}} \text{Alg}(T)$$

where $U_S$ and $U_T$ are the forgetful functors associated to the monads $S$ and $T$, respectively.
§4. Conversely, show that every functor
\[ \tilde{F} : \text{Alg}(S) \rightarrow \text{Alg}(T) \]
making the diagram (\*) commute induces a distributivity law \( \lambda : T \circ \tilde{F} \Rightarrow \tilde{F} \circ S \) making the two diagrams (\( a \)) and (\( b \)) commute.

§5. Conclude that a homomorphism \((F, \lambda) : (\mathcal{A}, S) \rightarrow (\mathcal{B}, T)\) between two monads may be equivalently defined as a pair \((\tilde{F}, F)\) of functors
\[ F : \mathcal{A} \rightarrow \mathcal{B} \quad \tilde{F} : \text{Alg}(S) \rightarrow \text{Alg}(T) \]
making the diagram (\*) commute.

§6. Deduce that there is a category \( \text{Mon} \) of monads and homomorphisms between them.

§7. Describe the free abelian group functor \( F : \text{Sets} \rightarrow \text{Sets} \) which transports every set \( A \) to the free abelian group \( FA \) generated by the set \( A \).

§8. Construct a family of functions
\[ \lambda_A : TF(A) \rightarrow FT(A) \]
parametrized by an object \( A \in \mathcal{A} \) and check that the family \( \lambda \) is natural in \( A \) and makes the diagrams (\( a \)) and (\( b \)) commute.

§9. From this, deduce the existence of a functor
\[ \tilde{F} : \text{Monoid} \rightarrow \text{Monoid} \]
from the category of monoids and homomorphisms, making the diagram below commute:

\[ \begin{array}{ccc}
\text{Monoid} & \xrightarrow{\tilde{F}} & \text{Monoid} \\
U \downarrow & \xrightarrow{(*)} & \downarrow U \\
\text{Sets} & \xrightarrow{F} & \text{Sets}
\end{array} \]

§10. Describe the natural transformations \( \mu_F \) and \( \eta_F \) equipping the functor \( F \) as a monad \((F, \mu_F, \eta_F)\).

§11. A distributivity law
\[ \lambda : T \circ S \Rightarrow S \circ T \]
between two monads on the same category
\[ (S, \mu_S, \eta_S) : \mathcal{A} \rightarrow \mathcal{A} \quad (T, \mu_T, \eta_T) : \mathcal{A} \rightarrow \mathcal{A} \]
is a natural transformation making the diagrams below commute

\[ \begin{array}{ccc}
T \circ T \circ S & \xrightarrow{T \circ \lambda} & T \circ S \circ T & \xrightarrow{\lambda \circ T} & S \circ T \circ T \\
\mu_T \circ S \downarrow & \xrightarrow{(a)} & \downarrow \mu_T \circ S & \xrightarrow{S \circ \mu_T} & \downarrow \eta_T \\
T \circ S & \xrightarrow{\lambda} & S \circ T & \xrightarrow{S \circ T} & \downarrow \eta_T \\
\end{array} \quad \begin{array}{ccc}
S & \xrightarrow{\eta_T} & T \circ S \xrightarrow{\lambda} S \circ T \\
\end{array} \]

\[ (b) \]

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$\lambda = T \circ S \Rightarrow S \circ T$ between two monads $S$ and $T$ on the same category $\mathcal{A}$ induces a monad structure on the composite functor $S \circ T : \mathcal{A} \rightarrow \mathcal{A}$.

§13. Show that the natural transformation $\lambda$ defined in §7 defines a distributivity law between the monads $S = F$ and $T$.

§14. Show that the monad $S \circ T : \text{Sets} \rightarrow \text{Sets}$ associated to the distributivity law $\lambda : T \circ S \Rightarrow S \circ T$ coincides with the free algebra monad (here, by algebra, we mean $\mathbb{Z}$-algebra).

2 Grothendieck construction and colimits computed in the category of sets and functions

We recall that a contravariant presheaf on a small category $\mathcal{C}$ is a functor

$$\varphi : \mathcal{C}^{\text{op}} \rightarrow \text{Sets}.$$

The purpose of this exercise is to compute the colimit of this functor, seen as a diagram in the category $\text{Sets}$. Every contravariant presheaf $\varphi$ induces a category $\text{Groth} [\varphi]$ together with a projection functor

$$\pi[\varphi] : \text{Groth} [\varphi] \rightarrow \mathcal{C}. \quad (2)$$

The objects of the category are the pairs $(c, x)$ with $c$ an object of $\mathcal{C}$ and $x$ an element of $\varphi(x)$; the maps

$$(c, x) \rightarrow (d, y)$$

of the category are maps $f : c \rightarrow d$ of the underlying category $\mathcal{C}$ such that

$$\varphi(f)(y) = x.$$

§1. Show that these data define a category $\text{Groth} [\varphi]$ together with a functor (2).

§2. Show that every natural transformation

$$\theta : \varphi \Rightarrow \phi : \mathcal{C}^{\text{op}} \rightarrow \text{Sets}$$

induces a functor

$$\text{Groth}[\theta] : \text{Groth}[\varphi] \rightarrow \text{Groth}[\psi]$$

making the diagram below commute

$$\begin{array}{ccc}
\text{Groth}[\varphi] & \xrightarrow{\text{Groth}[\theta]} & \text{Groth}[\psi] \\
\pi[\varphi] \downarrow & & \downarrow \pi[\psi] \\
\mathcal{C} & \xrightarrow{} & \mathcal{C}
\end{array}$$
§3. Conversely, show that every functor

\[ F : \text{Groth}[\varphi] \rightarrow \text{Groth}[\psi] \]

making the diagram below commute

\[
\begin{array}{ccc}
\text{Groth}[\varphi] & \xrightarrow{F} & \text{Groth}[\psi] \\
\downarrow{\pi[\varphi]} & & \downarrow{\pi[\psi]} \\
\mathcal{C} & \xrightarrow{\theta} & \pi[\mathcal{C}]
\end{array}
\]

is of the form \( F = \text{Groth}[\theta] \) for a unique natural transformation

\[ \theta : \varphi \Rightarrow \psi : \mathcal{C}^{\text{op}} \rightarrow \text{Sets} \]

§4. For every object \( c \) of the category \( \mathcal{C} \), construct a function

\[ \theta_c : \varphi(c) \rightarrow \pi_0(\text{Groth}[\varphi]) \]

where the set

\[ \pi_0(\text{Groth}[\varphi]) \]

denotes the set of connected components of the category \( \text{Groth}[\varphi] \), defined as the connected components of the underlying graph.

§5. Show that the diagram below commutes

\[
\begin{array}{ccc}
\varphi(c) & \xrightarrow{\varphi(f)} & \varphi(d) \\
\downarrow{\theta_c} & & \downarrow{\theta_d} \\
\pi_0(\text{Groth}[\varphi]) & & \pi_0(\text{Groth}[\varphi])
\end{array}
\]

for every map \( f : c \rightarrow d \) in the category \( \mathcal{C} \). Deduce from this that the family \( \theta \) defines a natural transformation

\[ \theta : \varphi \Rightarrow \pi_0(\text{Groth}[\varphi]) \]

and thus a cone.

§5. Show that the cone is a colimiting cone, and thus that the colimit of the diagram

\[ \varphi : \mathcal{C}^{\text{op}} \rightarrow \text{Sets} \]

coincides with the set

\[ \pi_0(\text{Groth}[\varphi]) \]

of connected components of the Grothendieck category \( \text{Groth}[\varphi] \). From this, deduce that the category \( \text{Sets} \) has all small colimits.