1 Pullbacks

In this exercise, we study the notion of pullback (called “produit fibré” in French), an important variation of the notion of “cartesian product” studied during the lectures and a previous TD. A commutative diagram in a category $\mathcal{C}$

\[
P \xrightarrow{p_2} Y \\
p_1 \downarrow \quad \ast \quad \downarrow g \\
X \xrightarrow{f} Z
\]

is called a pullback diagram when the following property holds: for every commutative diagram

\[
\begin{align*}
\begin{array}{c}
Q \\
q_1 \downarrow \\
X \xrightarrow{f} Z
\end{array}
\rightarrow
\begin{array}{c}
Y \\
g \downarrow \\
\end{array}
\rightarrow
\begin{array}{c}
P \\
p_2 \downarrow \\
X \xrightarrow{f} Z
\end{array}
\end{align*}
\]

there exists a unique morphism $h : Q \to P$ making the diagram below commute:

\[
\begin{align*}
\begin{array}{c}
Q \\
q_1 \downarrow \\
P \xrightarrow{p_2} Y \\
p_1 \downarrow \\
X \xrightarrow{f} Z
\end{array}
\rightarrow
\begin{array}{c}
P \\
p_2 \downarrow \\
Y \\
g \downarrow \\
\end{array}
\rightarrow
\begin{array}{c}
Q \\
q_2 \downarrow \\
X \xrightarrow{f} Z
\end{array}
\end{align*}
\]

\[
\begin{align*}
&f \circ p_1 = g \circ p_2 \\
f \circ q_1 = g \circ q_2 \\
p_1 & = q_1 \circ h \\
q_2 & = q_2 \circ h
\end{align*}
\]

§1. Given two functions $f : X \to Z$ and $g : Y \to Z$, describe explicitly a set $P$ and a pair of functions $p_1 : P \to X$ and $p_2 : P \to Y$ defining a pullback diagram of the form $(\ast)$ in the category Sets of sets and functions. Hint: the terminology “produit fibré” comes from this construction.
§2. Given two pullback diagrams

\[
\begin{array}{ccc}
Y'' & \xrightarrow{p'} & Y' \\
\downarrow{g'} & & \downarrow{g'} \\
X'' & \xrightarrow{f'} & X'
\end{array}
\quad\begin{array}{ccc}
Y'' & \xrightarrow{p} & Y \\
\downarrow{g'} & & \downarrow{g} \\
X' & \xrightarrow{f} & X
\end{array}
\]

in a category \(\mathcal{C}\), show that the commutative diagram

\[
\begin{array}{ccc}
Y'' & \xrightarrow{p'} & Y' & \xrightarrow{p} & Y \\
\downarrow{g'} & & \downarrow{g'} & \downarrow{g} & \\
X'' & \xrightarrow{f'} & X' & \xrightarrow{f} & X
\end{array}
\]

obtained by “glueing” the two diagrams (a) and (b) defines a pullback diagram in the category \(\mathcal{C}\).

§3. Suppose given three commutative diagrams (a)(b)(c) in a category \(\mathcal{C}\). We have seen in the previous question that when (b) is a pullback diagram,

\[(a) \text{ is a pullback diagram} \implies (c) \text{ is a pullback diagram}\]

Establish the converse property that

\[(c) \text{ is a pullback diagram} \implies (a) \text{ is a pullback diagram}\]

when (b) is a pullback diagram.

§4. Exhibit an example of three commutative diagrams (a)(b)(c) such that

\[(a) \text{ and (c) are pullback diagrams, but (b) is not a pullback diagram!}\]

Hint: one can take \(X = \{x\}\) et \(X'' = \{x''\}\) singleton sets and \(X' = \{x_1, x_2\}\) a two-element set in the category \(\mathcal{C} = \text{Sets}\).

2 Monomorphisms and epimorphisms

§1. An arrow \(m : A \to B\) of a category \(\mathcal{C}\) is called a monomorphism (mono for short) when \(m\) is left-simplifiable in the sense that

\[m \circ f = m \circ g \implies f = g\]

for every pair of arrows \(f, g : X \to A\). Show that a function \(m : A \to B\) is a mono in the category \(\text{Sets}\) precisely when it is an injective function.

§2. An arrow \(e : A \to B\) of a category \(\mathcal{C}\) is called an epimorphism (epi for short) when \(e\) is right-simplifiable in the sense that

\[f \circ e = g \circ e \implies f = g\]

for every pair of arrows \(f, g : B \to Y\). Show that a function \(e : A \to B\) is an epi in the category \(\text{Sets}\) precisely when it is a surjective function.
§3. Show that in any category \( \mathcal{C} \), the composite \( g \circ f : A \to C \) of two monos \( f : A \to B \) and \( g : B \to C \) is a mono, and that the composite of two epis is an epi.

§4. Show that an arrow \( m : A \to B \) is a mono precisely when the commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{id} & A \\
\downarrow & & \downarrow m \\
A & \xrightarrow{m} & B \\
\end{array}
\]

is a pullback diagram in the category \( \mathcal{C} \). Explain what the property means in the specific case of a function \( m : A \to B \) in the category Sets.

§5. Show that every pullback diagram

\[
\begin{array}{ccc}
V & \xrightarrow{p} & U \\
\downarrow m' & & \downarrow m \\
B & \xrightarrow{f} & A \\
\end{array}
\]

in a category \( \mathcal{C} \) satisfies the following property:

\[
m : U \to A \text{ is a mono } \Rightarrow \text{ } m' : V \to B \text{ is a mono.}
\]

Show that the converse property does not hold by constructing a counter-example in the category Sets.

3 Comma categories and subobject categories

§1. Every object \( A \) in a category \( \mathcal{C} \) induces a category \( \mathcal{C}/A \) called the comma category on the object \( A \), and defined in the following way. The objects of \( \mathcal{C}/A \) are the pairs \( (X, f) \) consisting of an object \( X \in \mathcal{C} \) and of an arrow \( f : X \to A \) with target \( A \). The arrows of the category \( \mathcal{C}/A \)

\[
h : (X, f) \longrightarrow (Y, g)
\]

are the morphisms

\[
h : X \longrightarrow Y
\]

of the underlying category \( \mathcal{C} \), making the diagram below commute:

\[
\begin{array}{ccc}
X & \xrightarrow{h} & Y \\
\downarrow f & & \downarrow g \\
A & & \\
\end{array}
\]

Establish our claim above that \( \mathcal{C}/A \) defines a category.
§2. Show that a commutative diagram

\[
\begin{array}{ccc}
P & \xrightarrow{p_2} & Y \\
\downarrow{p_1} & & \downarrow{g} \\
X & \xrightarrow{f} & Z
\end{array}
\]

in the category \(\mathcal{C}\) is the same thing as a diagram

\[
\begin{array}{ccc}
(P, u) & \xrightarrow{p_1} & (X, f) \\
\downarrow{p_2} & & \downarrow{g} \\
(Y, g)
\end{array}
\]

in the category \(\mathcal{C}/Z\). Show moreover that the commutative diagram \((\ast)\) is a pullback in the category \(\mathcal{C}\) precisely when the span diagram \((\ast\ast)\) defines a cartesian product of \((X, f)\) and \((Y, g)\) in the comma category \(\mathcal{C}/Z\). Deduce from this that the pullback diagram \((\ast)\) associated to a pair of morphisms \(f : X \to Z\) and \(g : Y \to Z\) is unique up to isomorphism.

§3. Every object \(A\) in a category \(\mathcal{C}\) induces a category \(\text{Sub}(A)\) called the category of subobjects of \(A\), and defined in the following way. Its objects \((U, m)\) are the pairs consisting of an object \(U \in \mathcal{C}\) and of a mono \(m : U \to A\) with target \(A\). Its morphisms \(h : (U, m) \to (V, n)\) are the morphisms \(h : U \to V\) of the underlying category \(\mathcal{C}\) making the diagram

\[
\begin{array}{ccc}
U & \xrightarrow{h} & V \\
\downarrow{m} & & \downarrow{n} \\
A & & \end{array}
\]

commute in the category \(\mathcal{C}\). The category \(\text{Sub}(A)\) is thus the full subcategory of monos in the comma category \(\mathcal{C}/A\). Show that the category \(\text{Sub}(A)\) is a preorder category, in the sense there exists at most one arrow \(h : (U, m) \to (V, n)\) between two objects \((U, m)\) and \((V, n)\).

§4. Show that in the case \(\mathcal{C} = \text{Sets}\), one recovers the powerset \((\mathcal{P}(A), \subseteq)\) with subsets \(U, V \subseteq A\) ordered by inclusion \(U \subseteq V\), as the ordered set of equivalence classes associated to the preorder \(\text{Sub}(A)\). A useful convention in category theory is to identify the preorder category \(\text{Sub}(A)\) with the ordered set \((\mathcal{P}(A), \subseteq)\) in that case.

§5. A category \(\mathcal{C}\) has pullbacks when there exists a pullback diagram \((\ast)\) for every pair of arrows \(f : X \to Z\) and \(g : Y \to Z\). Show that in a category \(\mathcal{C}\) with pullbacks, every arrow \(f : B \to A\) induces a monotone function

\[
f^* : \text{Sub}(A) \to \text{Sub}(B)
\]

defined by transporting every mono \(m : U \to A\) to the mono \(m' : V \to B\) using the pullback diagram \((\oslash)\) in Exercise 2.5. Give an explicit description of the resulting monotone function

\[
f^* : \mathcal{P}(A) \to \mathcal{P}(B)
\]

in the case when \(\mathcal{C} = \text{Sets}\) and when \(f : A \to B\) is a function between two sets \(A\) and \(B\).