1 Equalizers and coequalizers

In this exercise, we study the notion of equalizer (called “égalisateur” in French) and its dual notion of coequalizer. Suppose given a pair of cointitial and cofinal arrows

\[ f, g : X \longrightarrow Y \]

in a category \( \mathcal{C} \). An equalizer of \( f \) and \( g \) is an arrow \( m : E \longrightarrow X \) such that

\[ f \circ m = g \circ m \]

and such that, for every arrow \( n : F \longrightarrow X \) such that

\[ f \circ n = g \circ n \]

there exists a unique arrow \( h : F \longrightarrow E \) such that

\[ n = m \circ h. \]

§1. Show that every pair of functions \( f, g : X \longrightarrow Y \) has an equalizer \( m : E \longrightarrow X \) in the category \( \text{Sets} \) and describe this equalizer.

§2. Show that when it exists in a category \( \mathcal{C} \), the equalizer \( m : E \longrightarrow X \) of a pair of arrows \( f, g : X \longrightarrow Y \) is a mono.

§3. Formulate the dual notion of coequalizer \( e : Y \longrightarrow Q \) of two arrows

\[ f, g : X \longrightarrow Y \]

in a category \( \mathcal{C} \).

§4. Show that when it exists in a category \( \mathcal{C} \), the coequalizer \( e : Y \longrightarrow Q \) of two arrows \( f, g : X \longrightarrow Y \) is an epi.

§5. Show that every pair of functions \( f, g : X \longrightarrow Y \) have a coequalizer \( e : Y \longrightarrow Q \) in the category \( \text{Sets} \) and describe this coequalizer.

§6. One says that an epi \( e : Y \longrightarrow Q \) is regular when there exists a pair of arrows \( f, g : X \longrightarrow Y \) such that \( e \) is a coequalizer of \( f \) and \( g \) as in the diagram below:

\[ X \xrightarrow{f} Y \xrightarrow{g} Q \]

Show that every surjective function \( e : A \longrightarrow B \) is a regular epi in the category \( \text{Sets} \).
2 Epi-mono factorization

An arrow \( f : A \to B \) is orthogonal to an arrow \( g : X \to Y \) in a category \( \mathcal{C} \) when for every pair of arrows \( u : A \to X \) and \( v : B \to Y \) making the diagram below commute

\[
\begin{array}{ccc}
A & \xrightarrow{u} & X \\
\downarrow f & & \downarrow g \\
B & \xrightarrow{v} & Y
\end{array}
\]

there exists a unique arrow \( h : B \to X \) making the diagram below commute

\[
\begin{array}{ccc}
A & \xrightarrow{u} & X \\
\downarrow f & & \downarrow g \\
B & \xrightarrow{h} & X \\
\uparrow h & & \uparrow h \\
B & \xrightarrow{v} & Y
\end{array}
\]

in the sense that \( u = h \circ f \) and \( v = g \circ h \).

We write in that case \( f \perp g \).

A factorisation system \( (\mathcal{E}, \mathcal{M}) \) is a pair of collections \( \mathcal{E} \) and \( \mathcal{M} \) of arrows of the category \( \mathcal{C} \) satisfying the three properties below:

A. every arrow \( X \to Y \) of the category \( \mathcal{C} \) factors as

\[
\begin{array}{ccc}
X & \xrightarrow{e} & U \\
\downarrow f & & \downarrow m \\
Y
\end{array}
\]

where \( e \in \mathcal{E} \) and \( m \in \mathcal{M} \).

B. every arrow \( e \in \mathcal{E} \) is orthogonal to every arrow \( m \in \mathcal{M} \), what we write

\( \mathcal{E} \perp \mathcal{M} \).

C. both collections \( \mathcal{E} \) and \( \mathcal{M} \) are closed under composition and contain the isos.

The purpose of the exercise is to show that the category \( \text{Sets} \) is equipped with a factorisation system \( (\mathcal{E}, \mathcal{M}) \) where \( \mathcal{E} \) and \( \mathcal{M} \) are respectively the collections of surjective and of injective functions.

§1. Show that every function \( X \to Y \) factors as

\[
\begin{array}{ccc}
X & \xrightarrow{e} & U \\
\downarrow f & & \downarrow m \\
Y
\end{array}
\]

where \( e : X \to U \) is a surjective function and \( m : U \to Y \) is an injective function.
§2. Show that every surjective function \( e : A \to B \) is orthogonal to every injective function \( m : X \to Y \) in the category \( \text{Sets} \).

§3. Deduce from §1 and §2 that \( (E, M) \) defines a factorization system in \( \text{Sets} \), where \( E \) and \( M \) are respectively the collections of surjective and injective functions in \( \text{Sets} \).

§4. Suppose given a category \( \mathcal{C} \) equipped with a factorization system \( (E, M) \) and a commutative diagram

\[
\begin{array}{ccc}
X_1 & \xrightarrow{e_1} & U_1 & \xrightarrow{m_1} & Y_1 \\
\downarrow & & \downarrow v & & \\
X_2 & \xrightarrow{e_2} & U_2 & \xrightarrow{m_2} & Y_2
\end{array}
\]

where \( e_1, e_2 \in E \) and \( m_1, m_2 \in M \). Show that there exists a unique arrow \( h : U_1 \to U_2 \) making the diagram below commute:

\[
\begin{array}{ccc}
X_1 & \xrightarrow{e_1} & U_1 & \xrightarrow{m_1} & Y_1 \\
\downarrow & & \downarrow h & & \downarrow v \\
X_2 & \xrightarrow{e_2} & U_2 & \xrightarrow{m_2} & Y_2
\end{array}
\]

in the category \( \mathcal{C} \).

§5. Suppose given a category \( \mathcal{C} \) whose collections \( E \) of epis and \( M \) of monos define a factorization system \( (E, M) \). Show that every arrow \( f : X \to Y \) induces a subobject \( (U, m) \in \text{Sub}(Y) \) defined as the unique subobject of \( Y \) such that the arrow \( f : X \to Y \) factors as

\[
X \xrightarrow{e} U \xrightarrow{m} Y
\]

for a given epi \( e : X \to U \). Show that in the case of the category \( \text{Sets} \), the construction associates to every function \( f : X \to Y \) its image in the set \( Y \). For that reason, one often calls the subobject \( m : U \to Y \) the image of the arrow \( f : X \to Y \).

§6. Suppose that we are still in the situation of §5. Show that every arrow \( f : A \to B \) of the category \( \mathcal{C} \) induces a monotone function

\[
f_* : \text{Sub}(A) \longrightarrow \text{Sub}(B)
\]

which transports every subobject \( (U, m) \) to the image \( f_*(U, m) \) of the composite arrow

\[
U \xrightarrow{m} A \xrightarrow{f} B
\]

using the notion of “image” of an arrow \( f \circ m : U \to B \) formulated in §5.

§7. Show that in the particular case \( \mathcal{C} = \text{Sets} \), one associates in this way to every function \( f : A \to B \) the monotone function

\[
f_* : \mathcal{P}(A) \longrightarrow \mathcal{P}(B)
\]

which transports every subset \( U \subseteq A \) to its image \( f(U) \subseteq B \).
§8. Suppose that we are in the situation of §5 and that the category $\mathcal{C}$ has moreover pullbacks. We have seen in the previous TD that every arrow $f : A \to B$ induces in that case a monotone function $f^* : \text{Sub}(B) \to \text{Sub}(A)$ defined by “pulling back” subobjects $(V, n) \in \text{Sub}(B)$ into subobjects $(U, m) \in \text{Sub}(A)$. Show that the monotone function $f^*$ is left adjoint to $f^*$ in the sense that

$$f_*(U, m) \leq (V, n) \iff (U, m) \leq f^*(V, n)$$

for every pair of subobjects $(U, m) \in \text{Sub}(A)$ and $(V, n) \in \text{Sub}(B)$.

3 Application to first-order logic

Consider a family of sets $X_1, \ldots, X_n$ and their cartesian product $\Gamma = X_1 \times \ldots \times X_n$. As we will see in the course, every first-order formula $\varphi$ with free variables $x_1, \ldots, x_n$ induces a subset

$$[\varphi] \subseteq \Gamma$$

consisting of all the elements $(x_1, \ldots, x_n) \in \Gamma$ satisfying the formula $\varphi$. Note that the interpretation $[\varphi]$ of the formula $\varphi$ can be also seen as an element of the powerset:

$$[\varphi] \in \mathcal{P}(\Gamma).$$

§1. Every set $X$ induces a function

$$\pi : \Gamma \times X \to \Gamma$$

defined by the first projection. Given a first-order formula $\varphi$ with free variables $x_1, \ldots, x_n$, show that the subset

$$\pi^*[\varphi] = \{(x_1, \ldots, x_n, x) \in \Gamma \times X \mid \varphi(x_1, \ldots, x_n)\}$$

coinsides with the interpretation of the same formula $\varphi$ seen as a formula with free variables $x_1, \ldots, x_n, x$.

§2. Given a first-order formula $\psi$ with free variables $x_1, \ldots, x_n, x$ and with interpretation

$$[\psi] \in \mathcal{P}(\Gamma \times X)$$

show that

$$\pi_*[\psi] = \{(x_1, \ldots, x_n) \in \Gamma \mid \exists x \in X, \psi(x_1, \ldots, x_n, x)\}.$$

coinsides with the interpretation $[\exists_{x \in X} \psi]$ of the formula $\exists_{x \in X} \psi$.

§3. From this, deduce that

$$[\exists_{x \in X} \psi] \leq_{\Gamma} [\varphi] \iff [\psi] \leq_{\Gamma \times X} [\varphi]$$

where we write $U \leq_{\Gamma} V$ for the inclusion $U \subseteq V$ between subsets $U, V \in \mathcal{P}(\Gamma)$. Justify this equivalence from the point of view of first-order logic.