A Traced Monoidal Category of Relations (DRAFT)

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1 Introduction

This note is the third in a series concerned with specification via pre- and post-conditions. It sets up a generic framework for systems formed by wiring together primitive building blocks. Such systems will be modelled as relations using existentially quantified equations for the fixed points that correspond to feedback loops in the wiring diagram. The approach is based on the notion of traced monoidal categories, e.g., see [1]. The category-phobic reader should not despair as the discussion here is self-contained, and the relatively simple ideas that fall under this fancy name do provide a very nice general algebraic way of thinking about diagrammatic models.

2 TRACED MONOIDAL CATEGORIES

In this section we formalise the notion of a traced monoidal category. The reader can think of this as just providing a convenient syntax for expressing diagrams of components connected by "wires" of some sort, together with a set of algebraic relations expressing some evident identities that ought to hold in such diagrams.

As an example, consider what one might call the *Kirchhoff category* which models a class of electrical circuits. Some building blocks for these circuits are shown in figure 1 on page 2. Here one should think of the nodes labelled X_i as inputs and of those labelled Y_j as outputs. The values of these labels range over some set of names. Circuits are constructed by wiring together inputs and outputs (and deleting the labels on the resulting internal nodes). The resistor building block is labelled with a non-negative real number R giving its resistance in ohms.

The laws of Kirchhoff and Ohm imply relationships that must hold between the voltages, V_{X_i}, V_{Y_j}, and currents, I_{X_i}, I_{Y_j}, that obtain at the labelled nodes in a physical realisation of a circuit. Taking the sense of current flow so that the nodes labelled X_i act as positive current sources, the four building blocks of figure 1 impose the following constraints:

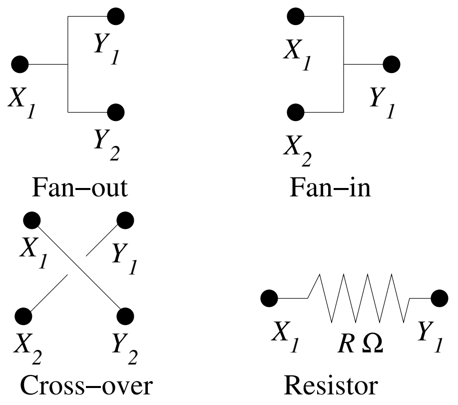


Figure 1: Kirchhoff Category: Some Building Blocks

Fan-out:	$V_{Y_1} = V_{Y_2} = V_{X_1}$ $I_{Y_1} + I_{Y_2} = I_{X_1}$	Fan-in:	$V_{Y_1} = V_{X_1} = V_{X_2}$ $I_{Y_1} = I_{X_1} + I_{X_2}$
Cross-over:	$V_{Y_i} = V_{X_{3-i}}$ $I_{Y_i} = I_{X_{3-i}}$	Resistor:	$V_{Y_1} = V_{X_1} - RI_{X_1}$ $I_{Y_1} = I_{X_1}$

On solving these equations for the circuits in figure 2 on page 2, one finds that the three circuits are equivalent. The idea of a traced monoidal category will formalise the process of building up the overall input-output relationship of a circuit from its constituent parts. The algebraic laws satisfied by a traced monoidal category capture equivalences like that of the first two circuits in figure 2 which are essentially independent of the semantics of the basic building blocks. The laws also provide a way of structuring proofs that do depend on the semantics (like the other equivalence in figure 2).

The operators that we are using to build diagrams are shown in figure 3 on page 4. The notion of traced monoidal category that we are now going to define may be viewed as an algevraic theory of these operators. The formalisation of this notion deals with the operators in turn.

The first stage in the formalisation deals with horizontal composition. Horizontal composition is a partial operation: in the Kirchoff category, two circuits f and g can only be composed horizontally if f has the same number of outputs as g has of inputs. The composition is associative in the sense that $f \fatsemi (g \fatsemi h) = (f \fatsemi g) \fatsemi h$ whenever these composites are defined. In the Kirchoff category, a vertical stack of an appropriate number of resistors with $R = 0$ provides a unit for the composition. All this means that horizontal composition is the composition operator of a *category* whose morphisms or *arrows* are the diagrams and whose objects are the vectors of inputs or outputs of a circuit.

Formally, we specify a category in Z to be a member of the generic schema CAT defined below. The generic parameters O and A give the sets of objects and arrows of the category respectively. In the Kirchhoff category, the domain and codomain operators correspond to the operations of reading off the vector of inputs or outputs of a circuit and the identity operator corresponds to the operation of constructing a vertical stacks of resistors with $R = 0$.

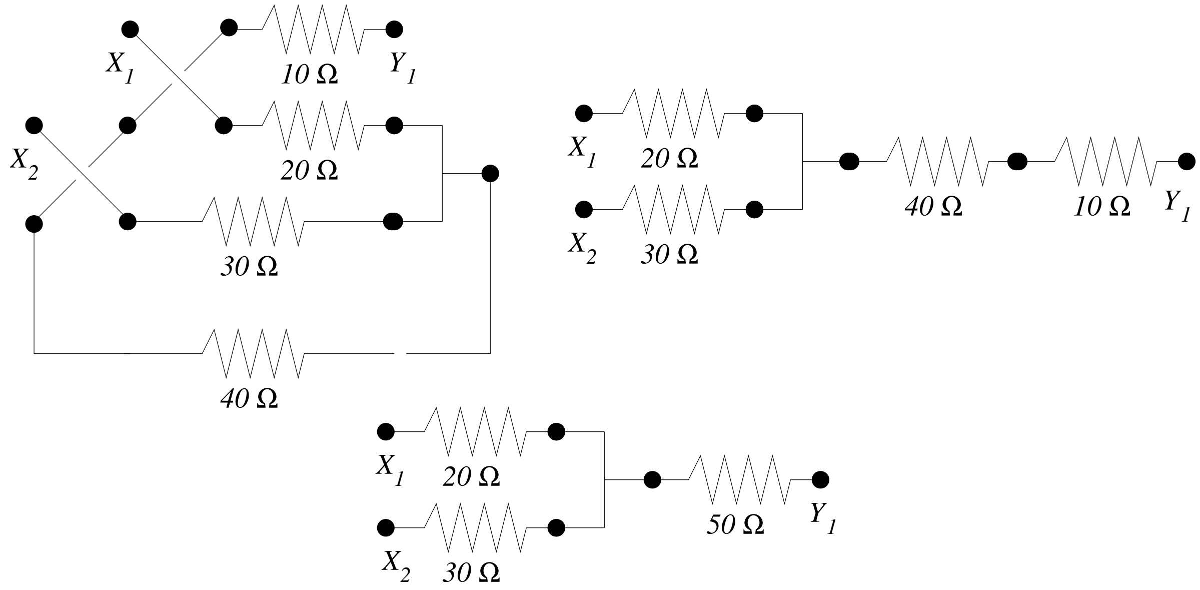


Figure 2: Kirchoff Category: Some Equivalent Circuits

$\mathit{function}\ 40\ \mathit{leftassoc}\ \ \_ \fatsemi_c \_$

$\boldsymbol{CAT}[O, A]$

$\_ \fatsemi_c \_ : A \times A \nrightarrow A$;

$dom_c,\ cod_c : A \rightarrow O$;

$id_c : O \rightarrow A$

$dom\ (\_ \fatsemi_c \_) = \{f,\ g : A \mid cod_c\ f = dom_c\ g\}$;

$\forall f,\ g,\ h : A \mid cod_c\ f = dom_c\ g \land cod_c\ g = dom_c\ h \bullet$

$\quad (f \fatsemi_c g) \fatsemi_c h = f \fatsemi_c (g \fatsemi_c h)$;

$\forall f,\ g : A \mid cod_c\ f = dom_c\ g \bullet$

$\quad dom_c\ (f \fatsemi_c g) = dom_c\ f \land cod_c\ (f \fatsemi_c g) = cod_c\ g$;

$\forall\ X : O \bullet$

$\quad dom_c\ (id_c\ X) = cod_c\ (id_c\ X) = X$;

$\forall\ f : A \bullet$

$\quad id_c(dom_c\ f) \fatsemi_c f = f \fatsemi_c id_c(cod_c\ f) = f$

We now look at vertical composition: in the Kirchhoff category, vertical composition is associative, has a unit given by an empty diagram, preserves identities and commutes with horizontal composition in the senses illustrated in figure 4 on page 5. In categorical terminology, the vertical composition is an associative bifunctor from the category to itself with a unit, and a category possessing this additional structure is called a monoidal category.

Formally, we specify a monoidal category by the following generic schema *MON_CAT*. In which $\oplus_o$ and $\oplus_a$ give the vertical composition operations for objects and arrows respectively and $I_\oplus$ gives the unit. (One could do without $\oplus_o$ here, but it helps to make the definition succinct). In informal narrative, we will often omit the subscripts from $\oplus_o$ and $\oplus_a$.

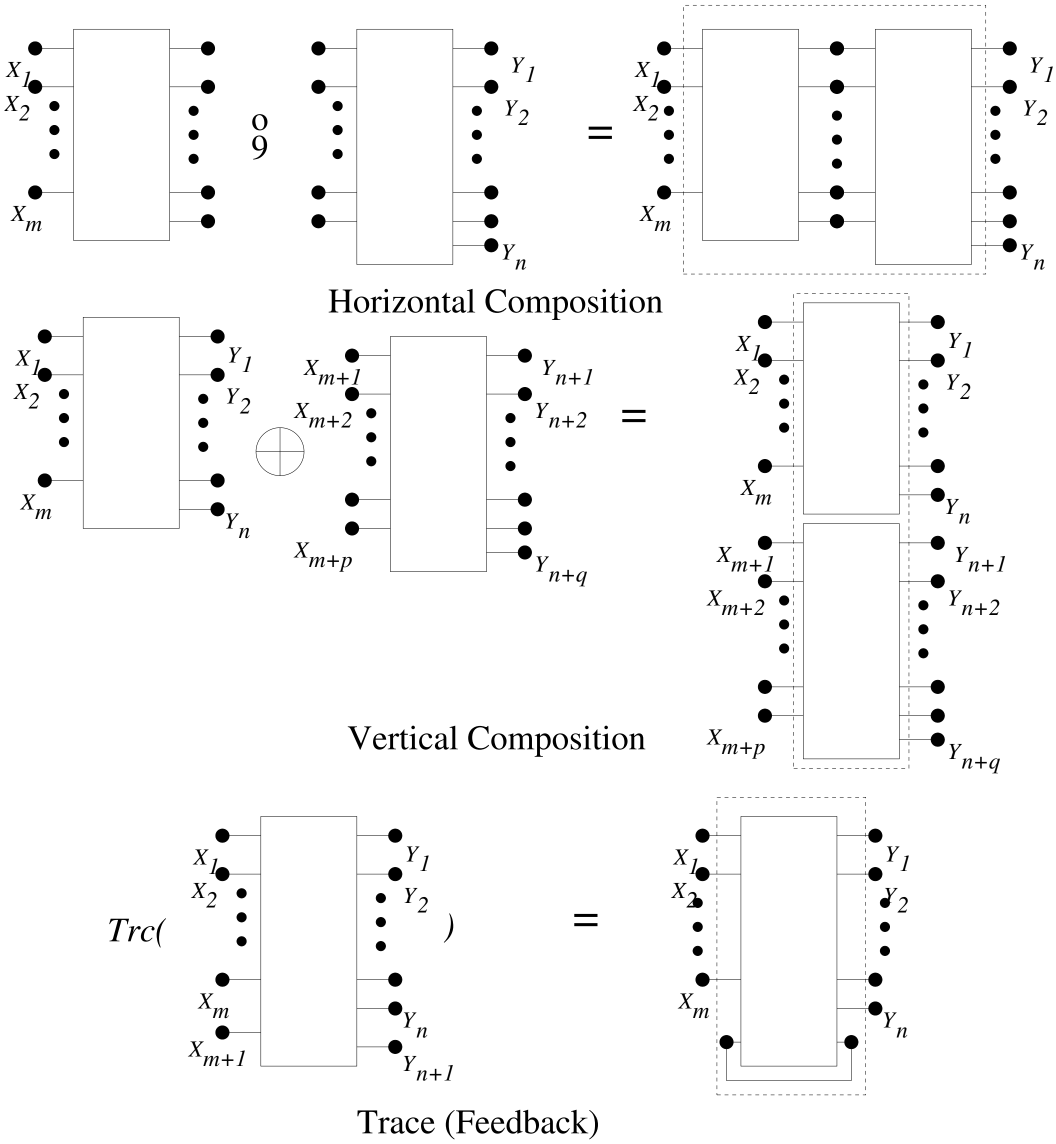
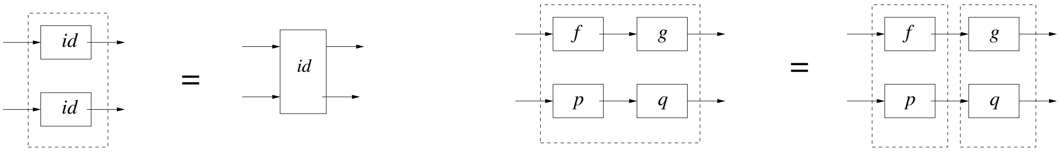


Figure 3: Building Diagrams



Bifunctoriality 1

Bifunctoriality 2

Figure 4: Monoidal Categories: Bifunctoriality

| *function 40 leftassoc* $\_ \oplus_o \_, \_ \oplus_a \_$

$\boldsymbol{MON\_CAT}$ $[O, A]$

$CAT[O,\ A]$;
$\_ \oplus_o \_ : O \times O \rightarrow O;$
$\_ \oplus_a \_ : A \times A \rightarrow A;$
$I_\oplus : A$

$\forall f,\ g,\ h : A \bullet (f \oplus_a g) \oplus_a h = f \oplus_a (g \oplus_a h)$;
$\forall f : A \bullet (f \oplus_a I_\oplus) = (I_\oplus \oplus_a f) = f$;
$\forall\ X,\ Y : O \bullet X \oplus_o Y = dom_c\ (id_c\ X \oplus_a id_c\ Y);$
$\forall\ X,\ Y : O \bullet id_c\ X \oplus_a id_c\ Y = id_c\ (X \oplus_o Y)$;
$\forall\ f,\ g,\ p,\ q : A\ |$
$\quad cod_c\ f = dom_c\ g \wedge cod_c\ p = dom_c\ q \bullet$
$\quad (f \fatsemi_c g) \oplus_a (p \fatsemi_c q) = (f \oplus_a p) \fatsemi_c (g \oplus_a q)$

In the Kirchoff category, we included a cross-over for two wires as a basic building block. Since any permutation of a vector of elements can be written as a product of permutations that exchange adjacent elements, this can be used to implement any permutation of the inputs. Any such permutation is an isomorphism in the category, i.e., an arrow with an inverse with respect to horizontal composition. In particular, if X and Y are objects in the category (vectors of voltage-current pairs), one has an isomorphism or *symmetry* between the vertical compositions $X \oplus Y$ and $Y \oplus X$. Subject to a condition which ensures that symmetries compose just like transpositions of elements in a vector, this means that our category is what is called a *symmetric monoidal category.*

Formally, a symmetric monoidal category is to be a member of the following generic schema. Here, for objects X and X, the arrow $Sym(X, Y)$ is the isomorphism between $X \oplus Y$ and $Y \oplus X$.

SYM_MON_CAT$[O, A]$

$$MON\_CAT[O,\ A]\ ;$$
$$Sym : O \times O \to A$$

$$\forall\ X,\ Y : O \bullet dom_c(Sym(X,\ Y)) = X \oplus_o Y \land cod_c(Sym(X,\ Y)) = Y \oplus_o X\ ;$$
$$\forall\ X,\ Y : O \bullet Sym(X,\ Y) \mathbin{⨟_c} Sym(Y,\ X) = id_c\ (X \oplus_o Y)\ ;$$
$$\forall\ X,\ Y,\ Z : O \bullet$$
$$\quad Sym(X \oplus_o Y,\ Z) =$$
$$\quad (id_c\ X \oplus_a Sym(Y,\ Z)) \mathbin{⨟_c} (Sym(X,\ Z) \oplus_a id_c\ Y)$$

Finally, we deal with the feeback or *trace* operator.

We now give the definition of a traced monoidal category in stages. A traced monoidal category will be a symmetric monoidal category equipped with a trace operator which comprises a partial function from arrows to arrows parameterised by three objects:

TRA_MON_CAT_SIG$[O, A]$

$$SYM\_MON\_CAT[O,\ A]\ ;$$
$$Trc : O \times O \times O \to A \mathrel{\ooalign{$\to$\cr\hidewidth$+$\hidewidth}} A$$

Our first requirement on the trace operator says that $Trc(X, Y, Z)$ maps an arrow from $X \oplus Z$ to $Y \oplus Z$ to an arrow from X to Y. Typically one thinks of the trace operator as taking a fixed point with respect to to the part of an arrow from $X \oplus Z$ to $Y \oplus Z$ that lies over Z.

TRA_MON_CAT_DOM$[O, A]$

$$TRA\_MON\_CAT\_SIG[O,\ A]$$

$$\forall\ X,\ Y,\ Z : O \bullet$$
$$\quad dom\ (Trc(X,\ Y,\ Z)) = \{f : A \mid dom_c\ f = X \oplus_o Z \land cod_c\ f = Y \oplus_o Z\}\ ;$$
$$\forall\ X,\ Y,\ Z : O;\ f : A \mid$$
$$\quad dom_c\ f = X \oplus_o Z \land cod_c\ f = Y \oplus_o Z \bullet$$
$$\quad dom_c\ (Trc(X,\ Y,\ Z)\ f) = X \land cod_c\ (Trc(X,\ Y,\ Z)\ f) = Y$$

The following are the naturality properties required of $Trc(X, Y, Z)$. Hasegawa calls these three properties *left tightening*, *right tightening* and *sliding*. The properties are illustrated in figure 5 on page 7.

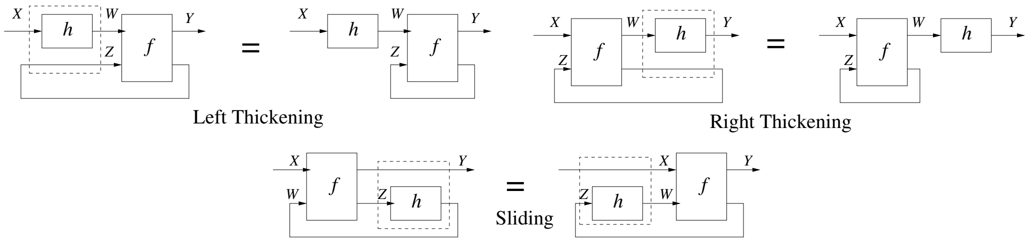


Figure 5: Traced Monoidal Categories: Naturality

Figure 6: Traced Monoidal Categories: Vanishing

TRA_MON_CAT_NAT$[O, A]$

$TRA\_MON\_CAT\_SIG[O,\ A]$

$\forall\ W,\ X,\ Y,\ Z\ :\ O;\ f,\ h\ :\ A\ |$

$\quad dom_c\ f\ =\ W\ \oplus_o\ Z\ \wedge\ cod_c\ f\ =\ Y\ \oplus_o\ Z\ ;$

$\quad dom_c\ h\ =\ X\ \wedge\ cod_c\ h\ =\ W\ \bullet$

$\quad Trc(X,\ Y,\ Z)\ ((h\ \oplus_a\ id_c\ Z)\ \fatsemi_c\ f)\ =$

$\quad h\ \fatsemi_c\ Trc(W,\ Y,\ Z)\ f\ ;$

$\forall\ W,\ X,\ Y,\ Z\ :\ O;\ f,\ h\ :\ A\ |$

$\quad dom_c\ f\ =\ X\ \oplus_o\ Z\ \wedge\ cod_c\ f\ =\ W\ \oplus_o\ Z\ ;$

$\quad dom_c\ h\ =\ W\ \wedge\ cod_c\ h\ =\ Y\ \bullet\ (*\ Y\ not\ X!\ *)$

$\quad Trc(X,\ Y,\ Z)\ (f\ \fatsemi_c\ (h\ \oplus_a\ id_c\ Z))\ =$

$\quad Trc(X,\ W,\ Z)\ f\ \fatsemi_c\ h;\ (*\ in\ that\ order!\ *)$

$\forall\ W,\ X,\ Y,\ Z\ :\ O;\ f,\ h\ :\ A\ |$

$\quad dom_c\ f\ =\ X\ \oplus_o\ W\ \wedge\ cod_c\ f\ =\ Y\ \oplus_o\ Z\ ;$

$\quad dom_c\ h\ =\ Z\ \wedge\ cod_c\ h\ =\ W\ \bullet$

$\quad Trc(X,\ Y,\ Z)\ ((id_c\ X\ \oplus_a\ h)\ \fatsemi_c\ f)\ =$

$\quad Trc(X,\ Y,\ W)\ (f\ \fatsemi_c\ (id_c\ Y\ \oplus_a\ h))$

The following property is called *vanishing* by Hasegawa. It captures the idea that taking a fixed point with respect to an identity arrow does nothing and that fixed points over a vertical product $W\ \ominus\ Z$ can be taken by taking fixed points over Z then W in turn. This property is illustrated in figure 6 on page 7.

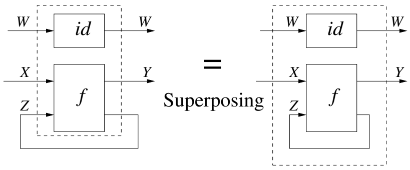


Figure 7: Traced Monoidal Categories: Superposing

TRA_MON_CAT_VAN$[O, A]$

TRA_MON_CAT_SIG$[O,\ A]$

$\forall\ X,\ Y\ :\ O;\ f\ :\ A\ |$
$\quad dom_c\ f\ =\ X\ \ \wedge\ cod_c\ f\ =\ Y\ \bullet$
$\quad Trc(X,\ Y,\ dom_c\ I_{\oplus})\ (f\ \oplus_a\ I_{\oplus})\ =\ f\ ;$
$\forall\ X,\ Y,\ W,\ Z\ :\ O;\ f\ :\ A\ |$
$\quad dom_c\ f\ =\ X\ \oplus_o\ W\ \oplus_o\ Z\ \wedge\ cod_c\ f\ =\ Y\ \oplus_o\ W\ \oplus_o\ Z\ \bullet$
$\quad Trc(X,\ Y,\ W\ \oplus_o\ Z)\ f\ =$
$\quad Trc(X,\ Y,\ W)\ (Trc(X\ \oplus_o\ W,\ Y\ \oplus_o\ W,\ Z)\ f)$

The following property says that traces commute with the operation of vertical composition with an identity arrow. This is called *superposing* by Hasegawa. This property is illustrated in figure 7 on page 8.

TRA_MON_CAT_SUP$[O, A]$

TRA_MON_CAT_SIG$[O,\ A]$

$\forall\ W,\ X,\ Y,\ Z\ :\ O;\ f\ :\ A\ |$
$\quad dom_c\ f\ =\ X\ \oplus_o\ Z\ \wedge\ cod_c\ f\ =\ Y\ \oplus_o\ Z\ \bullet$
$\quad Trc(W\ \oplus_o\ X,\ W\ \oplus_o\ Y,\ Z)\ (id_c\ W\ \oplus_a\ f)\ =$
$\quad id_c\ W\ \oplus_a\ Trc(X,\ Y,\ Z)\ f$

The final property says that the trace of the symmetry on $X \oplus X$ is the identity arrow on X. This is called *yanking* by Hasegawa. This property is illustrated in figure 8 on page 9.

TRA_MON_CAT_YAN$[O, A]$

TRA_MON_CAT_SIG$[O,\ A]$

$\forall\ X\ :\ O\ \bullet$
$\quad Trc\ (X,\ X,\ X)\ (Sym\ (X,\ X))\ =\ id_c\ X$

Finally, we put all these pieces together to complete the definition of a traced monoidal category:

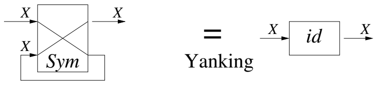


Figure 8: Traced Monoidal Categories: Yanking

TRA_MON_CAT$[O, A]$

TRA_MON_CAT_SIG$[O,\ A]$

TRA_MON_CAT_DOM$[O,\ A]$;
TRA_MON_CAT_NAT$[O,\ A]$;
TRA_MON_CAT_VAN$[O,\ A]$;
TRA_MON_CAT_SUP$[O,\ A]$;
TRA_MON_CAT_YAN$[O,\ A]$

3 A CATEGORY OF RELATIONS

We will now formalise a generic category of relations in Z. The Kirchhoff category discussed informally in the previous section may be formalised as an instance of this category. As we go we state theorems which build up to the assertion that the generic category is in fact a traced monoidal category.

If V is any set, V^m gives the set of all m-tuples with values ranging over V.

function 50 _ ^ _

$[U]$

$\_ \hat{} \_ : \mathbb{P}U \times \mathbb{N} \rightarrow \mathbb{P}(seq\ U)$

$\forall V : \mathbb{P}U;\ m{:}\mathbb{N} \bullet\ V \hat{}\ m =\ \{s : seq\ V \mid \#s = m\}$

For each set V, there is a category, which we shall call $RelCat[V]$, whose objects are the natural numbers and whose morphisms (a.k.a. arrows) from m to n are the relations between V^m and V^n composed via ordinary relational composition[1]. An arrow in the category is given as labelled triple whose components give the domain (d), codomain (c), and the relation (r).

$\boldsymbol{Rel}[V] \mathrel{\widehat=} \bigcup\{m,\ n : \mathbb{N} \bullet \{R : V\hat{}m \leftrightarrow V\hat{}n \bullet (d \mathrel{\widehat=} m,\ c \mathrel{\widehat=} n,\ r \mathrel{\widehat=} R)\}\}$

The composition of arrows that makes $Rel[V]$ into a category is the "horizontal" composition defined below.

---

[1]We take the objects to be the natural numbers, m, rather than the sets V^m for technical convenience to avoid having the category collapse when $V = \{\}$.

$$
\begin{array}{l}
\text{[V]} \\
\boldsymbol{HCompose} : Rel[V] \times Rel[V] \nrightarrow Rel[V] \\
\hline
dom\ HCompose = \{R, S : Rel[V] \mid R.c = S.d\} ; \\
\forall R, S : Rel[V] \mid R.c = S.d \bullet \\
\quad HCompose(R, S) = (d \mathrel{\widehat=} R.d, c \mathrel{\widehat=} S.c, r \mathrel{\widehat=} R.r \mathbin{;} S.r)
\end{array}
$$

The following function gives the identity arrows in the category.

$$
\begin{array}{l}
\text{[V]} \\
\boldsymbol{HId} : \mathbb{N} \rightarrow Rel[V] \\
\hline
\forall\ m : \mathbb{N} \bullet HId\ m = (d \mathrel{\widehat=} m, c \mathrel{\widehat=} m, r \mathrel{\widehat=} id(V \hat{}\ m))
\end{array}
$$

We can now define $RelCat[V]$ and state the theorem that it is indeed a category.

$$
\begin{array}{l}
\boldsymbol{RelCat}\ [V] \mathrel{\widehat=} \\
\quad (\_ \mathbin{;_c} \_ \mathrel{\widehat=} HCompose[V], \\
\quad dom_c \mathrel{\widehat=} (\lambda R{:}Rel[V] \bullet R.d), \\
\quad cod_c \mathrel{\widehat=} (\lambda R{:}Rel[V] \bullet R.c), \\
\quad id_c \mathrel{\widehat=} HId[V])
\end{array}
$$

$$
rel\_cat\_thm\ ?\vdash [V](RelCat[V] \in CAT[\mathbb{N}, Rel[V]])
$$

The category becomes a monoidal category under a product given by addition on objects and by the following "vertical composition" on morphisms.

$$
\begin{array}{l}
\text{[V]} \\
\boldsymbol{VCompose} : Rel[V] \times Rel[V] \rightarrow Rel[V] \\
\hline
\forall R, S : Rel[V] \bullet \\
\quad VCompose(R, S) = (d \mathrel{\widehat=} R.d + S.d, c \mathrel{\widehat=} R.c + S.c, \\
\qquad r \mathrel{\widehat=} \{a, b, c, d : seq\ V \mid (a, c) \in R.r \land (b, d) \in S.r \bullet (a \frown b, c \frown d)\})
\end{array}
$$

Adding in the necessary structure we get the monoidal category $RelMonCat[V]$.

$$\textbf{\textit{RelMonCat}}\ [V] \mathrel{\widehat{=}}$$
$$(\_ \mathbin{ {}_{9}^{\circ}{}_c} \_ \mathrel{\widehat{=}} HCompose[V],$$
$$dom_c \mathrel{\widehat{=}} (\lambda R{:}Rel[V] \bullet\ R.d),$$
$$cod_c \mathrel{\widehat{=}} (\lambda R{:}Rel[V] \bullet\ R.c),$$
$$id_c \mathrel{\widehat{=}} HId[V],$$
$$\_ \oplus_o \_ \mathrel{\widehat{=}} (\lambda m,\ n{:}\mathbb{N} \bullet\ m + n),$$
$$\_ \oplus_a \_ \mathrel{\widehat{=}} VCompose[V],$$
$$I_{\oplus} \mathrel{\widehat{=}} HId[V](0))$$

$$rel\_mon\_cat\_thm\ ?\vdash\ [V](RelMonCat[V] \in MON\_CAT[\mathbb{N},\ Rel[V]])$$

To use the relation category, we will generally need to know something about the value domain, V. However, one can describe some generic constructions that just impose equational constraints. The most important is the following operator that makes the monoidal category into a symmetric monoidal category:

$[V]$

$$\textbf{\textit{Transpose}} : \mathbb{N} \times \mathbb{N} \rightarrow Rel[V]$$

$$\forall m,\ n : \mathbb{N} \bullet$$
$$Transpose(m,\ n) = (d \mathrel{\widehat{=}} m + n,\ c \mathrel{\widehat{=}} n + m,$$
$$r \mathrel{\widehat{=}} \{a : V\hat{\ }m;\ b : V\hat{\ }n \bullet (a \frown b,\ b \frown a)\})$$

We continue to add in the new structure and prove that it gives a symmetric monoidal category.

$$\textbf{\textit{RelSymMonCat}}\ [V] \mathrel{\widehat{=}}$$
$$(\_ \mathbin{ {}_{9}^{\circ}{}_c} \_ \mathrel{\widehat{=}} HCompose[V],$$
$$dom_c \mathrel{\widehat{=}} (\lambda R{:}Rel[V] \bullet\ R.d),$$
$$cod_c \mathrel{\widehat{=}} (\lambda R{:}Rel[V] \bullet\ R.c),$$
$$id_c \mathrel{\widehat{=}} HId[V],$$
$$\_ \oplus_o \_ \mathrel{\widehat{=}} (\lambda m,\ n{:}\mathbb{N} \bullet\ m + n),$$
$$\_ \oplus_a \_ \mathrel{\widehat{=}} VCompose[V],$$
$$I_{\oplus} \mathrel{\widehat{=}} HId[V](0),$$
$$Sym \mathrel{\widehat{=}} Transpose[V])$$

$$rel\_sym\_mon\_cat\_thm\ ?\vdash\ [V](RelSymMonCat[V] \in SYM\_MON\_CAT[\mathbb{N},\ Rel[V]])$$

Finally, we get a traced monoidal category by defining the following operator that corresponds to adding k feedback loops into a diagram.

$[V]$

$$\boldsymbol{Trace} : (\mathbb{N} \times \mathbb{N} \times \mathbb{N}) \to Rel[V] \mathrel{+\!\!\!\rightarrow} Rel[V]$$

$$\forall m,\ n,\ k : \mathbb{N} \bullet$$
$$dom\ (Trace(m,\ n,\ k)) = \{R : Rel[V] \mid R.d = m + k \wedge R.c = n + k\}\ ;$$
$$\forall m,\ n,\ k : \mathbb{N};\ R : Rel[V] \mid R.d = m + k \wedge R.c = n + k \bullet$$
$$Trace\ (m,\ n,\ k)\ R = (d \mathrel{\widehat{=}} m,\ c \mathrel{\widehat{=}} n,$$
$$r \mathrel{\widehat{=}} \{a{:}\ V\hat{\ }m;\ b : V\hat{\ }n \mid \exists c{:}V\hat{\ }k \bullet (a \frown c,\ b \frown c) \in R.r\})$$

This enables us to complete the specification of our relational traced monoidal category $RelTraMonCat[V]$:

$$\boldsymbol{RelTraMonCat}\ [V] \mathrel{\widehat{=}}$$
$$(\_ \mathbin{;}_c \_ \mathrel{\widehat{=}} HCompose[V],$$
$$dom_c \mathrel{\widehat{=}} (\lambda R{:}Rel[V] \bullet R.d),$$
$$cod_c \mathrel{\widehat{=}} (\lambda R{:}Rel[V] \bullet R.c),$$
$$id_c \mathrel{\widehat{=}} HId[V],$$
$$\_ \oplus_o \_ \mathrel{\widehat{=}} (\lambda m,\ n{:}\mathbb{N} \bullet m + n),$$
$$\_ \oplus_a \_ \mathrel{\widehat{=}} VCompose[V],$$
$$I_\oplus \mathrel{\widehat{=}} HId[V](0),$$
$$Sym \mathrel{\widehat{=}} Transpose[V],$$
$$Trc \mathrel{\widehat{=}} Trace[V])$$

$$rel\_tra\_mon\_cat\_thm\ ?\vdash [V](RelTraMonCat[V] \in TRA\_MON\_CAT[\mathbb{N},\ Rel[V]])$$

Finally, just for completeness let us instantiate the definition to give the Kirchhoff category (whose objects are voltage-current) pairs):

$\boldsymbol{VI}$

$$V,\ I : \mathbb{R}$$

$$\boldsymbol{KirchhoffCategory} \mathrel{\widehat{=}} RelTraMonCat\ [VI]$$

References

[1] M. Hasegawa. The Uniformity Principle on Traced Monoidal Categories. *Electronic Notes in Theoretical Computer Science*, 60, 2003. Proc. CICS'02.