Lambda calculs et catégories

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Synopsis of the lecture

1 – Lambda-calculus

2 – Categories and 2-categories

3 – String diagrams

4 – Adjunctions

5 – Monads

First part

Lambda-calculus

The calculus of functions

The pure λ-calculus

Terms $$M \quad ::= \quad x \quad | \quad M\,N \quad | \quad \lambda x.\,M$$

The β-reduction:

$$(\lambda x.M)\,N \quad \longrightarrow \quad M\,[x := N]$$

The η-expansion:

$$M \quad \longrightarrow \quad \lambda x.\,(M\,x)$$

Remark: every term is considered up to renaming $\equiv_\alpha$ of the bound variables, typically:

$$\lambda x.\lambda y.x \quad \equiv_\alpha \quad \lambda z.\lambda y.z$$

Occurrences

The set of occurrences of a λ-term M is defined by induction:

- ▷ $\mathbf{occ}\,(x) \quad = \quad \{\,\varepsilon\,\}$
- ▷ $\mathbf{occ}\,(MN) \quad = \quad \{\,\varepsilon\,\} \cup \{\,1 \cdot o \,|\, o \in \mathbf{occ}\,(M)\,\} \cup \{\,2 \cdot o \,|\, o \in \mathbf{occ}\,(N)\,\}$
- ▷ $\mathbf{occ}\,(\lambda x.M) \quad = \quad \{\,\varepsilon\,\} \cup \{\,1 \cdot o \,|\, o \in \mathbf{occ}\,(M)\,\}$

Note that every occurrence of the λ-term M is labelled by

- ▷ an application node App
- ▷ a binder λx
- ▷ a variable x

Free variables

The set of **free variables** of a λ-term is defined by induction:

- $FV(x) = \{x\}$
- $FV(MN) = FV(M) \cup FV(N)$
- $FV(\lambda x.M) = FV(M) \setminus \{x\}$

Every occurrence of a variable x in a λ-term is

- either free
- or bound by a binder λx above it in the λ-term.

Church-Rosser theorem

Also called confluence theorem.

Given two β-rewriting paths

$$f \; : \; M \xrightarrow{*} P \qquad\qquad g \; : \; M \xrightarrow{*} Q$$

there exists a λ-term N and two β-rewriting paths f' and g' completing the diagram as

$$\begin{tikzcd}
 & M \arrow[dl, "f"'] \arrow[dr, "g"] & \\
P \arrow[dr, "g'"'] & & Q \arrow[dl, "f'"] \\
 & N &
\end{tikzcd}$$

The simply-typed λ-calculus

It is possible to **type** the expressions of the λ-calculus using simple types A, B constructed by the grammar:

$$A, B \ ::= \ \alpha \mid A \Rightarrow B.$$

A **typing context** Γ is a finite sequence

$$\Gamma = (x_1 : A_1, ..., x_n : A_n)$$

where each x_i is a variable and each A_i is a simple type.

A **sequent** is a triple

$$x_1 : A_1, ..., x_n : A_n \vdash P : B$$

where

$$x_1 : A_1, ..., x_n : A_n$$

is a typing context, P is a λ-term and B is a simple type.

The simply-typed λ-calculus

Variable	$\dfrac{}{x : A \vdash x : A}$
Abstraction	$\dfrac{\Gamma, x : A \vdash P : B}{\Gamma \vdash \lambda x.P : A \Rightarrow B}$
Application	$\dfrac{\Gamma \vdash P : A \Rightarrow B \qquad \Delta \vdash Q : A}{\Gamma, \Delta \vdash PQ : B}$
Weakening	$\dfrac{\Gamma \vdash P : B}{\Gamma, x : A \vdash P : B}$
Contraction	$\dfrac{\Gamma, x : A, y : A \vdash P : B}{\Gamma, z : A \vdash P[x, y \leftarrow z] : B}$
Exchange	$\dfrac{\Gamma, x : A, y : B, \Delta \vdash P : C}{\Gamma, y : B, x : A, \Delta \vdash P : C}$

Subject reduction

A λ-term P is simply typed when there exists a sequent

$$\Gamma \quad \vdash \quad P : A$$

which may be obtained by a derivation tree.

One establishes that the set of simply typed λ-terms is closed under β-réduction:

Subject Reduction:

If $\Gamma \vdash P : A$ and $P \longrightarrow_{\beta} Q$, then $\Gamma \vdash Q : A$.

Strong normalization

A λ-term P is strongly normalizing when there exists no infinite sequence of β-reductions:

$$P \longrightarrow_{\beta} P_1 \longrightarrow_{\beta} P_2 \longrightarrow_{\beta} \cdots \longrightarrow_{\beta} P_n \longrightarrow_{\beta} \cdots$$

Strong normalization:

Every simply typed λ-term P is strongly normalizing.

In particular, the λ-term $\Delta\Delta$ loops:

$$\Delta\Delta \quad \longrightarrow_{\beta} \quad \Delta\Delta \quad \longrightarrow_{\beta} \quad \cdots$$

is not simply typed.

Curry-Howard (1)

minimal logic

Variable

$$\frac{}{A \vdash \quad A}$$

Abstraction

$$\frac{\Gamma, \quad A \vdash \quad B}{\Gamma \vdash \qquad A \Rightarrow B}$$

Application

$$\frac{\Gamma \vdash \quad A \Rightarrow B \qquad \Delta \vdash \quad A}{\Gamma, \Delta \vdash \quad B}$$

Weakening

$$\frac{\Gamma \vdash \quad B}{\Gamma, \quad A \vdash \quad B}$$

Contraction

$$\frac{\Gamma, \quad A, \quad A \vdash \quad B}{\Gamma, \quad A \vdash \qquad B}$$

Exchange

$$\frac{\Gamma, \quad A, \quad B, \Delta \vdash \quad C}{\Gamma, \quad B, \quad A, \Delta \vdash \quad C}$$

Curry-Howard (1)

simply typed λ-calculus

Variable

$$\frac{}{x : A \vdash x : A}$$

Abstraction

$$\frac{\Gamma, x : A \vdash P : B}{\Gamma \vdash \lambda x.P : A \Rightarrow B}$$

Application

$$\frac{\Gamma \vdash P : A \Rightarrow B \qquad \Delta \vdash Q : A}{\Gamma, \Delta \vdash PQ : B}$$

Weakening

$$\frac{\Gamma \vdash P : B}{\Gamma, x : A \vdash P : B}$$

Contraction

$$\frac{\Gamma, x : A, y : A \vdash P : B}{\Gamma, z : A \vdash P[x, y \leftarrow z] : B}$$

Exchange

$$\frac{\Gamma, x : A, y : B, \Delta \vdash P : C}{\Gamma, y : B, x : A, \Delta \vdash P : C}$$

Algebraic Church-Rosser Theorem

Given two β-rewriting paths

$$f \; : \; M \xrightarrow{*} P \qquad\qquad g \; : \; M \xrightarrow{*} Q$$

there exists a λ-term N and two β-rewriting paths f' and g' completing the diagram as

$$\begin{tikzcd}
 & M \arrow[dl, "f"'] \arrow[dr, "g"] & \\
P \arrow[dr, "g'"'] & \sim & Q \arrow[dl, "f'"] \\
 & N &
\end{tikzcd}$$

where $\sim$ denotes the permutation equivalence on rewriting paths.

Theorem established by Jean-Jacques Lévy in 1978

Redex

Definition. A β-redex is a pair

$$(M, o)$$

consisting of

- ▷ a λ-term M
- ▷ an occurrence of the λ-term M such that

$$M_{|o} \quad = \quad (\lambda x.P)\, Q$$

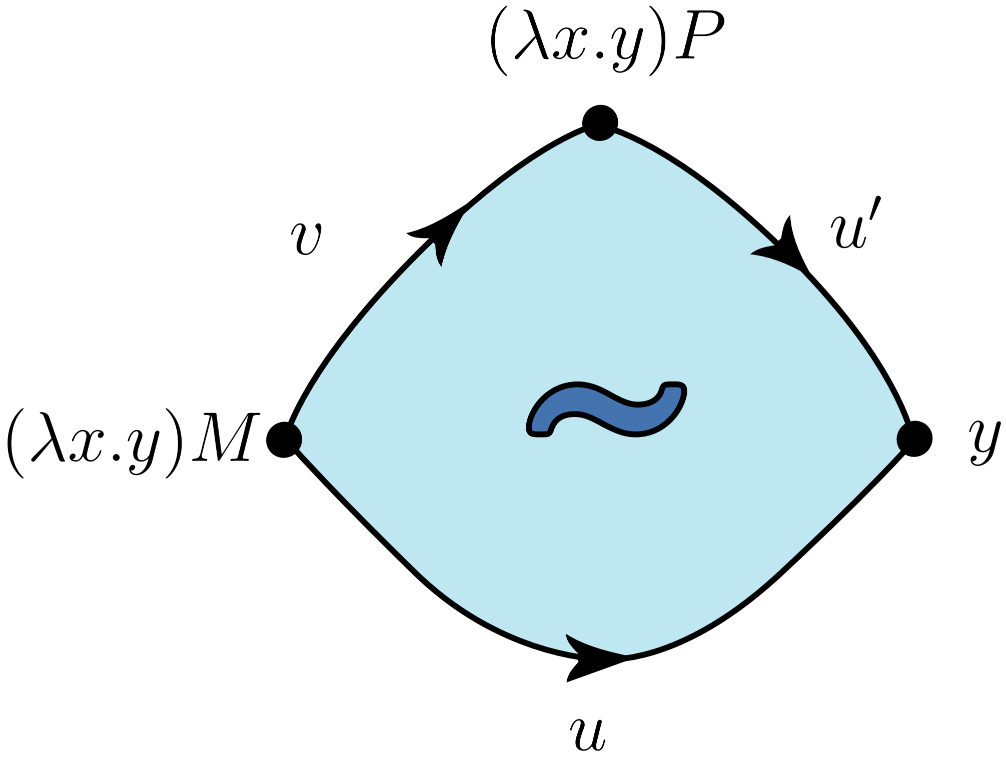
is a β-reduction pattern.

Redex permutations

$$\begin{tikzcd}
 & MQ \arrow[dr, "u'"] & \\
MN \arrow[ur, "v"] \arrow[dr, "u"'] & \sim & PQ \\
 & PN \arrow[ur, "v'"'] &
\end{tikzcd}$$

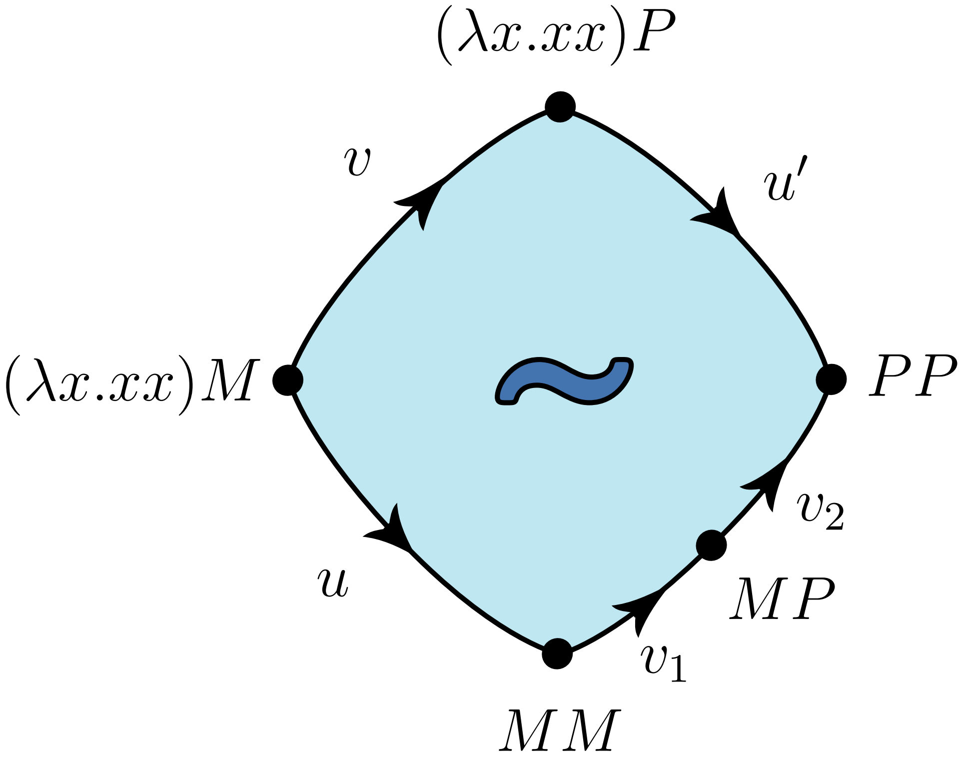
The two redexes $u : M \to P$ and $v : N \to Q$ are disjoint.

Redex permutations



The redex u erases the redex $v : M \to P$.

Redex permutations



The redex u duplicates the redex $v : M \to P$.

Rewriting paths modulo permutations

An important problem of rewriting theory: compare the several paths which rewrite **a λ-term P into its normal form Q.**

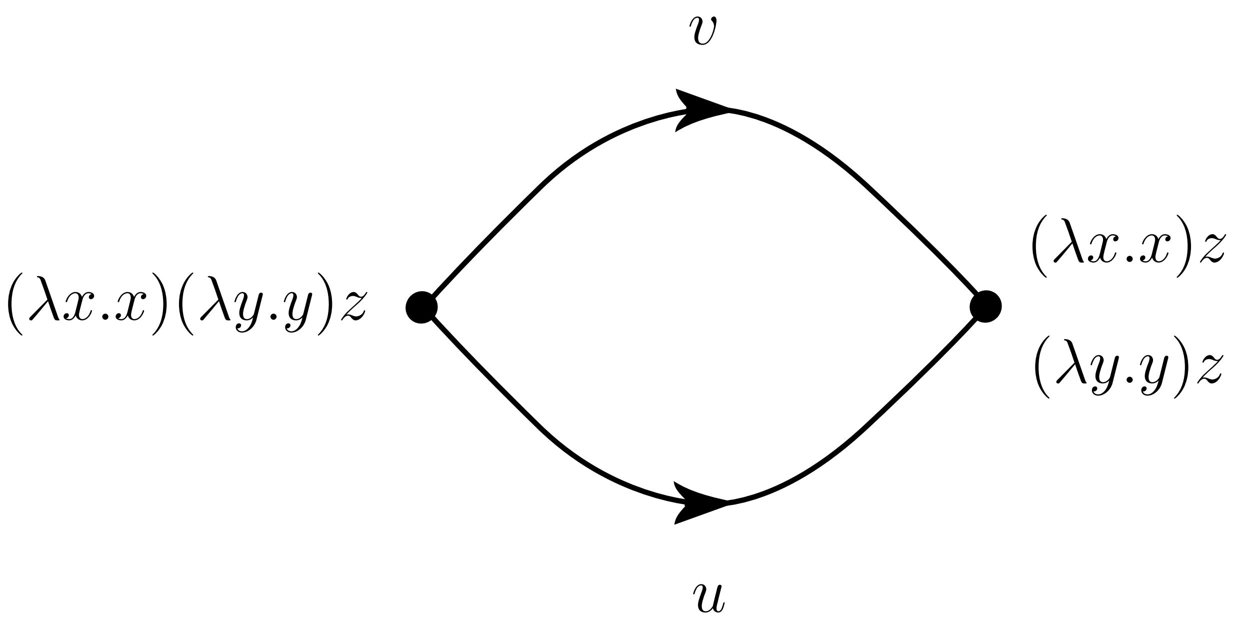
Corollary

Every two rewriting paths to the normal form

$$f, g \quad : \quad P \quad \longrightarrow \quad Q$$

are equal modulo a series of redex permutations.

A 2-dimensional hole



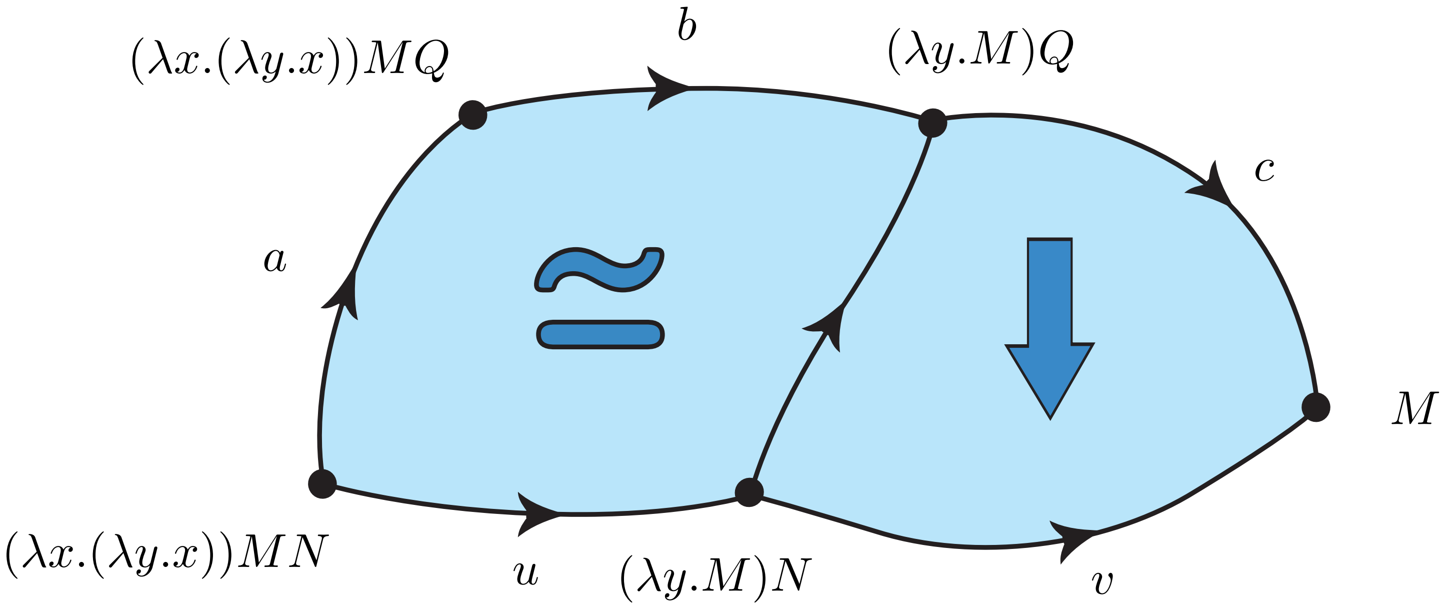
The two redexes u and v are not equivalent modulo permutation.

The 2-dimensional hole continued

$$\begin{tikzcd}
(\lambda x.x)(\lambda y.y)z \arrow[r, bend left, "v"] \arrow[r, bend right, "u"'] & (\lambda x.x)z \arrow[r, "w"] & z
\end{tikzcd}$$

The two paths $u \cdot w$ and $v \cdot w$ are equivalent modulo permutation.

Geometry of rewriting



A standardization theorem will be established in the course

The λ-calculus with de Bruijn indices

Variable
$$\frac{}{\Gamma, A \vdash 1 : A}$$

Abstraction
$$\frac{\Gamma, A \vdash P : B}{\Gamma \vdash \lambda\, P : A \Rightarrow B}$$

Application
$$\frac{\Gamma \vdash P : A \Rightarrow B \qquad \Gamma \vdash Q : A}{\Gamma \vdash P\, Q : B}$$

Weakening
$$\frac{\Gamma \vdash P : B}{\Gamma, A \vdash P\,[\uparrow]\ : B}$$

where $P\,[\uparrow]$ denotes the λ-term P where each free variable has been incremented.

The λ-calculus with explicit substitutions

Terms	M	$::=$	$\mathbf{1} \mid MN \mid \lambda M \mid M[s]$
Substitutions	s	$::=$	$id \mid \uparrow \mid M \cdot s \mid s \circ t$

Key idea: replace the β-rule of the λ-calculus

$$(\lambda x.M)\, N \quad \longrightarrow \quad M\,[x := N]$$

by the Beta-rule of the $\lambda\sigma$-calculus

$$(\lambda\, M)\, N \quad \longrightarrow \quad M\,[N \cdot id]$$

where the substitution is explicit – and thus similar to a closure.

The eleven rewriting rules of the $\lambda\sigma$-calculus

$Beta$	$(\lambda M)N$	$\longrightarrow$	$M[N \cdot id]$
App	$(MN)[s]$	$\longrightarrow$	$M[s]N[s]$
Abs	$(\lambda M)[s]$	$\longrightarrow$	$\lambda(M[\mathbf{1} \cdot (s \circ \uparrow)])$
$Clos$	$M[s][t]$	$\longrightarrow$	$M[s \circ t]$
$VarCons$	$\mathbf{1}[M \cdot s]$	$\longrightarrow$	M
$VarId$	$\mathbf{1}[id]$	$\longrightarrow$	$\mathbf{1}$
Map	$(M \cdot s) \circ t$	$\longrightarrow$	$M[t] \cdot (s \circ t)$
IdL	$id \circ s$	$\longrightarrow$	s
Ass	$(s_1 \circ s_2) \circ s_3$	$\longrightarrow$	$s_1 \circ (s_2 \circ s_3)$
$ShiftCons$	$\uparrow \circ (M \cdot s)$	$\longrightarrow$	s
$ShiftId$	$\uparrow \circ id$	$\longrightarrow$	$\uparrow$

The eleven critical pairs of the $\lambda\sigma$-calculus

$\mathit{App} + \mathit{Beta}$	$(\lambda M)[s](N[s])$	$\xleftarrow{\mathit{App}}$	$((\lambda M)N)[s]$	$\xrightarrow{\mathit{Beta}}$	$M[N \cdot \mathit{id}][s]$
$\mathit{Clos} + \mathit{App}$	$(MN)[s \circ t]$	$\xleftarrow{\mathit{Clos}}$	$(MN)[s][t]$	$\xrightarrow{\mathit{App}}$	$(M[s](N[s]))[t]$
$\mathit{Clos} + \mathit{Abs}$	$(\lambda M)[s \circ t]$	$\xleftarrow{\mathit{Clos}}$	$(\lambda M)[s][t]$	$\xrightarrow{\mathit{Abs}}$	$(\lambda(M[\mathbf{1} \cdot s \circ \uparrow]))[t]$
$\mathit{Clos} + \mathit{VarId}$	$\mathbf{1}[\mathit{id} \circ s]$	$\xleftarrow{\mathit{Clos}}$	$\mathbf{1}[\mathit{id}][s]$	$\xrightarrow{\mathit{VarId}}$	$\mathbf{1}[s]$
$\mathit{Clos} + \mathit{VarCons}$	$\mathbf{1}[(M \cdot s) \circ t]$	$\xleftarrow{\mathit{Clos}}$	$\mathbf{1}[M \cdot s][t]$	$\xrightarrow{\mathit{VarCons}}$	$M[t]$
$\mathit{Clos} + \mathit{Clos}$	$M[s][t \circ t']$	$\xleftarrow{\mathit{Clos}}$	$M[s][t][t']$	$\xrightarrow{\mathit{Clos}}$	$M[s \circ t][t']$
$\mathit{Ass} + \mathit{Map}$	$(M \cdot s) \circ (t \circ t')$	$\xleftarrow{\mathit{Ass}}$	$((M \cdot s) \circ t) \circ t'$	$\xrightarrow{\mathit{Map}}$	$(M[t] \cdot s \circ t) \circ t'$
$\mathit{Ass} + \mathit{IdL}$	$\mathit{id} \circ (s \circ t)$	$\xleftarrow{\mathit{Ass}}$	$(\mathit{id} \circ s) \circ t$	$\xrightarrow{\mathit{IdL}}$	$s \circ t$
$\mathit{Ass} + \mathit{ShiftId}$	$\uparrow \circ (\mathit{id} \circ s)$	$\xleftarrow{\mathit{Ass}}$	$(\uparrow \circ \mathit{id}) \circ s$	$\xrightarrow{\mathit{ShiftId}}$	$\uparrow \circ s$
$\mathit{Ass} + \mathit{ShiftCons}$	$\uparrow \circ ((M \cdot s) \circ t)$	$\xleftarrow{\mathit{Ass}}$	$(\uparrow \circ (M \cdot s)) \circ t$	$\xrightarrow{\mathit{ShiftCons}}$	$s \circ t$
$\mathit{Ass} + \mathit{Ass}$	$(s \circ s') \circ (t \circ t')$	$\xleftarrow{\mathit{Ass}}$	$((s \circ s') \circ t) \circ t'$	$\xrightarrow{\mathit{Ass}}$	$(s \circ (s' \circ t)) \circ t'$

A dangerous critical pair

$$\begin{tikzcd}
((\lambda P)Q)[s] \arrow[r, "App"] \arrow[d, "Beta"'] & (\lambda P)[s]Q[s] \arrow[r, "Lam"] & (\lambda(P[\mathbf{1}\cdot s\circ\uparrow]))Q[s] \arrow[d, "Beta"] \\
P[Q\cdot id][s] \arrow[d, "Clos+Map"'] & & P[\mathbf{1}\cdot s\circ\uparrow][Q[s]\cdot id] \arrow[d, "Clos+Map"] \\
P[Q[s]\cdot id\circ s] \arrow[d, "IdL"'] & & P[Q[s]\cdot(s\circ\uparrow)\circ(Q[s]\cdot id)] \arrow[d, "Ass+Shift"] \\
P[Q[s]\cdot s] & & P[Q[s]\cdot(s\circ id)] \arrow[ll, dotted, "when\ s=M_1\cdot M_2\cdot\cdots\cdot M_n\cdot id", "and\ the\ M_i's\ are\ in\ \sigma-normal\ form"']
\end{tikzcd}$$

This critical pair leads to a counter-example to **strong normalization** of the simply-typed $\lambda\sigma$-calculus.

Second part

Categories and 2-categories

Fonctors and natural transformations

Categories

A category $\mathscr{C}$ is given by

$[0]$ a class of **objects**

$[1]$ a set $\mathbf{Hom}(A,B)$ of **morphisms**

$$f \quad : \quad A \quad \longrightarrow \quad B$$

for every pair of objects (A,B)

$[2]$ a **composition law** $\circ : \mathbf{Hom}(B,C) \times \mathbf{Hom}(A,B) \longrightarrow \mathbf{Hom}(A,C)$

$[2]$ an **identity** morphism

$$id_A \quad : \quad A \quad \longrightarrow \quad A$$

for every object A,

Categories

satisfying the following properties:

[3] the composition law $\circ$ is associative:

$$\begin{array}{ll} \forall f \in \mathbf{Hom}(A,B) & \\ \forall g \in \mathbf{Hom}(B,C) & f \circ (g \circ h) = (f \circ g) \circ h \\ \forall h \in \mathbf{Hom}(C,D) & \end{array}$$

[3] the morphisms id are neutral elements

$$\forall f \in \mathbf{Hom}(A,B) \qquad f \circ id_A \ = \ f \ = \ id_B \circ f$$

A hint of higher-dimensional wisdom

$$\begin{tikzcd}
& B \arrow[dr, "g"] & \\
A \arrow[ur, "f"] \arrow[rr, "f \circ g"'] & & C
\end{tikzcd}$$

The composition law hides a 2-dimensional simplex

A hint of higher-dimensional wisdom

$$\begin{tikzcd}
& B \arrow[r, "g"] & C \arrow[dr, "f"] & & & & B \arrow[r, "g"] \arrow[drr, "f \circ g"'] & C \arrow[dr, "f"] & \\
A \arrow[ur, "h"] \arrow[urr, "g \circ h"'] \arrow[rrr, "f \circ (g \circ h)"'] & & & D & = & A \arrow[ur, "h"] \arrow[rrr, "(f \circ g) \circ h"'] & & & D
\end{tikzcd}$$

The associativity rule hides a 3-dimensional simplex

Functors

A **functor** between categories

$$F \quad : \quad \mathscr{C} \quad \longrightarrow \quad \mathscr{D}$$

is defined as the following data:

$[0]$ an object FA of $\mathscr{D}$ for every object A of $\mathscr{C}$,

$[1]$ a function

$$F_{A,B} \quad : \quad \mathbf{Hom}_{\mathscr{C}}(A,B) \quad \longrightarrow \quad \mathbf{Hom}_{\mathscr{D}}(FA,FB)$$

for every pair of objects (A,B) of the category $\mathscr{C}$.

Functors

One requires moreover

[2] that F preserves composition

$$FA \xrightarrow{Ff} FB \xrightarrow{Fg} FC \qquad = \qquad FA \xrightarrow{F(g \circ f)} FC$$

[2] that F preserves the identities

$$FA \xrightarrow{Fid_A} FA \qquad = \qquad FA \xrightarrow{id_{FA}} FA$$

Illustration [*orders*]

Every ordered set

$$(X, \leq)$$

defines a category

$$[X, \leq]$$

- ▷ whose objects are the elements of X
- ▷ whose hom-sets are defined as

$$\mathbf{Hom}(x, y) \quad = \quad \begin{cases} \{*\} & \text{if } \; x \leq y \\ \varnothing & \text{otherwise} \end{cases}$$

In this category, there exists at most one map between two objects

Illustration [*orders*]

Exercise: given two ordered sets

$$(X, \leq) \qquad (Y, \leq)$$

a functor

$$F \quad : \quad [X, \leq] \quad \longrightarrow \quad [Y, \leq]$$

is the same thing as a monotonic function

$$F \quad : \quad (X, \leq) \quad \longrightarrow \quad (Y, \leq)$$

between the underlying ordered sets.

Illustration [*monoids*]

A monoid $(M, \cdot, e)$ is a set M equipped with a binary operation

$$\cdot \quad : \quad M \times M \quad \longrightarrow \quad M$$

and a neutral element

$$e \quad : \quad \{*\} \quad \longrightarrow M$$

satisfying the two properties below:

Associativity law	$\forall x, y, z \in M,$	$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
Unit law	$\forall x \in M,$	$x \cdot e = x = e \cdot x.$

Illustration [*monoids*]

Key observation: there is a one-to-one relationship

$$(M, \cdot, e) \quad \mapsto \quad \Sigma\,(M, \cdot, e)$$

between

- ▷ monoids
- ▷ categories with one object $*$

obtained by defining $\Sigma(M, \cdot, e)$ as the category with unique hom-set

$$\Sigma(M, \cdot, e)\ (*, *) \quad = \quad M$$

and composition law and unit defined as

$$g \circ f \quad = \quad g \cdot f \qquad\qquad\qquad id_{\,*} = e$$

Illustration [*monoids*]

Key observation: given two monoids

$$(M, \cdot, e) \qquad\qquad (N, \bullet, u)$$

a functor

$$F \quad : \quad \Sigma\,(M, \cdot, e) \quad \longrightarrow \quad \Sigma\,(N, \bullet, u)$$

is the same thing as a homomorphism

$$f \quad : \quad (M, \cdot, e) \quad \longrightarrow \quad (N, \bullet, u)$$

between the underlying monoids.

Recall that a homomorphism is a function f such that

$$\forall x, y \in M, \quad f(x \cdot y) \;=\; f(x) \bullet f(y) \qquad\qquad f(e) \;=\; u$$

Illustration [*actions*]

The action of a monoid

$$(M, \cdot, e)$$

on a set

$$X$$

is the same thing as a functor

$$\Sigma\,(M, \cdot, e) \quad \longrightarrow \quad \mathbf{Set}$$

Illustration [*representations*]

The action of a monoid

$$(M, \cdot, e)$$

on a vector space

$$V$$

is the same thing as a functor

$$\Sigma\,(M, \cdot, e) \quad \longrightarrow \quad \mathbf{Vect}$$

Transformations

A **transformation**

$$\theta \quad : \quad F \overset{\cdot}{\longrightarrow} G$$

between two functors

$$F, G \quad : \quad \mathscr{A} \longrightarrow \mathscr{B}$$

is a family of morphisms

$$(\theta_A : FA \longrightarrow GA)_{A \in Obj(\mathscr{A})}$$

of the category $\mathscr{B}$ indexed by the objects of the category $\mathscr{A}$.

Vertical composition of transformations

The transformations compose vertically

$$\begin{tikzcd}
\mathscr{A} \arrow[r, bend left=60, "F"] \arrow[r, bend right=60, "H"'] \arrow[r, phantom, "\Downarrow\, \theta_1 * \theta_2"] & \mathscr{B} & = & \mathscr{A} \arrow[r, bend left=60, "F"] \arrow[r, "G" description] \arrow[r, bend right=60, "H"'] \arrow[r, phantom, bend left=30, "\Downarrow\, \theta_1"] \arrow[r, phantom, bend right=30, "\Downarrow\, \theta_2"] & \mathscr{B}
\end{tikzcd}$$

and thus define a category

$$\mathbf{Trans}\,(\,\mathscr{A}\,,\,\mathscr{B}\,)$$

for all categories $\mathscr{A}$ and $\mathscr{B}$.

Left action

In the following situation:

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=D}] \arrow[from=U, to=D, phantom, "\Downarrow\theta"] & \mathscr{B} \arrow[r, "H"] & \mathscr{C} \end{tikzcd}$$

the **left action** of the functor H on the transformation

$$\theta \quad : \quad F \quad \longrightarrow \quad G \quad : \quad \mathscr{A} \quad \longrightarrow \quad \mathscr{B}$$

is defined as the transformation

$$H \circ_L \theta \quad : \quad H \circ F \quad \longrightarrow \quad H \circ G \quad : \quad \mathscr{A} \quad \longrightarrow \quad \mathscr{C}$$

whose instance at object A is defined as the morphism

$$H \circ F(A) \quad \xrightarrow{\quad H(\theta_A) \quad} \quad H \circ G(A).$$

Properties of the left action [1]

From a diagrammatic point of view, the two equations

$$H \circ_L (\theta_2 * \theta_1) \;=\; (H \circ_L \theta_2) * (H \circ_L \theta_1) \qquad\qquad H \circ_L 1_F \;=\; 1_{H \circ F}$$

mean that

$$\begin{tikzcd}
\mathscr{A} \arrow[r, bend left=70] \arrow[r] \arrow[r, bend right=70] \arrow[r, phantom, shift left=3ex, "\Downarrow \theta_1"] \arrow[r, phantom, shift right=3ex, "\Downarrow \theta_2"] & \mathscr{B} \arrow[r, "H"] & \mathscr{C}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathscr{A} \arrow[r, bend left=70] \arrow[r] \arrow[r, phantom, shift left=3ex, "\Downarrow \theta_1"] \arrow[d, dotted, no head] & \mathscr{B} \arrow[r, "H"] \arrow[d, dotted, no head] & \mathscr{C} \arrow[d, dotted, no head] \\
\mathscr{A} \arrow[r] \arrow[r, bend right=70] \arrow[r, phantom, shift right=3ex, "\Downarrow \theta_2"] & \mathscr{B} \arrow[r, "H"'] & \mathscr{C}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathscr{A} \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "F"'] \arrow[r, phantom, "\Downarrow 1_F"] & \mathscr{A} \arrow[r, "H"] & \mathscr{B}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathscr{A} \arrow[r, bend left=50, "H \circ F"] \arrow[r, bend right=50, "H \circ F"'] \arrow[r, phantom, "\Downarrow 1_{H \circ F}"] & \mathscr{B}
\end{tikzcd}$$

Properties of the left action (2)

These two equations mean that

$$\boxed{\begin{array}{cccc} H \circ_L - & : & \mathbf{Trans}(\mathscr{A},\mathscr{B}) & \longrightarrow & \mathbf{Trans}(\mathscr{A},\mathscr{C}) \\ & & \theta & \mapsto & H \circ_L \theta \end{array}}$$

defines a functor, while the two equations

$$(H_1 \circ H_2) \circ_L F = H_1 \circ_L (H_2 \circ_L F) \qquad id_{\mathscr{B}} \circ_L \theta = \theta$$

mean that $\circ_L$ defines an action.

Right action

In the following situation:

$$\begin{tikzcd} \mathscr{A} \arrow[r, "H"] & \mathscr{B} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=D}] & \mathscr{C} \arrow[from=U, to=D, phantom, "\Downarrow\theta"] \end{tikzcd}$$

the functor H acts on the transformation

$$\theta \quad : \quad F \quad \longrightarrow \quad G \quad : \quad \mathscr{B} \quad \longrightarrow \quad \mathscr{C}$$

and transports it into the transformation:

$$\theta \circ_R H \quad : \quad F \circ H \quad \longrightarrow \quad G \circ H \quad : \quad \mathscr{A} \quad \longrightarrow \quad \mathscr{C}$$

whose instance at A is defined as the morphism

$$F \circ H(A) \quad \xrightarrow{\theta_{H(A)}} \quad G \circ H(A).$$

Properties of the right action (1)

From a diagrammatic point of view, the two equations

$$(\theta_2 * \theta_1) \circ_R H \;=\; (\theta_2 \circ_R H) * (\theta_1 \circ_R H) \qquad\qquad 1_F \circ_R H \;=\; 1_{F \circ H}$$

mean that

$$\begin{tikzcd}
\mathscr{A} \arrow[r, "H"] & \mathscr{B} \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=3ex, "\Downarrow \theta_1"] \arrow[r, phantom, shift right=3ex, "\Downarrow \theta_2"] & \mathscr{C}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathscr{A} \arrow[r, "H"] \arrow[d, dash, dotted] & \mathscr{B} \arrow[r] \arrow[r, bend left=60] \arrow[r, phantom, shift left=3ex, "\Downarrow \theta_1"] \arrow[d, dash, dotted] & \mathscr{C} \arrow[d, dash, dotted] \\
\mathscr{A} \arrow[r, "H"'] & \mathscr{B} \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift right=3ex, "\Downarrow \theta_2"] & \mathscr{C}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathscr{A} \arrow[r, "H"] & \mathscr{B} \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "F"'] \arrow[r, phantom, "\Downarrow 1_F"] & \mathscr{B}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathscr{A} \arrow[r, bend left=50, "F \circ H"] \arrow[r, bend right=50, "F \circ H"'] \arrow[r, phantom, "\Downarrow 1_{F \circ H}"] & \mathscr{B}
\end{tikzcd}$$

Properties of the right action (2)

The two equations mean that

$$\begin{array}{rcccc} - \circ_R H & : & \mathbf{Trans}(\mathscr{B},\mathscr{C}) & \longrightarrow & \mathbf{Trans}(\mathscr{A},\mathscr{C}) \\ & & \theta & \mapsto & \theta \circ_R H \end{array}$$

defines a functor, while the two equations

$$\theta \circ_R (H_2 \circ H_1) \quad = \quad (\theta \circ_R H_2) \circ_R H_1 \qquad\qquad \theta \circ_R id_{\mathscr{A}} \quad = \quad \theta$$

mean that $\circ_R$ defines an action.

Compatibility of the left and right actions

Last equation: in the situation

$$\begin{tikzcd}
\mathscr{A}' \arrow[r, "H_1"] & \mathscr{A} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=D}] & \mathscr{B} \arrow[r, "H_2"] & \mathscr{B}' \arrow[from=U, to=D, phantom, "\Downarrow\theta"]
\end{tikzcd}$$

the order in which one makes the functors

$$H_1 \; : \; \mathscr{A}' \longrightarrow \mathscr{A} \qquad\qquad H_2 \; : \; \mathscr{B} \longrightarrow \mathscr{B}'$$

act on the transformation θ does not matter:

$$(H_2 \circ_L \theta) \circ_R H_1 \quad = \quad H_2 \circ_L (\theta \circ_R H_1)$$

Sesqui-category

A sesqui-category $\mathscr{D}$ is

$[0]$ a class of objects

$[1,2]$ equipped with a category

$$\mathscr{D}(A,B)$$

for every pair of objects (A,B) of the sesqui-category, where

$$\text{the objects of } \mathscr{D}(A,B) \quad = \quad \text{the morphisms from } A \text{ to } B$$

equipped with a pair of actions $\circ_L$ and $\circ_R$ satisfying...

Sesqui-categories

equipped with a pair of actions $\circ_L$ and $\circ_R$ satisfying the equations

$$\begin{array}{rcl@{\qquad}rcl}
h \circ_L (\theta_2 * \theta_1) & = & (h \circ_L \theta_2) * (h \circ_L \theta_1) & h \circ_L 1_f & = & 1_{h \circ f} \\
(h_1 \circ h_2) \circ_L f & = & h_1 \circ_L (h_2 \circ_L f) & id_{\mathscr{B}} \circ_L \theta & = & \theta \\
(\theta_2 * \theta_1) \circ_R h & = & (\theta_2 \circ_R h) * (\theta_1 *_R h) & 1_f \circ_R h & = & 1_{f \circ h} \\
\theta \circ_R (h_2 \circ h_1) & = & (\theta \circ_R h_2) \circ_R h_1 & \theta \circ_R id_{\mathscr{A}} & = & \theta
\end{array}$$

$$(h_2 \circ_L \theta) \circ_R h_1 = h_2 \circ_L (\theta \circ_R h_1)$$

Theorem.
Categories, functors and transformations define a sesqui-category.

The sesqui-category of categories and transformations

Let θ_1 and θ_2 be two transformations in

$$\begin{tikzcd}
\mathscr{A} \arrow[r, bend left, "F_1"] \arrow[r, bend right, "G_1"'] \arrow[r, phantom, "\Downarrow\theta_1"] & \mathscr{B} \arrow[r, bend left, "F_2"] \arrow[r, bend right, "G_2"'] \arrow[r, phantom, "\Downarrow\theta_2"] & \mathscr{C}
\end{tikzcd}$$

In general, the transformation obtained by applying θ_1 then θ_2

$$\begin{tikzcd}
\mathscr{A} \arrow[r, bend left, "F_1"] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\theta_1"] \arrow[d, dash, dotted] & \mathscr{B} \arrow[r, bend left, "F_2"] \arrow[d, dash, dotted] & \mathscr{C} \arrow[d, dash, dotted] \\
\mathscr{A} \arrow[r, bend right, "G_1"'] & \mathscr{B} \arrow[r, bend left] \arrow[r, bend right, "G_2"'] \arrow[r, phantom, "\Downarrow\theta_2"] & \mathscr{C}
\end{tikzcd}$$

is not the same as the transformation obtained by applying θ_1 then θ_2:

$$\begin{tikzcd}
\mathscr{A} \arrow[r, bend left, "F_1"] \arrow[d, dash, dotted] & \mathscr{B} \arrow[r, bend left, "F_2"] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\theta_2"] \arrow[d, dash, dotted] & \mathscr{C} \arrow[d, dash, dotted] \\
\mathscr{A} \arrow[r, bend left] \arrow[r, bend right, "G_1"'] \arrow[r, phantom, "\Downarrow\theta_1"] & \mathscr{B} \arrow[r, bend right, "G_2"'] & \mathscr{C}
\end{tikzcd}$$

Natural transformations

A transformation $\theta \quad : \quad F \Rightarrow G \quad : \quad \mathscr{A} \longrightarrow \mathscr{B}$
is **natural** when the diagram

$$\begin{tikzcd}
FA \arrow[r, "\theta_A"] \arrow[d, "Ff"'] & GA \arrow[d, "Gf"] \\
FB \arrow[r, "\theta_B"] & GB
\end{tikzcd}$$

commutes for every morphism $f : A \longrightarrow B$.

Notation. we write

$$\mathbf{Nat}(\mathscr{A}, \mathscr{B})$$

for the category of functors and natural transformations

$$\theta \quad : \quad F \quad \Rightarrow \quad G \quad : \quad \mathscr{A} \quad \longrightarrow \quad \mathscr{B}$$

Exchange law

A pair of 2-cells θ_1 and θ_2 in a sesqui-categorie $\mathscr{D}$

$$\begin{tikzcd}
A \arrow[r, bend left, "F_1"] \arrow[r, bend right, "G_1"'] \arrow[r, phantom, "\Downarrow\theta_1"] & B \arrow[r, bend left, "F_2"] \arrow[r, bend right, "G_2"'] \arrow[r, phantom, "\Downarrow\theta_2"] & C
\end{tikzcd}$$

satisfy the exchange law when the equality

$$\begin{tikzcd}
A \arrow[r, bend left, "F_1"] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\theta_1"] \arrow[d, dotted, no head] & B \arrow[r, bend left, "F_2"] \arrow[d, dotted, no head] & C \arrow[d, dotted, no head] \\
A \arrow[r, bend right, "G_1"'] & B \arrow[r, bend left] \arrow[r, bend right, "G_2"'] \arrow[r, phantom, "\Downarrow\theta_2"] & C
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, bend left, "F_1"] \arrow[d, dotted, no head] & B \arrow[r, bend left, "F_2"] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\theta_2"] \arrow[d, dotted, no head] & C \arrow[d, dotted, no head] \\
A \arrow[r, bend left] \arrow[r, bend right, "G_1"'] \arrow[r, phantom, "\Downarrow\theta_1"] & B \arrow[r, bend right, "G_2"'] & C
\end{tikzcd}$$

holds.

The order in which one applies θ_1 and θ_2 does not matter.

Definition

A 2-cell

$$\begin{tikzcd}
B \arrow[r, bend left, "f_2"] \arrow[r, bend right, "g_2"'] \arrow[r, phantom, "\Downarrow\theta_2"] & C
\end{tikzcd}$$

is called **central on the left** when the exchange law

$$\begin{tikzcd}
A \arrow[r, bend left, "f_1"] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\theta_1"] \arrow[d, dotted, no head] & B \arrow[r, bend left, "f_2"] \arrow[d, dotted, no head] & C \arrow[d, dotted, no head] & & A \arrow[r, bend left, "f_1"] \arrow[d, dotted, no head] & B \arrow[r, bend left, "f_2"] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\theta_2"] \arrow[d, dotted, no head] & C \arrow[d, dotted, no head] \\
A \arrow[r, bend right, "g_1"'] & B \arrow[r, bend left] \arrow[r, bend right, "g_2"'] \arrow[r, phantom, "\Downarrow\theta_2"] & C & = & A \arrow[r, bend left] \arrow[r, bend right, "g_1"'] \arrow[r, phantom, "\Downarrow\theta_1"] & B \arrow[r, bend right, "g_2"'] & C
\end{tikzcd}$$

is satisfied for every 2-cell θ_1 of the sesqui-category $\mathscr{D}$.

Exercise

Show that in the sesqui-category with

- categories as objects
- functors as 1-cells
- transformations as 2-cells

the natural transformations are the 2-cells central on the left.

Deduce the existence of a functor

$$\mathbf{Nat}(\mathscr{B},\mathscr{C}) \times \mathbf{Nat}(\mathscr{A},\mathscr{B}) \quad \longrightarrow \quad \mathbf{Nat}(\mathscr{A},\mathscr{C})$$

2-categories

A 2-category $\mathscr{D}$ is a sesqui-category such that the **exchange law** is satisfied for every pair of 2-cells

$$\begin{tikzcd}
A \arrow[r, bend left, "f_1"] \arrow[r, bend right, "g_1"'] \arrow[r, phantom, "\Downarrow\theta_1"] & B \arrow[r, bend left, "f_2"] \arrow[r, bend right, "g_2"'] \arrow[r, phantom, "\Downarrow\theta_2"] & C
\end{tikzcd}$$

2-categories (alternative definition)

A 2-category $\mathscr{D}$ is given by

$[0]$ a class of **objects**

$[1,2]$ a category $\mathscr{D}(A,B)$ for every pair of objects (A,B)

$[2,3,4]$ a **composition law** defined as a functor

$$\circ : \mathscr{D}(B,C) \times \mathscr{D}(A,B) \longrightarrow \mathscr{D}(A,C)$$

$[2,3,4]$ an **identity** defined as a functor

$$id_A : \mathbb{1} \longrightarrow \mathscr{D}(A,A)$$

this for all objects A, B, C of the 2-category,

2-categories (alternative definition)

1— such that the composition law $\circ$ is associative in the sense that

$$\begin{tikzcd}
\mathscr{D}(C,D) \times \mathscr{D}(B,C) \times \mathscr{D}(A,B) \arrow[r, "\circ \times \mathscr{D}(A,B)"] \arrow[d, "\mathscr{D}(C,D) \times \circ"'] & \mathscr{D}(B,D) \times \mathscr{D}(A,B) \arrow[d, "\circ"] \\
\mathscr{D}(C,D) \times \mathscr{D}(A,C) \arrow[r, "\circ"] & \mathscr{D}(A,D)
\end{tikzcd}$$

commutes.

2-categories (alternative definition)

2— such that id is a neutral element of $\circ$ in the sense that

$$\begin{tikzcd}
\mathscr{D}(A,B) \arrow[rr, equal] \arrow[d, "\cong"'] & & \mathscr{D}(A,B) \\
\mathscr{D}(A,B) \times \mathbb{1} \arrow[rr, "\mathscr{D}(A{,}B) \times id_A"] & & \mathscr{D}(A,B) \times \mathscr{D}(A,A) \arrow[u, "\circ"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathscr{D}(A,B) \arrow[rr, equal] \arrow[d, "\cong"'] & & \mathscr{D}(A,B) \\
\mathbb{1} \times \mathscr{D}(A,B) \arrow[rr, "id_B \times \mathscr{D}(A{,}B)"] & & \mathscr{D}(B,B) \times \mathscr{D}(A,B) \arrow[u, "\circ"']
\end{tikzcd}$$

commute for all A and B.

Notation

One writes

$$\theta \quad : \quad f \quad \Rightarrow \quad g \quad : \quad A \longrightarrow B$$

when

$$\theta \quad : \quad f \longrightarrow g$$

is a morphism of the category $\mathscr{D}(A, B)$.

Godement law

In a 2-category

$$\mathscr{D}(\mathscr{A},\mathscr{B})$$

the two canonical ways to compose the 2-cells

$$\begin{tikzcd}[column sep=huge]
\mathscr{A} \arrow[r, bend left=60, "f_1"] \arrow[r, "g_1" description] \arrow[r, bend right=60, "h_1"'] \arrow[r, phantom, bend left=30, "\Downarrow\alpha_1"] \arrow[r, phantom, bend right=30, "\Downarrow\beta_1"] & \mathscr{B} \arrow[r, bend left=60, "f_2"] \arrow[r, "g_2" description] \arrow[r, bend right=60, "h_2"'] \arrow[r, phantom, bend left=30, "\Downarrow\alpha_2"] \arrow[r, phantom, bend right=30, "\Downarrow\beta_2"] & \mathscr{C}
\end{tikzcd}$$

coincide:

$$(\beta_2 * \alpha_2) \circ (\beta_1 * \alpha_1) \quad = \quad (\beta_2 \circ \beta_1) * (\alpha_2 \circ \alpha_1)$$

Suspension

The notion of monoidal category will be defined very soon.

Every strict monoidal category $\mathscr{C}$ may be seen as the 2-category $\Sigma(\mathscr{C})$

▷ which contains only one 0-cell,

▷ whose 1-cells are the 0-cells of $\mathscr{C}$

▷ whose 2-cells are the 1-cells of $\mathscr{C}$

equipped with the induced composition laws.

A sesqui-category $\Sigma(\mathscr{C})$ with one object is
the same thing as a premonoidal category $(\mathscr{C}, \otimes, I)$.

Useful equality

In a 2-category $\mathscr{D}(\mathscr{A},\mathscr{B})$, the two canonical ways to compose the 2-cells

$$\begin{tikzcd}[column sep=huge]
\mathscr{A} \arrow[r, bend left=70, "f_1"] \arrow[r, phantom, bend left=35, "\Downarrow\alpha_1"] \arrow[r, "g_1" description] \arrow[r, phantom, bend right=35, "\Downarrow\beta_1"] \arrow[r, bend right=70, "h_1"'] & \mathscr{B} \arrow[r, bend left=70, "f_2"] \arrow[r, phantom, bend left=35, "\Downarrow\alpha_2"] \arrow[r, "g_2" description] \arrow[r, phantom, bend right=35, "\Downarrow\beta_2"] \arrow[r, bend right=70, "h_2"'] & \mathscr{C}
\end{tikzcd}$$

commute:

$$(\beta_2 * \alpha_2) \circ (\beta_1 * \alpha_1) \quad = \quad (\beta_2 \circ \beta_1) * (\alpha_2 \circ \alpha_1)$$

The 2-category of sets and relations

The 2-category $\mathscr{Rel}$ is defined as follows:

- its 0-cells are the sets,
- its 1-cells are the relations between sets,

$$\begin{tikzcd} A \arrow[r, "f\cdot g"] & B \end{tikzcd} \quad = \quad \begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, "g"] & C \end{tikzcd}$$

relationally composed:

$$a\,[f \cdot g]\,c \qquad \Longleftrightarrow \qquad \exists b \in B, \qquad a\,[f]\,b \quad \text{et} \quad b\,[g]\,c.$$

- its 2-cells are inclusions:

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow"] & B \end{tikzcd} \qquad \Longleftrightarrow \qquad f \subseteq g$$

In particular, the categories $\mathscr{Rel}(A, B)$ are order categories.

String diagrams

A notation introduced by Roger Penrose

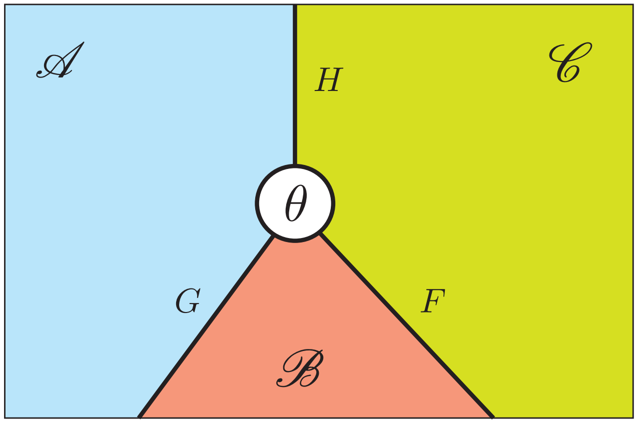
String diagrams

Two key ideas

1. apply the Poincaré duality on the original pasting diagrams:

$$\begin{tikzcd} & \mathscr{B} \arrow[dr, bend left, "G"] \arrow[d, phantom, "\Downarrow \theta"] & \\ \mathscr{A} \arrow[ur, bend left, "F"] \arrow[rr, bend right, "H"'] & {} & \mathscr{C} \end{tikzcd}$$

is depicted as



$$\boxed{\theta \quad : \quad G \circ F \quad \Rightarrow \quad H}$$

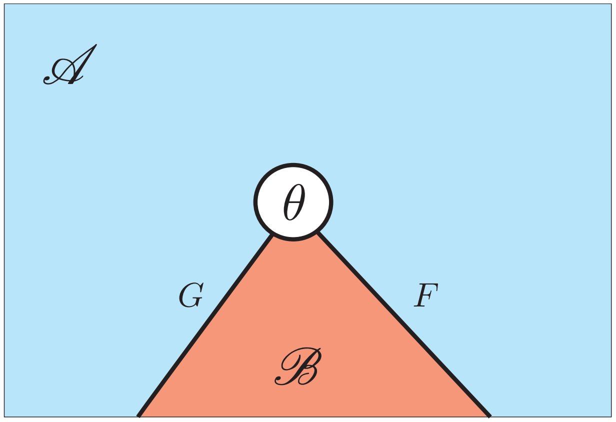
String diagrams

Two key ideas

2. hide the identity 1-cells in the picture:

$$\begin{tikzcd}
& \mathscr{B} \arrow[dr, bend left=20, "G"] & \\
\mathscr{A} \arrow[ur, bend left=20, "F"] \arrow[rr, bend right=15, "id"'] \arrow[rr, phantom, shift left=4, "\Downarrow\ \theta"] & & \mathscr{A}
\end{tikzcd}$$

is depicted as



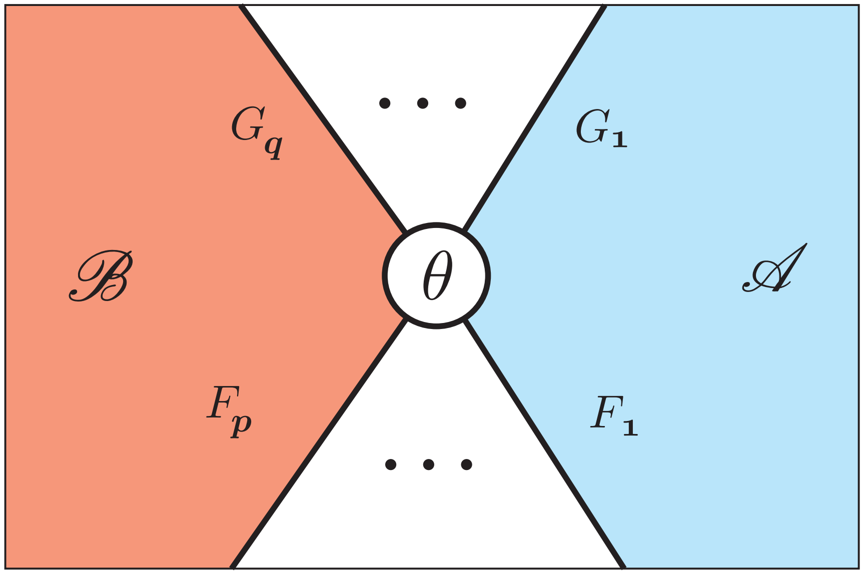
$$\boxed{\theta \quad : \quad G \circ F \quad \Rightarrow \quad id}$$

String diagrams

More generally, a 2-dimensional cell

$$\theta \quad : \quad F_1 \circ \cdots \circ F_p \quad \Rightarrow \quad G_1 \circ \cdots \circ G_q \quad : \quad \mathscr{A} \quad \longrightarrow \quad \mathscr{B}$$

is depicted as



Exercise

Draw the exchange law and explain the connection to concurrency

Adjunctions

A notion of duality between functors

Adjunction

An **adjunction** is a triple (L, R, ϕ) where L and R are two functors

$$L : \mathscr{A} \longrightarrow \mathscr{B} \qquad\qquad R : \mathscr{B} \longrightarrow \mathscr{A}$$

and ϕ is a family of bijections, for all objects A in $\mathscr{A}$ and B in $\mathscr{B}$,

$$\phi_{A,B} : \mathscr{B}(LA, B) \;\cong\; \mathscr{A}(A, RB)$$

natural in A et B. One also writes

$$\frac{LA \longrightarrow_{\mathscr{B}} B}{A \longrightarrow_{\mathscr{A}} RB} \quad \phi_{A,B}$$

One says that L **is left adjoint to** R, noted $L \dashv R$.

The 2-dimensional version of isomorphism

The naturality of the bijection ϕ

Natural in A and B means that the family of bijections

$$\phi_{A,B} \quad : \quad \mathscr{B}(LA, B) \cong \mathscr{A}(A, RB)$$

transforms every commutative diagram

$$\begin{tikzcd}
LA \arrow[r, "g"] & B \arrow[d, "h_B"] \\
LA' \arrow[u, "Lh_A"] \arrow[r, "f"'] & B'
\end{tikzcd}$$

into a commutative diagram

$$\begin{tikzcd}
A \arrow[r, "\phi_{A,B}(g)"] & RB \arrow[d, "Rh_B"] \\
A' \arrow[u, "h_A"] \arrow[r, "\phi_{A',B'}(f)"'] & RB'
\end{tikzcd}$$

Example: the free vector space

$$\begin{tikzcd}
\mathbf{Set} \arrow[rr, bend left, "L"] \arrow[rr, phantom, "\perp"] & & \mathbf{Vect} \arrow[ll, bend left, "R"]
\end{tikzcd}$$

where

$\mathscr{A} = \mathbf{Set}$: the category of sets and functions
$\mathscr{B} = \mathbf{Vect}$: the category of vector spaces on a field k

R : the « forgetful » functor $V \mapsto U(V)$
L : the « free vector space » functor $X \mapsto kX$

$$kX := \left\{ \sum_{x \in X} \lambda_x \, x \mid \lambda_x \in k \text{ null almost everywhere.} \right\}$$

Illustration: the tensor algebra

$$\begin{tikzcd}
\mathbf{Vect} \arrow[rr, bend left, "L"] \arrow[rr, phantom, "\perp"] & & \mathbf{Alg} \arrow[ll, bend left, "R"]
\end{tikzcd}$$

where

$\mathscr{A} = \mathbf{Vect}$: the category of vector spaces

$\mathscr{B} = \mathbf{Alg}$: the category of algebras and homomorphisms,

R : the « forgetful » functor $A \mapsto U(A)$.

L : the « free algebra » functor $V \mapsto TV$.

$$TV \quad := \quad \bigoplus_{n \in \mathbb{N}} V^{\otimes n}$$

Definition of a Lie algebra

Vector space $\mathfrak{g}$ equipped with a Lie bracket

Anti-symmetry:

$$[x, y] \quad = \quad -[y, x]$$

Jacobi identity:

$$[x, [y, z]] \quad + \quad [y, [z, x]] \quad + \quad [z, [x, y]] \quad = \quad 0$$

Example: the vector space of vector fields on a smooth manifold.

Illustration: the enveloping algebra of a Lie algebra

$$\begin{tikzcd}
\mathbf{Lie} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathbf{Alg} \arrow[l, bend left, "R"]
\end{tikzcd}$$

where

$\mathscr{A} = \mathbf{Lie}$: the category of Lie algebras,
$\mathscr{B} = \mathbf{Alg}$: the category of algebras,

R : equips A with the canonical Lie bracket $[a,b] = ab - ba$,
L : « enveloping algebra » functor $\mathfrak{g} \mapsto U(\mathfrak{g})$.

$$U(\mathfrak{g}) \quad := \quad T\mathfrak{g} \quad / \quad I(\mathfrak{g})$$

where $I(\mathfrak{g})$ is the ideal generated by $ab - ba - [a,b]$.

Illustration: the free category

$$\begin{tikzcd}
\mathbf{Graph} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathbf{Cat} \arrow[l, bend left, "R"]
\end{tikzcd}$$

where

$\mathscr{A} = \mathbf{Graph}$: the category of graphs,
$\mathscr{B} = \mathbf{Cat}$: the category of categories and functors,

R : the « forgetful » functor
L : the « free category » functor

Illustration : the terminal object

$$\begin{tikzcd}
\mathscr{C} \arrow[rr, bend left, "L"] \arrow[rr, phantom, "\perp"] & & \mathbb{1} \arrow[ll, bend left, "R"]
\end{tikzcd}$$

where

$\mathscr{A} = \mathscr{C}$: any category equipped with a terminal object 1
$\mathscr{B} = \mathbb{1}$: the singleton category

R : the functor whose image is the terminal object 1
L : the canonical (and unique) functor

Adjunction in the 2-category Cat

A bijection ϕ between the natural transformations

$$\begin{tikzcd}
 & \mathscr{A} \arrow[dd, "L"] \\
\mathbb{1} \arrow[ur, bend left, "{[A]}"] \arrow[dr, bend right, "{[B]}"'] \arrow[r, phantom, "\Downarrow"] & \\
 & \mathscr{B}
\end{tikzcd}
\quad \xmapsto{\phi_{A,B}} \quad
\begin{tikzcd}
 & \mathscr{A} \\
\mathbb{1} \arrow[ur, bend left, "{[A]}"] \arrow[dr, bend right, "{[B]}"'] \arrow[r, phantom, "\Downarrow"] & \\
 & \mathscr{B} \arrow[uu, "R"']
\end{tikzcd}$$

Here, a morphism $X \longrightarrow Y$ in the category $\mathscr{C}$
is seen as a natural transformation $[X] \longrightarrow [Y]$.

$$\begin{tikzcd}
\mathbb{1} \arrow[r, bend left, "{[X]}"] \arrow[r, bend right, "{[Y]}"'] \arrow[r, phantom, "\Downarrow"] & \mathscr{C}
\end{tikzcd}$$

Adjunction in the 2-category $\mathbf{Cat}$

A bijection ϕ between the natural transformations

$$\begin{tikzcd}
& \mathscr{A} \arrow[dd, "L"] & & & \mathscr{A} \\
\mathbb{1} \arrow[ru, bend left, "A"] \arrow[rd, bend right, "B"'] \arrow[r, phantom, "\Downarrow"] & {} & \overset{\phi_{A,B}}{\longmapsto} & \mathbb{1} \arrow[ru, bend left, "A"] \arrow[rd, bend right, "B"'] \arrow[r, phantom, "\Downarrow"] & {} \\
& \mathscr{B} & & & \mathscr{B} \arrow[uu, "R"']
\end{tikzcd}$$

Here, a morphism $X \longrightarrow Y$ in the category $\mathscr{C}$ seen as a natural transformation $[X] \longrightarrow [Y]$.

$$\begin{tikzcd}
\mathbb{1} \arrow[r, bend left, "X"] \arrow[r, bend right, "Y"'] \arrow[r, phantom, "\Downarrow"] & \mathscr{C}
\end{tikzcd}$$

A 2-dimensional naturality condition

One reformulates the naturality conditionin that way:

The bijection ϕ is natural with respect to the natural transformations α and β.

$$\begin{tikzcd}
 & & \mathscr{A} \arrow[dd, "L"] & & & & & \mathscr{A} \\
\mathbb{1} \arrow[r, dotted] \arrow[urr, bend left, "A'"] \arrow[drr, bend right, "B'"'] \arrow[ur, phantom, "\Downarrow\alpha"] \arrow[dr, phantom, "\Downarrow\beta"] & \mathbb{1} \arrow[ur, dotted, "A"] \arrow[dr, dotted, "B"'] \arrow[r, phantom, "\Downarrow\theta"] & {} & {} \arrow[r, mapsto, "\phi_{A',B'}"] & {} & \mathbb{1} \arrow[r, dotted] \arrow[urr, bend left, "A'"] \arrow[drr, bend right, "B'"'] \arrow[ur, phantom, "\Downarrow\alpha"] \arrow[dr, phantom, "\Downarrow\beta"] & \mathbb{1} \arrow[ur, dotted, "A"] \arrow[dr, dotted, "B"'] \arrow[r, phantom, "\Downarrow\zeta"] & {} \\
 & & \mathscr{B} & & & & & \mathscr{B} \arrow[uu, "R"']
\end{tikzcd}$$

Adjunction in the 2-category Cat

This point of view leads to a more satisfactory definition of adjunction:

A bijection ϕ between the natural transformations

$$\begin{tikzcd}
 & & \mathscr{A} \arrow[dd, "L"] & & & & \mathscr{A} \\
\mathscr{C} \arrow[urr, bend left, "A"] \arrow[drr, bend right, "B"'] & \Downarrow & & \xmapsto{\phi_{A,B}} & \mathscr{C} \arrow[urr, bend left, "A"] \arrow[drr, bend right, "B"'] & \Downarrow & \\
 & & \mathscr{B} & & & & \mathscr{B} \arrow[uu, "R"']
\end{tikzcd}$$

Adjunction in the 2-category Cat

One reformulates the naturality condition as follows:

The bijection ϕ is natural with respect to the natural transformations α et β.

$$\begin{tikzcd}
 & & \mathscr{A} \arrow[dd, "L"] & & & & \mathscr{A} \\
\mathscr{D} \arrow[r, dotted] \arrow[urr, bend left=50, "A'"] \arrow[drr, bend right=50, "B'"'] \arrow[ur, phantom, "\Downarrow\alpha"] \arrow[dr, phantom, "\Downarrow\beta"] & \mathscr{C} \arrow[ur, dotted, bend left=15, "A"] \arrow[dr, dotted, bend right=15, "B"'] \arrow[r, phantom, "\Downarrow\theta"] & & \xmapsto{\phi_{A',B'}} & \mathscr{D} \arrow[r, dotted] \arrow[urr, bend left=50, "A'"] \arrow[drr, bend right=50, "B'"'] \arrow[ur, phantom, "\Downarrow\alpha"] \arrow[dr, phantom, "\Downarrow\beta"] & \mathscr{C} \arrow[ur, dotted, bend left=15, "A"] \arrow[dr, dotted, bend right=15, "B"'] \arrow[r, phantom, "\Downarrow\zeta"] & \\
 & & \mathscr{B} & & & & \mathscr{B} \arrow[uu, "R"']
\end{tikzcd}$$

Algebraic presentation of the adjunction

An **adjonction** is a quadruple $(L, R, \eta, \varepsilon)$ where L and R are functors

$$L \quad : \quad \mathscr{A} \longrightarrow \mathscr{B} \qquad\qquad R \quad : \quad \mathscr{B} \longrightarrow \mathscr{A}$$

and η and ε are natural transformations:

$$\eta \quad : \quad Id_{\mathscr{A}} \overset{\cdot}{\longrightarrow} RL \qquad\qquad \varepsilon \quad : \quad LR \overset{\cdot}{\longrightarrow} Id_{\mathscr{B}}$$

such that the composite are the identities: (of L and R respectively).

$$\begin{tikzcd} R \arrow[r, "\eta R"] & RLR \arrow[r, "R\varepsilon"] & R \end{tikzcd} \qquad \begin{tikzcd} L \arrow[r, "L\eta"] & LRL \arrow[r, "\varepsilon F"] & L \end{tikzcd}$$

The situation is depicted as follows:

$$\begin{tikzcd}
\mathscr{A} \arrow[r, "L" description] \arrow[rr, bend left=50, "Id_{\mathscr{A}}"] \arrow[rr, phantom, shift left=4, "\Downarrow\eta"] & \mathscr{B} \arrow[r, "R" description] \arrow[rr, bend right=50, "Id_{\mathscr{B}}"'] \arrow[rr, phantom, shift right=4, "\Downarrow\varepsilon"] & \mathscr{A} \arrow[r, "L" description] & \mathscr{B}
\end{tikzcd}$$

Dual definition (but equivalent) of adjunction

By duality, an adjunction is given by a family of bijections ψ between the sets of 2-cells

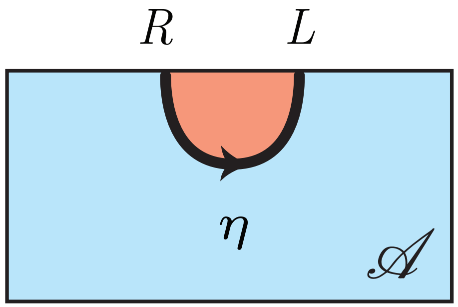
$$\begin{tikzcd}
\mathscr{A} \arrow[drr, bend left, "A"] \arrow[dd, "L"'] & & & & \mathscr{A} \arrow[drr, bend left, "A"] & & \\
 & \Downarrow\theta & \mathscr{C} & \overset{\psi_{A,B}}{\longmapsto} & & \Downarrow\zeta & \mathscr{C} \\
\mathscr{B} \arrow[urr, bend right, "B"'] & & & & \mathscr{B} \arrow[urr, bend right, "B"'] \arrow[uu, "R"] & &
\end{tikzcd}$$

natural in A and B.

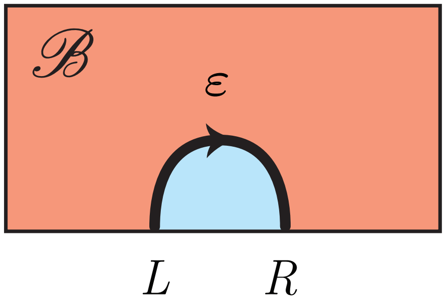
The 2-dimensional topology of adjunctions

The **unit** and **counit** of the adjunction $L \dashv R$ are depicted as

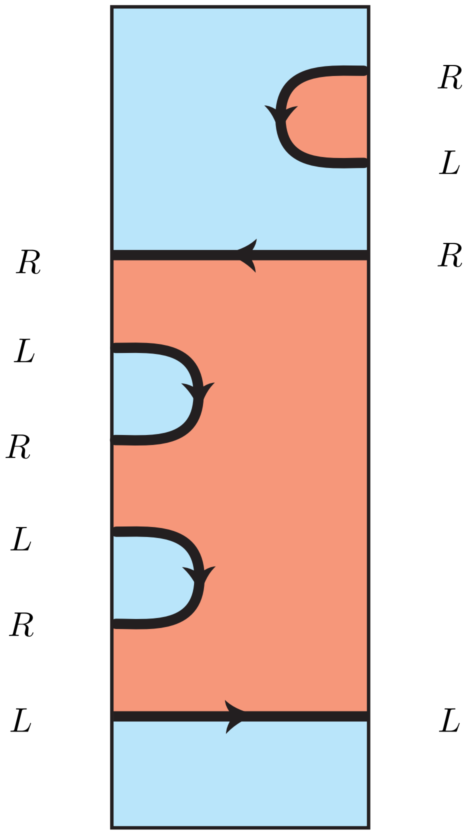
$$\eta : Id \Rightarrow R \circ L$$



$$\varepsilon : L \circ R \Rightarrow Id$$

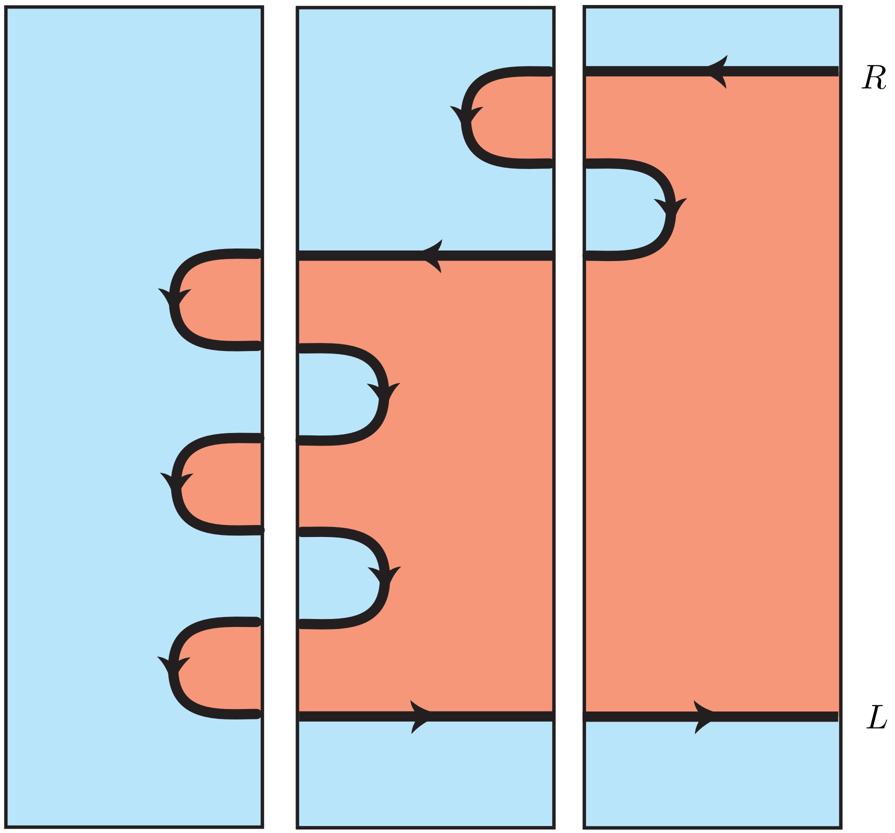


A typical 2-cell generated by an adjunction

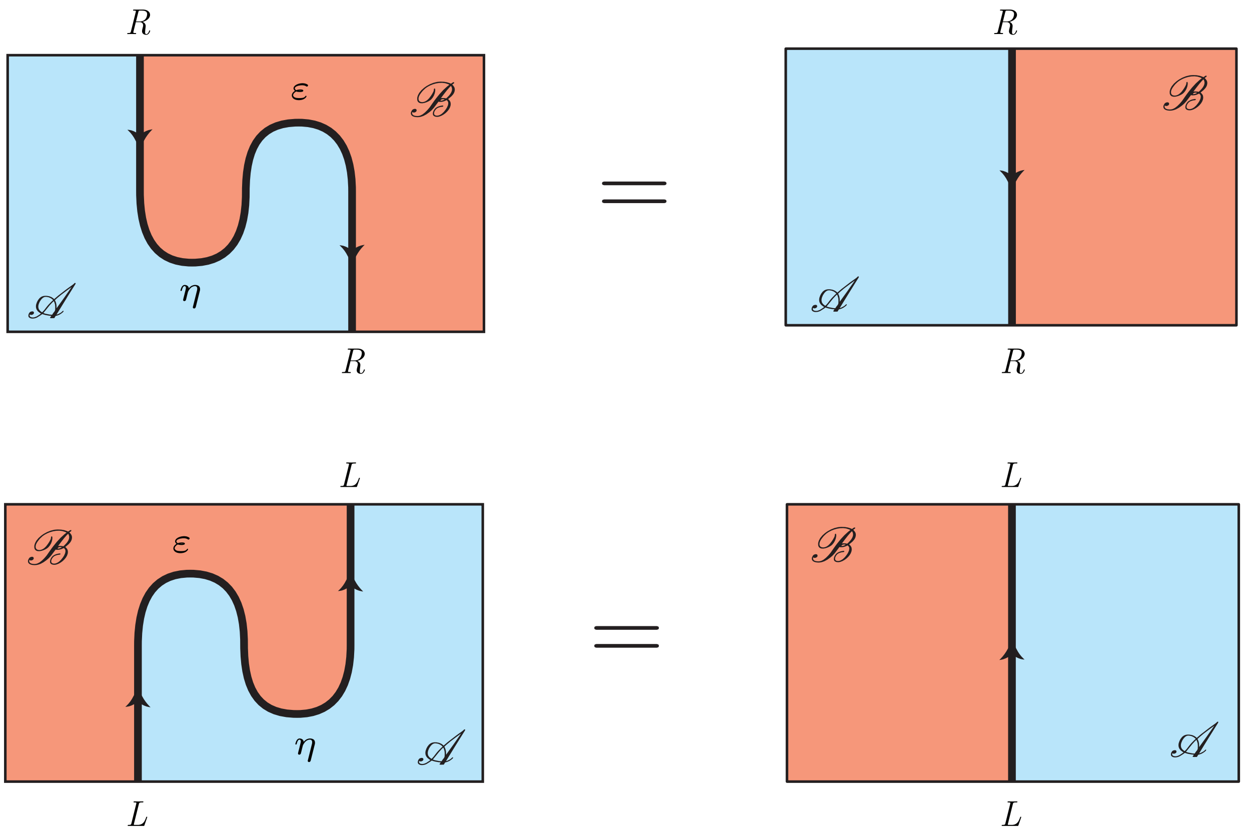


As we will see, deep connections with game semantics

A purely diagrammatic composition



The 2-dimensional dynamics of adjunctions



As we will see, deep connections with knot theory

Illustration: the 2-category of sets and relations

Show that a relation

$$f \quad : \quad A \quad \longrightarrow \quad B$$

is left adjoint if and only if it is functional:

$$\forall a \in A. \quad \exists ! b \in B. \qquad a\,[f]\,b$$

Show that its right adjoint g is the relation defined as

$$\forall a \in A. \quad \forall b \in B. \qquad a\,[f]\,b \quad \Longleftrightarrow \quad b\,[g]\,a.$$

Monads

Kleisli category, Eilenberg-Moore category

Monads

Suppose given a 0-cell $\mathscr{C}$ in a 2-category $\mathcal{W}$.

A monad T on a 0-cell $\mathscr{C}$ is a 1-cell

$$T \quad : \quad \mathscr{C} \quad \longrightarrow \quad \mathscr{C}$$

equipped with a multiplication

$$\mu \quad : \quad T \circ T \quad \Rightarrow \quad T \quad : \quad \mathscr{C} \quad \longrightarrow \quad \mathscr{C}$$

and with a unit

$$\eta \quad : \quad Id_{\mathscr{C}} \quad \Rightarrow \quad T \quad : \quad \mathscr{C} \quad \longrightarrow \quad \mathscr{C}$$

satisfying the expected associativity and unit laws.

Monads

- ▷ Associativity law:

$$\begin{tikzcd}
T \circ T \circ T \arrow[r, Rightarrow, "T\circ\mu"] \arrow[d, Rightarrow, "\mu\circ T"'] & T \circ T \arrow[d, Rightarrow, "\mu"] \\
T \circ T \arrow[r, Rightarrow, "\mu"] & T
\end{tikzcd}$$

- ▷ Left and right unit laws:

$$\begin{tikzcd}
& T \circ T \arrow[dr, Rightarrow, "\mu"] & \\
T \arrow[ur, Rightarrow, "\eta\circ T"] \arrow[rr, Rightarrow, "id"] & & T
\end{tikzcd}
\qquad
\begin{tikzcd}
& T \circ T \arrow[dr, Rightarrow, "\mu"] & \\
T \arrow[ur, Rightarrow, "T\circ\eta"] \arrow[rr, Rightarrow, "id"] & & T
\end{tikzcd}$$

Every adjunction defines a monad

(with a graphical proof)

Illustration: the state monad

Every set S induces a monad

$$X \quad \mapsto \quad S \Rightarrow (S \times X) \quad : \quad \mathbf{Set} \quad \longrightarrow \quad \mathbf{Set}$$

called **the state monad**. This monad is induced by the adjunction

$$\begin{tikzcd} \mathbf{Set} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathbf{Set} \arrow[l, bend left, "R"] \end{tikzcd}$$

where

$$\begin{array}{lcl} L & : & X \mapsto S \times X \\ R & : & X \mapsto S \Rightarrow X. \end{array}$$

Algebra

Suppose given a monad T on a category $\mathscr{C}$.

An algebra of the monad (T, μ, η) is a pair (A, h) consisting of

- an object A of the category $\mathscr{C}$
- a morphism

$$h \quad : \quad TA \longrightarrow A$$

making the diagrams

$$\begin{tikzcd}
 & TA \arrow[dr, "h"] & \\
A \arrow[ur, "\eta_A"] \arrow[rr, "id"'] & & A
\end{tikzcd}
\qquad
\begin{tikzcd}
T^2A \arrow[r, "\mu_A"] \arrow[d, "Th"'] & TA \arrow[d, "h"] \\
TA \arrow[r, "h"'] & A
\end{tikzcd}$$

commute.

Algebra homomorphism

An algebra homomorphism

$$f \quad : \quad (A, h_A) \quad \longrightarrow \quad (B, h_B)$$

is a morphism

$$f \quad : \quad A \quad \longrightarrow \quad B$$

making the diagram

$$\begin{tikzcd}
TA \arrow[r, "Tf"] \arrow[d, "h_A"'] & TB \arrow[d, "h_B"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

commute in the category $\mathscr{C}$.

Kleisli category

The Kleisli category $\mathscr{C}_T$ of a monad (T, μ, η) is the category $\mathscr{C}$

- ▷ with the same objects as the category $\mathscr{C}$,
- ▷ with the morphisms

$$A \longrightarrow TB$$

in the category $\mathscr{C}$ as morphisms

$$A \longrightarrow\!\!\!\!\!\rightarrow B$$

in the Kleisli category.

Kleisli category

The identities

$$id_A \quad : \quad A \twoheadrightarrow A$$

are given by the morphisms

$$\eta_A \quad : \quad A \longrightarrow TA.$$

The two morphisms

$$f : A \twoheadrightarrow B \qquad\qquad g : B \twoheadrightarrow C$$

are composed as follows

$$g \circ_K f \quad := \quad \begin{tikzcd}
 & & TTC \arrow[d, "\mu_C"] \\
 & TB \arrow[ur, "Tg"] & TC \\
A \arrow[ur, "f"] & B \arrow[ur, "g"] &
\end{tikzcd}$$

Exercise

Show that:

- that the identities of the Kleisli category are identities
- that its composition is associative.

Remark: checking associativity requires to consider the diagram

$$\begin{tikzcd}
& & & T^3D \arrow[d, "T\mu"'] \\
& & T^2C \arrow[ur, "T^2h"] \arrow[d, "\mu"'] & T^2D \arrow[d, "\mu"] \\
& TB \arrow[ur, "Tg"] & TC \arrow[ur, "Th"] & TD \\
A \arrow[ur, "f"] & B \arrow[ur, "g"] & C \arrow[ur, "h"] & D
\end{tikzcd}$$

and to show that the two maps from A to TD coincide.

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