Lambda calculs et catégories

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Synopsis of the lecture

1 – Lambda-calculus

2 – Concurrent graphs

3 – Proof of the finite developments theorem

First part

Lambda-calculus

The calculus of functions

The pure λ-calculus

Terms $$M \quad ::= \quad x \quad | \quad M\,N \quad | \quad \lambda x.\,M$$

The β-reduction:

$$(\lambda x.M)\,N \quad \longrightarrow \quad M\,[x := N]$$

The η-expansion:

$$M \quad \longrightarrow \quad \lambda x.\,(M\,x)$$

Remark: every term is considered up to renaming $\equiv_\alpha$ of the bound variables, typically:

$$\lambda x.\lambda y.x \quad \equiv_\alpha \quad \lambda z.\lambda y.z$$

Occurrences

The set of occurrences of a λ-term M is defined by induction:

- $\mathbf{occ}\,(x) \quad = \quad \{\,\varepsilon\,\}$
- $\mathbf{occ}\,(MN) \quad = \quad \{\,\varepsilon\,\} \cup \{\,1 \cdot o \,|\, o \in \mathbf{occ}\,(M)\,\} \cup \{\,2 \cdot o \,|\, o \in \mathbf{occ}\,(N)\,\}$
- $\mathbf{occ}\,(\lambda x.M) \quad = \quad \{\,\varepsilon\,\} \cup \{\,1 \cdot o \,|\, o \in \mathbf{occ}\,(M)\,\}$

Note that every occurrence of the λ-term M is labelled by

- an application node App
- a binder λx
- a variable x

Free variables

The set of **free variables** of a λ-term is defined by induction:

- $FV(x) = \{x\}$
- $FV(MN) = FV(M) \cup FV(N)$
- $FV(\lambda x.M) = FV(M) \setminus \{x\}$

Every occurrence of a variable x in a λ-term is

- either free
- or bound by a binder λx above it in the λ-term.

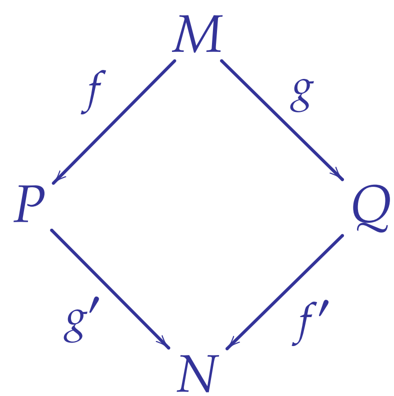
Church-Rosser theorem

Also called confluence theorem.

Given two β-rewriting paths

$$f \ : \ M \xrightarrow{*} P \qquad\qquad g \ : \ M \xrightarrow{*} Q$$

there exists a λ-term N and two β-rewriting paths f' and g' completing the diagram as



Redex

Definition. A β-redex is a pair

$$(M, o)$$

consisting of

- ▷ a λ-term M
- ▷ an occurrence of the λ-term M such that

$$M_{|o} \quad = \quad (\lambda x.P)\, Q$$

is a β-reduction pattern.

Residuals

Given a pair of β-redexes

$$u : M \to P \qquad\qquad v : M \to Q$$

from the same λ-term M with occurrences o_u and o_v

$$M_{|o_u} = (\lambda x.A)B \qquad\qquad M_{|o_v} = (\lambda y.C)D$$

we would like to define the **residuals** of v along u in the λ-term P.

We write in that case

$$\boxed{v [\![u]\!] w}$$

The idea is that the computation of v in M has been postponed to the computation of its residuals in Q.

Residuals [case 1]

There are three possibilities to consider:

▷ the β-redex v occurs inside the function A of the β-redex u.

In that case, the occurrence o_v factors as

$$o_v = o_u \cdot 1 \cdot 1 \cdot o'$$

and the β-redex v has a unique residual w with occurrence

$$o_w = o_u \cdot o'$$

Residuals [case 2]

▷ the β-redex v occurs inside the argument B of the β-redex u.

In that case, the occurrence o_v factors as

$$o_v \quad = \quad o_u \cdot 2 \cdot o'$$

Let

$$\{ o_1 , \ldots , o_k \}$$

denote the set of occurrences of the variable x in the λ-term A.

In that case, the β-redex v has a residual w_i with occurrence

$$o_{w_i} \quad = \quad o_u \cdot o_i \cdot o'$$

for each occurrence of the variable x in the λ-term A.

Residuals [case 3]

▷ the β-redex v does not occur inside the β-redex u.

In that case, the β-redex v has a unique residual w along u.

The β-redex w has occurrence

$$o_w \quad = \quad o_v$$

in the λ-term P.

Unique ancestor property

Property.

Suppose that u, v, v' are β-redexes

$$u : M \to P \qquad v : M \to Q \qquad v : M \to Q'$$

and that w is a β-redex in P.

The residual relation satisfies

$$v \llbracket\, u \,\rrbracket\, w \quad \text{and} \quad v' \llbracket\, u \,\rrbracket\, w \quad \Rightarrow \quad v = v'$$

Every path $f : M \to N$ thus defines a partial function from the redexes of N to the redexes of M

Residuals along a path

Given a β-rewriting path

$$f \quad : \quad M \quad \xrightarrow{*} \quad P$$

and a β-redex

$$v \quad : \quad M \quad \longrightarrow \quad Q$$

from the same λ-term M, one defines

$$v \, [\![\, f \,]\!] \, w$$

by induction on the length of the path f, as follows:

- $\triangleright \qquad v \, [\![\, id_M \,]\!] \, w \quad \Longleftrightarrow \quad v = w$
- $\triangleright \qquad v \, [\![\, u \cdot f \,]\!] \, w \quad \Longleftrightarrow \quad \exists \, v', \quad v \, [\![\, u \,]\!] \, v'$ and $v' \, [\![\, f \,]\!] \, w$

Sets of residuals

Given a β-rewriting path

$$f \quad : \quad M \quad \xrightarrow{*} \quad P$$

and a finite set

$$V \quad = \quad \{\, v_1 \,, \, \ldots \,, \, v_n \,\}$$

of β-redexes from the same λ-term M, one defines

$$V [\![f]\!] \quad = \quad \{ \quad w \quad | \quad \exists v \in V, \quad v [\![f]\!] w \quad \}$$

Development

A development

$$P_1 \xrightarrow{u_1} P_2 \xrightarrow{u_2} \cdots \xrightarrow{u_{n-1}} P_n \xrightarrow{u_n} \cdots$$

of a finite set of β-redexes

$$V$$

is a possibly infinite rewriting path where

$$\forall n, \qquad u_n \in V [\![f_n]\!]$$

for the rewriting path f_n defined as

$$f_n \quad = \quad u_1 \cdots u_{n-1}$$

Finite developments

Suppose given a finite set

$$V \quad = \quad \{ \, v_1 \, , \, \ldots \, , \, v_n \, \}$$

of β-redexes starting from the same λ-term M.

Key property [**Finite developments - termination**]

Every development of V is finite.

Permutation tiles

Definition.

A permutation tile consists of a pair of paths

$$f = u \cdot h_v : M \stackrel{*}{\longrightarrow} N \qquad\qquad g = v \cdot h_u : M \stackrel{*}{\longrightarrow} N$$

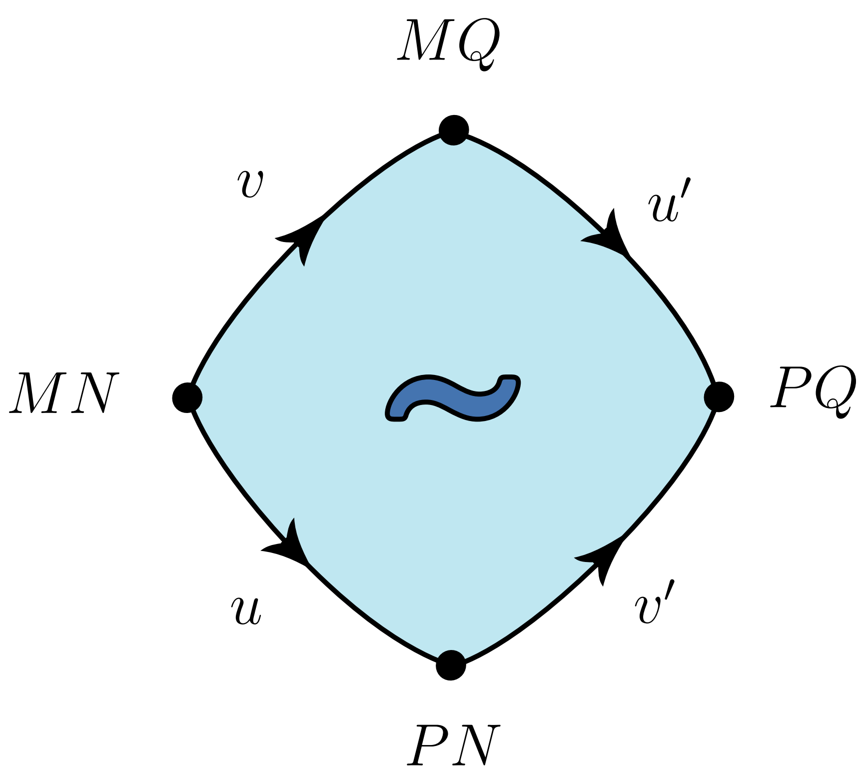
such that

$$u : M \to P \qquad\qquad v : M \to Q$$

are two β-redexes starting from the same λ-term M and moreover

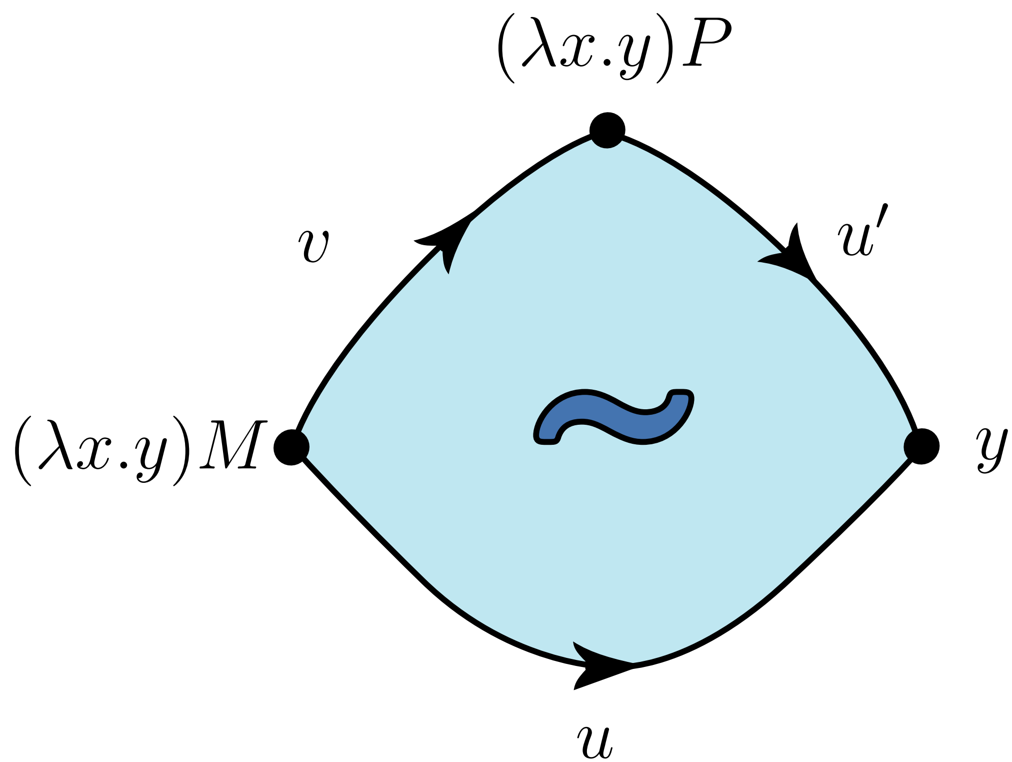
- h_u is a development of the residuals of u along v
- h_v is a development of the residuals of v along u
- h_u and h_v have the same target N.

Illustration



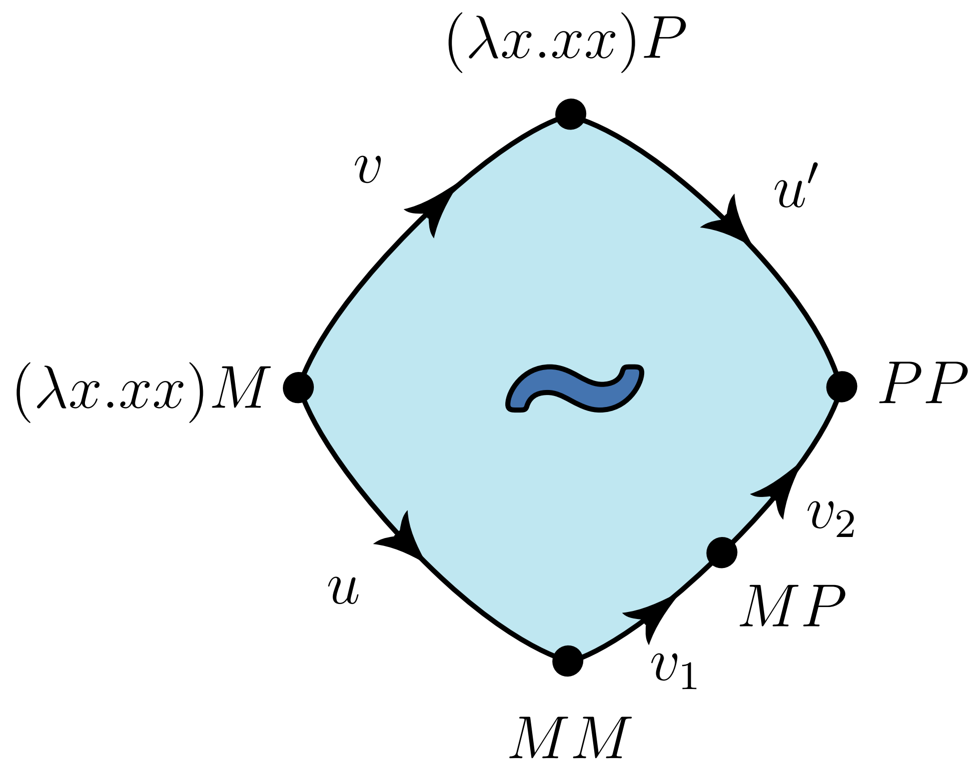
The two redexes $u : M \rightarrow P$ and $v : N \rightarrow Q$ are disjoint.

Illustration



The redex u erases the redex $v : M \to P$.

Illustration



The redex u duplicates the redex $v : M \to P$.

Local confluence

Key property [**Finite developments - confluence**]

For every pair of different β-redexes

$$u : M \to P \qquad\qquad v : M \to Q$$

in the same term M, there exists a permutation tile

$$f = u \cdot h_v \qquad\qquad g = v \cdot h_u$$

such that, moreover, the two paths

$$f, g \quad : \quad M \stackrel{*}{\longrightarrow} N$$

define the same residual relation:

$$[\![f]\!] \quad = \quad [\![g]\!]$$

between the β-redexes of M and the β-redexes of N.

Finite developments

Theorem [**Finite developments**]

Every two developments

$$f : M \xrightarrow{*} N \qquad\qquad g : M \xrightarrow{*} N'$$

of the same finite set of β-redexes

$$V \quad = \quad \{ v_1 \, , \, \ldots \, , \, v_n \}$$

reach the same λ-term

$$N \quad = \quad N'$$

and define the same residual relation

$$[\![\, f \,]\!] \quad = \quad [\![\, g \,]\!] \, .$$

Permutation equivalence

Given two rewriting paths

$$d, e \quad : \quad P \quad \stackrel{*}{\longrightarrow} \quad Q$$

we write

$$d \quad \stackrel{1}{\sim} \quad e$$

where there exists a permutation tile

$$f = u \cdot h_v : M \stackrel{*}{\longrightarrow} N \qquad g : v \cdot h_u : M \stackrel{*}{\longrightarrow} N$$

such that

$$d = P \stackrel{d_1}{\longrightarrow} M \stackrel{f}{\longrightarrow} N \stackrel{d_2}{\longrightarrow} Q \qquad e = P \stackrel{d_1}{\longrightarrow} M \stackrel{g}{\longrightarrow} N \stackrel{d_2}{\longrightarrow} Q$$

Permutation equivalence

Given two rewriting paths

$$d, e : P \xrightarrow{*} Q$$

we write

$$d \sim e$$

when there exists a sequence of permutations

$$d \quad \overset{1}{\sim} \quad f_1 \quad \overset{1}{\sim} \quad \cdots \quad \overset{1}{\sim} \quad f_n \quad \overset{1}{\sim} \quad e$$

transforming the rewriting path d into the path e.

Permutation equivalence continued

Proof of the finite developments theorem.

Every two developments

$$f, g \quad : \quad M \quad \xrightarrow{*} \quad N$$

of the same set V of β-redexes are equivalent

$$f \quad \sim \quad g.$$

Main consequence.

Every two equivalent paths define the same residual relation

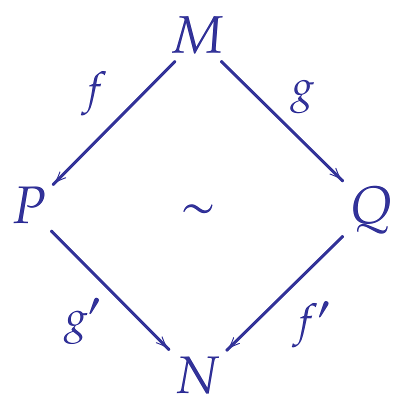
$$f \sim g \quad \Rightarrow \quad [\![f]\!] = [\![g]\!]$$

Algebraic Church-Rosser Theorem

Given two β-rewriting paths

$$f \; : \; M \xrightarrow{*} P \qquad\qquad g \; : \; M \xrightarrow{*} Q$$

there exists a λ-term N and two β-rewriting paths f' and g' completing the diagram as



Key property established by Jean-Jacques Lévy in 1978

Rewriting paths modulo permutations

An important problem of rewriting theory: compare the several paths which rewrite **a λ-term P into its normal form Q.**

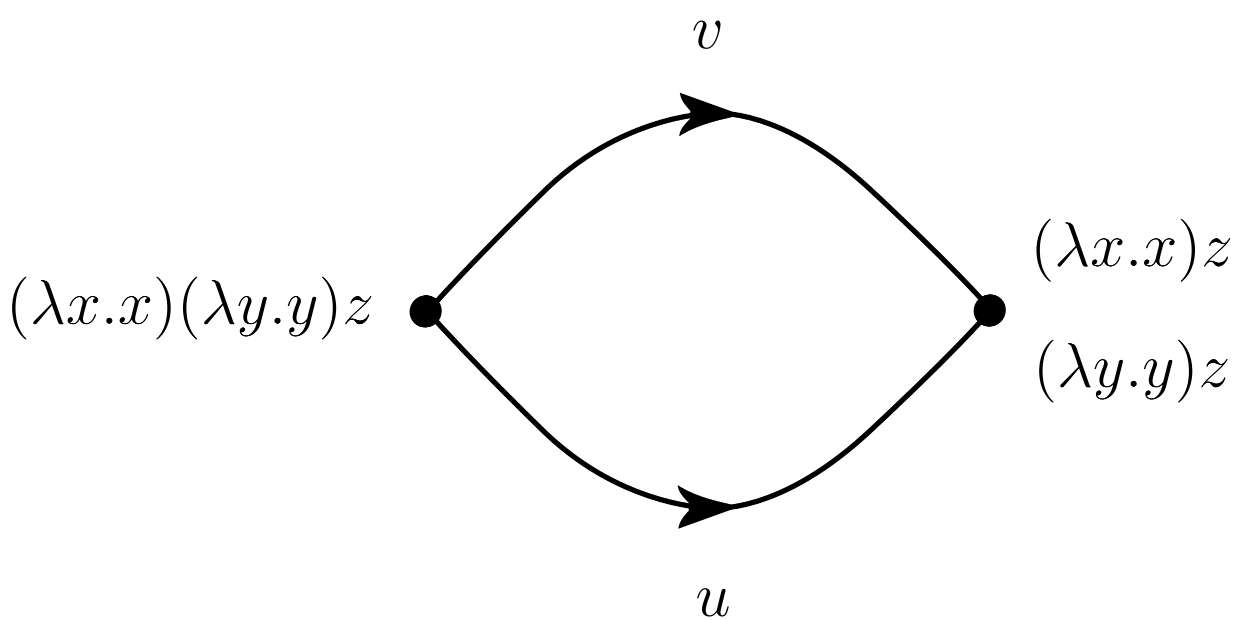
Corollary

Every two rewriting paths to the normal form

$$f, g \quad : \quad P \quad \longrightarrow \quad Q$$

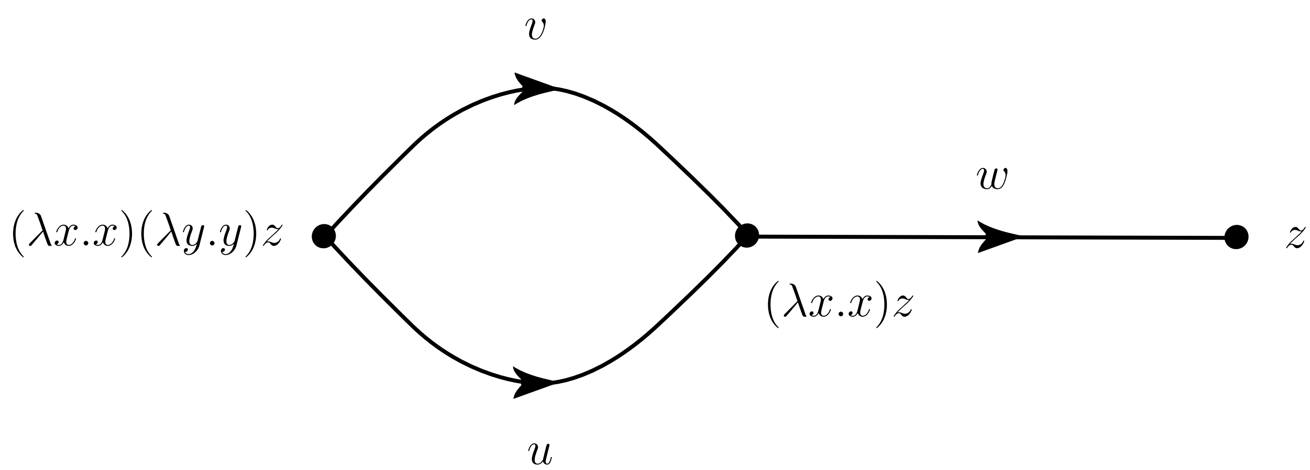
are equal modulo a series of redex permutations.

A 2-dimensional hole



The two redexes u and v are not equivalent modulo permutation.

The 2-dimensional hole continued



The two paths $u \cdot w$ and $v \cdot w$ are equivalent modulo permutation.

Pushouts in categories

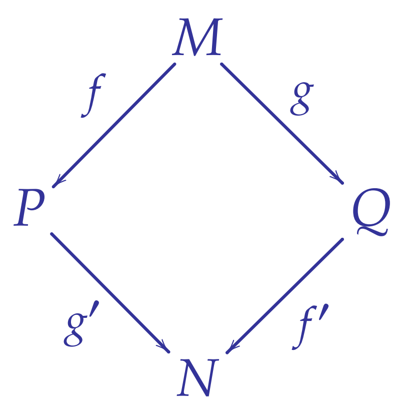
In a category $\mathscr{C}$, the pushout of a pair of morphisms

$$f : M \longrightarrow P \qquad g : M \longrightarrow Q$$

is a pair of morphisms

$$g' : P \longrightarrow N \qquad f' : Q \longrightarrow N$$

such that the resulting diagram



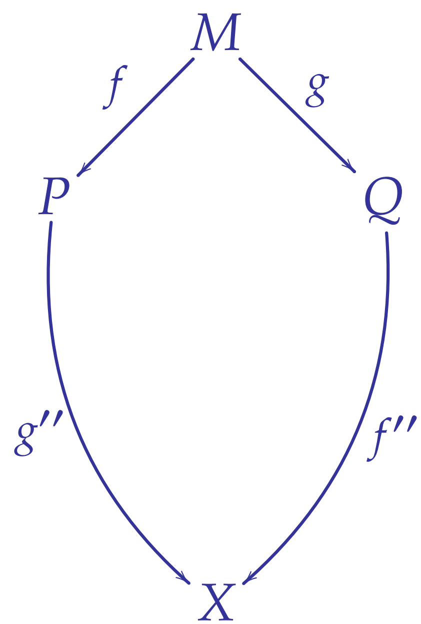
commutes and moreover...

Pushouts in categories

... for every pair of morphisms

$$g'' : P \longrightarrow X \qquad\qquad f'' : Q \longrightarrow X$$

making the diagram



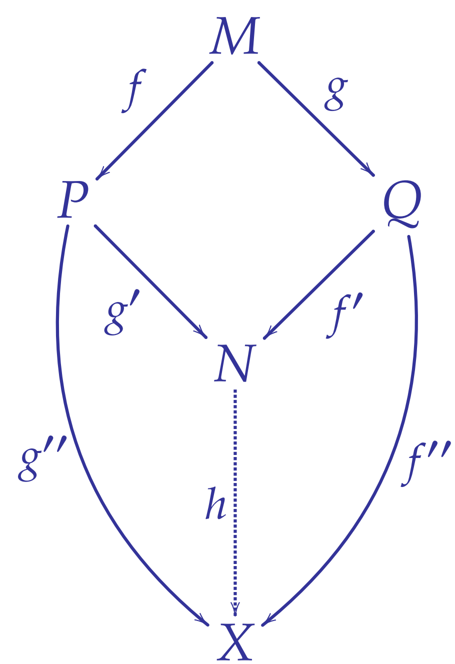
commute in the category $\mathscr{C}$ ...

Pushouts in categories

... there exists a **unique** morphism

$$h \quad : \quad N \quad \longrightarrow \quad X$$

making the diagram



commute in the category $\mathscr{C}$.

A reformulation of the Church-Rosser theorem

We will consider the category $\mathscr{C}_\lambda$ with

- ▷ λ-terms as objects,
- ▷ rewriting paths modulo permutation $\sim$ as morphisms.

Theorem [**Levy 1978, Huet-Levy 1981**]

The category $\mathscr{C}_\lambda$ has pushouts.

This property holds for every rewriting system without critical pairs.

Second part

Concurrent graphs

Confluence formulated as a pushout property

Reflexive graphs

A reflexive graph $\mathscr{G}$ is given by

- ▷ a set of vertices V
- ▷ a set of edges E
- ▷ a source and a target function $\partial_0, \partial_1 : E \to V$
- ▷ an identity function $\varnothing : V \to E$ such that

$$\partial_0\,(\varnothing_A) = A \qquad\qquad \partial_1\,(\varnothing_A) = A$$

this meaning that the edge $\varnothing_A$ connects the vertex A to itself:

$$\varnothing_A \quad : \quad A \quad \longrightarrow \quad A$$

Concurrent graphs

A **concurrent graph** is a reflexive graph $\mathscr{G}$ with a symmetric relation

$$f \quad \diamond \quad g$$

between coinitial and cofinal paths f and g of length 2, satisfying the following axioms.

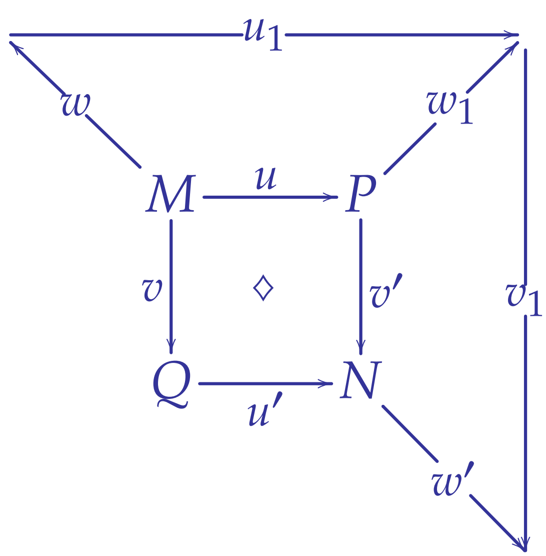
Axiom 1. [**Unique residual**]

$$\begin{array}{ccc} M & \xrightarrow{u} & P \\ {\scriptstyle v}\downarrow & \diamond & \downarrow{\scriptstyle v'} \\ Q & \xrightarrow[u']{} & N \end{array} \quad \text{and} \quad \begin{array}{ccc} M & \xrightarrow{u} & P \\ {\scriptstyle v}\downarrow & \diamond & \downarrow{\scriptstyle v''} \\ Q & \xrightarrow[u'']{} & O \end{array} \quad \text{implies} \quad u' = u'' \text{ and } v' = v''$$

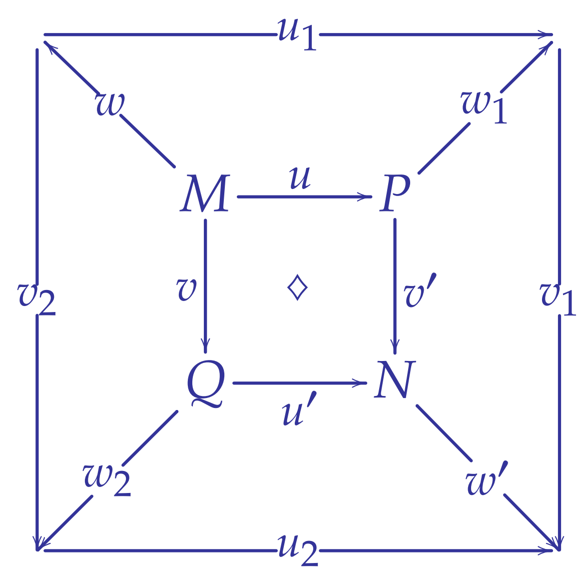
If $u \cdot v' \diamond v \cdot u'$ and $u \cdot v'' \diamond v \cdot u''$ then $v' = v''$ and $u' = u''$

Concurrent graphs

Axiom 2. **[Cube axiom]**



implies



If $u \cdot v' \diamond v \cdot u'$ and $w \cdot u_1 \diamond u \cdot w_1$ and $w_1 \cdot v_1 \diamond v' \cdot w'$,
then there exists edges w_2, v_2, u_2
such that $w \cdot v_2 \diamond v \cdot w_2$ and $w_2 \cdot u_2 \diamond u' \cdot w'$ and $u_1 \cdot v_1 \diamond v_2 \cdot u_2$

Concurrent graphs

Axiom 3.

$$\begin{array}{ccc} M & \xrightarrow{u} & N \\ {\scriptstyle u}\downarrow & \diamond & \downarrow \\ N & \longrightarrow & P \end{array} \quad \text{implies} \quad \begin{array}{ccc} M & \xrightarrow{u} & N \\ {\scriptstyle u}\downarrow & \diamond & \downarrow{\scriptstyle \varnothing_N} \\ N & \xrightarrow[\varnothing_N]{} & N \end{array}$$

If $u : M \longrightarrow N$ and $u \cdot v \diamond u \cdot v$ then $v = \varnothing_N$

Concurrent graphs

Axiom 4.

$$\begin{array}{ccc} M & \xrightarrow{\;u\;} & N \\ {\scriptstyle \varnothing_M}\Big\downarrow & \diamond & \Big\downarrow \\ M & \longrightarrow & N \end{array} \qquad \text{implies} \qquad \begin{array}{ccc} M & \xrightarrow{\;u\;} & N \\ {\scriptstyle \varnothing_M}\Big\downarrow & \diamond & \Big\downarrow{\scriptstyle \varnothing_N} \\ M & \xrightarrow[\;u\;]{} & N \end{array}$$

If $u : M \longrightarrow N$ and $u \cdot v \diamond \varnothing_M \cdot u'$ then $u = u'$ and $v = \varnothing_N$

Conflict-free graphs

Axiom 5.

For every pair of coinitial edges

$$u : M \longrightarrow P \qquad\qquad v : M \longrightarrow Q$$

there exists a pair of cofinal edges

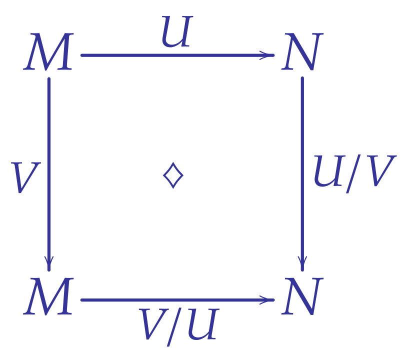
$$v' : P \longrightarrow N \qquad\qquad u' : Q \longrightarrow N$$

such that

$$u \cdot v' \quad \diamond \quad v \cdot u'$$

The conflict-free graph $\mathscr{G}_\lambda$

- its vertices are the λ-terms
- its edges are the multi-redexes.
- its diamonds are of the form



where

$$U/V \quad = \quad U[\![\, V \,]\!] \quad = \quad \{ \quad u' \quad | \quad \exists\, u \in U, \quad u\,[\![\, V \,]\!]\, u' \quad \}$$

Residual of a path after an edge

Every edge

$$u \quad : \quad M \quad \longrightarrow \quad N$$

defines a function

$$f \quad \mapsto \quad f \,/\, u$$

from M-paths to N-paths, defined by induction on the length of f as

▷ $\qquad id_M \,/\, u \quad = \quad id_N$

▷ $\qquad (v \cdot g) \,/\, u \quad = \quad (v/u) \,\cdot\, g \,/\, (u/v)$

Residual of a path after a path

Every path

$$f \quad : \quad M \quad \longrightarrow \quad N$$

defines a function

$$h \quad \mapsto \quad h \,/\, f$$

from M-paths to N-paths, defined as

- ▷ $$h \,/\, id_M \quad = \quad h$$

- ▷ $$h \,/\, (v \cdot g) \quad = \quad (h/v) \,/\, g$$

Permutation equivalence

Given two paths

$$d \;,\; e \quad : \quad P \quad \xrightarrow{*} \quad Q$$

one writes

$$d \quad \overset{1}{\approx} \quad e$$

when

$$d = P \xrightarrow{d_1} M \xrightarrow{f} N \xrightarrow{d_2} Q \qquad\qquad e = P \xrightarrow{d_1} M \xrightarrow{g} N \xrightarrow{d_2} Q$$

and

$$f \diamond g$$

or when

$$f = id_M \qquad \text{and} \qquad g = \varnothing_M.$$

Permutation equivalence

Given two paths

$$d \;,\; e \quad : \quad P \quad \xrightarrow{*} \quad Q$$

one writes

$$d \quad \approx \quad e$$

when there exists a sequence of permutations

$$d \quad \overset{1}{\approx} \quad f_1 \quad \overset{1}{\approx} \quad \cdots \quad \overset{1}{\approx} \quad f_n \quad \overset{1}{\approx} \quad e$$

transforming the rewriting path d into the path e.

Two structural properties

Suppose given three paths f, g, h starting from the same edge M.

First property.

$$f \approx g \quad \Rightarrow \quad h/f = h/g$$

Second property.

$$f \approx g \quad \Rightarrow \quad f/h \approx g/h$$

Algebraic confluence theorem

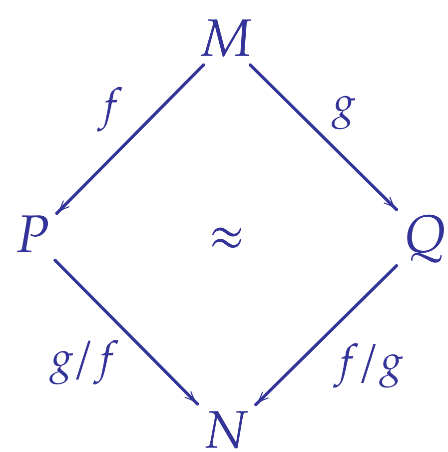
Given two rewriting paths

$$f \;:\; M \xrightarrow{*} P \qquad\qquad g \;:\; M \xrightarrow{*} Q$$

the two rewriting paths $f\,/\,g$ and $g\,/\,f$ satisfy the equation

$$f \cdot (g/f) \quad \approx \quad g \cdot (f/g)$$

and thus complete the diagram as



The rewriting category

The rewriting category $\mathscr{C}$ is defined as the category

- ▷ the vertices of $\mathscr{G}$ as objects
- ▷ the rewriting paths of $\mathscr{G}$ modulo $\approx$ as morphisms.

Every rewriting path is an epi

Property.

Given an edge

$$u \quad : \quad M \quad \longrightarrow \quad P$$

and two rewriting paths

$$f \ , \ g \quad : \quad M \quad \longrightarrow \quad P$$

one has

$$u \cdot f \approx u \cdot g \quad \Rightarrow \quad f \approx g$$

This means that u is an epimorphism in the category $\mathscr{C}$.

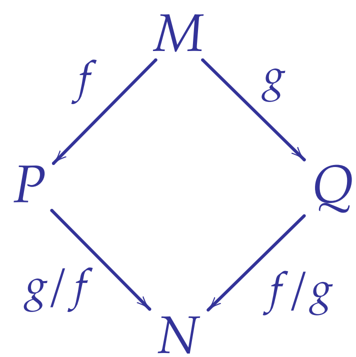
Pushouts

Theorem.

Every pair of morphisms (= rewriting paths modulo permutation)

$$f : M \longrightarrow P \qquad\qquad g : M \longrightarrow Q$$

defines a pushout diagram



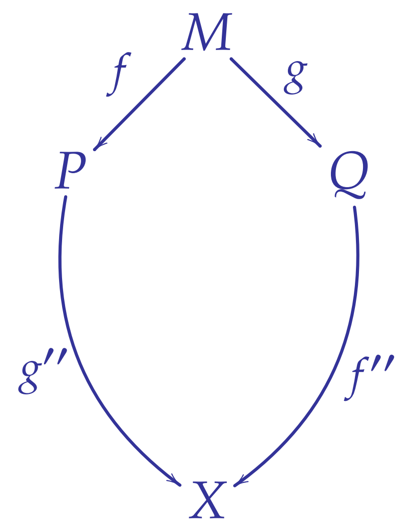
in the rewriting category $\mathscr{C}$.

Pushouts in categories

Indeed, for every pair of morphisms

$$g'' : P \longrightarrow X \qquad\qquad f'' : Q \longrightarrow X$$

making the diagram



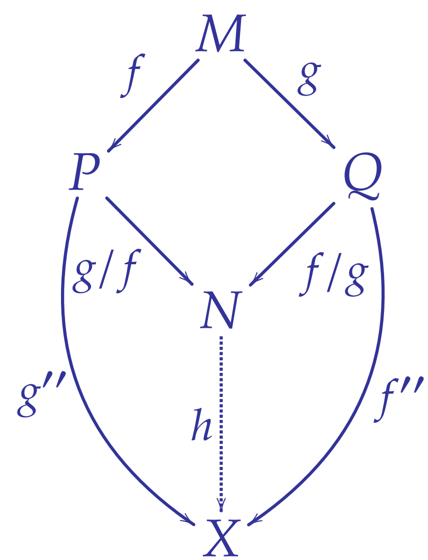
commute in the category $\mathscr{C}$...

Pushouts in categories

... there exists a **unique** morphism

$$h \quad : \quad N \quad \longrightarrow \quad X$$

making the diagram



commute in the category $\mathscr{C}$, given by the equation

$$h \quad \approx \quad (f \cdot g'') \,/\, (f \cdot (g/f)) \quad \approx \quad (g \cdot f'') \,/\, (g \cdot (f/g))$$

A reformulation of the Church-Rosser theorem

The category associated to $\mathscr{G}_\lambda$ coincides with $\mathscr{C}_\lambda$ with

- ▷ λ-terms as objects
- ▷ rewriting paths modulo permutation $\sim$ as morphisms.

Theorem [**Levy 1978, Huet-Levy 1981**]

The category $\mathscr{C}_\lambda$ has pushouts.

This property holds for every rewriting system without critical pairs.

Illustration : Jordan-Hölder theorem

A subgroup H of a group G is **normal** when

$$aH \quad = \quad Ha$$

for every element $a \in G$. One writes in that

$$H \lhd G$$

A group is **simple** when it contains no normal subgroup except $\{e\}$ and itself.

Fact. Groups form a conflict-free graph where an edge

$$G \longrightarrow H$$

indicates that

▷ H is (isomorphic to) a normal subgroup of G

▷ the subgroup G/H is simple.

Shapes of the Jordan-Hölder tiles

$$\begin{array}{ccc} G & \xrightarrow{G/H} & H \\ {\scriptstyle G/K}\big\downarrow & \diamond & \big\downarrow{\scriptstyle H/(H\cap K)} \\ K & \xrightarrow[K/(H\cap K)]{} & H \cap K \end{array}$$

Note in particular that

$$G/H \quad = \quad K/(H \cap K) \qquad\qquad G/K \quad = \quad H/(H \cap K)$$

Butterfly lemma

Let H, H', K, K' be four subgroups of a group G such that

$$H' \lhd H \qquad\qquad K' \lhd K$$

In that case

$$H' \cdot (H \cap K') \quad \lhd \quad H' \cdot (H \cap K)$$

$$K' \cdot (K \cap H') \quad \lhd \quad K' \cdot (K \cap H)$$

Moreover, the quotient groups

$$H' \cdot (H \cap K) \quad / \quad H' \cdot (H \cap K')$$

$$K' \cdot (H \cap K) \quad / \quad K' \cdot (H' \cap K)$$

are isomorphic.

Jordan-Hölder theorem

Deduce the Jordan-Hölder theorem that two normal towers

$$\{e\} \triangleleft G_n \triangleleft \cdots \triangleleft G_1 \triangleleft G$$

$$\{e\} \triangleleft H_n \triangleleft \cdots \triangleleft H_1 \triangleleft H$$

have the same length when

$$G = H$$

and every quotient group

$$G_i/G_{i+1} \qquad H_i/H_{i+1}$$

is simple and non trivial.

Third part

A proof of the finite developments theorem

A purely combinatorial argument

Nesting ordering

Given a pair of β-redexes

$$u : M \longrightarrow P \qquad\qquad v : M \longrightarrow Q$$

from the same λ-term M with occurrences o_u and o_v

$$M_{|o_u} = (\lambda x.A)B \qquad\qquad M_{|o_v} = (\lambda y.C)D$$

we declare that v is nested by u and write

$$u \quad \prec \quad v$$

when v lies in the argument B of the redex u. In that case,

$$o_v \quad = \quad o_u \cdot 2 \cdot o'$$

for some occurrence o'.

Gripping

We declare that u grips v and write

$$u \quad \ll \quad v$$

when

- ▷ the β-redex v lies in the functional body A of the β-redex u.
- ▷ the argument D of the β-redex v contains an occurrence
  . of the variable x bound by the β-redex u.

Property 1

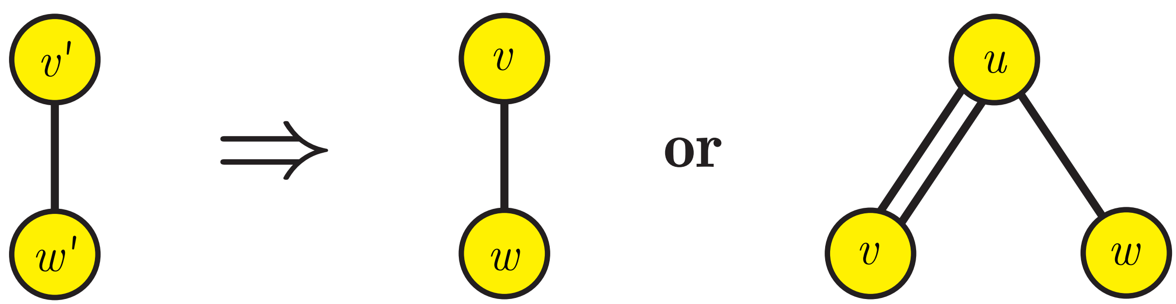
Given a β-redex

$$u \quad : \quad P \quad \longrightarrow \quad Q$$

and four β-redexes v, w, v', w' such that

$$v [\![u]\!] v' \qquad \text{and} \qquad w [\![u]\!] w'$$

satisfy the following property:



$$v' < w' \quad \Rightarrow \quad v < w \quad \text{or} \quad (\, u \ll v \quad \text{and} \quad u < w \,)$$

Property 2

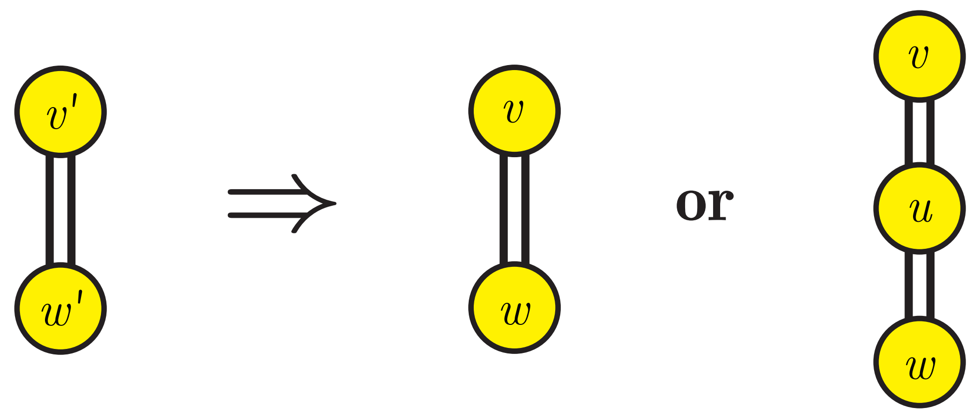
Given a β-redex

$$u \quad : \quad P \quad \longrightarrow \quad Q$$

and four β-redexes v, w, v', w' such that

$$v \llbracket u \rrbracket v' \qquad \text{and} \qquad w \llbracket u \rrbracket w'$$

satisfy the following property:



$$v' \ll w' \quad \Rightarrow \quad v \ll w \quad \text{or} \quad v \ll u \ll w$$

A depth on β-redexes

Definition

The depth of a β-redex u in a finite set V of β-redexes

$$|u|_V$$

is defined as the maximal length of gripping sequences

$$u \quad \ll \quad v_1 \quad \ll \quad \cdots \ll \quad v_n$$

starting from u and consisting of elements of V.

The depth decreases

Suppose that u is a β-redex of V with

$$v \llbracket u \rrbracket w$$

and with set of residuals

$$W \quad = \quad V \llbracket u \rrbracket .$$

Property. The depth of v decreases in the sense that

$$|w|_W \quad \leq \quad |v|_V.$$

Proof. By applying Property 2.

A norm on β-redexes

Definition

The norm of a β-redex u in a finite set V of β-redexes

$$\| u \|_V$$

is defined as the multiset of depths

$$\{ \ |u|_V \quad | \quad u \prec v \quad \}$$

of all the β-redexes u in V which nest the β-redex v.

The norm decreases

Suppose that u is a β-redex of V with

$$v \llbracket u \rrbracket w$$

and with set of residuals

$$W \quad = \quad V \llbracket u \rrbracket \,.$$

Property. The norm of v decreases in the sense that

$$\| w \|_W \quad \leq_{mset} \quad \| v \|_V .$$

Moreover, this norm strictly decreases when $u \prec v$.

Proof. By applying Property 1.

Finite developments

The norm $\| V \|$ of a finite set of β-redexes V is defined as the multiset

$$\{ \quad \| v \|_V \quad | \quad v \in V \quad \}$$

Property. Suppose that u is a β-redex of a finite set

$$V$$

of β-redexes with set of residuals

$$W \quad = \quad V [\![u]\!] .$$

In that case,

$$\| W \| \quad <_{mset} \quad \| V \|.$$

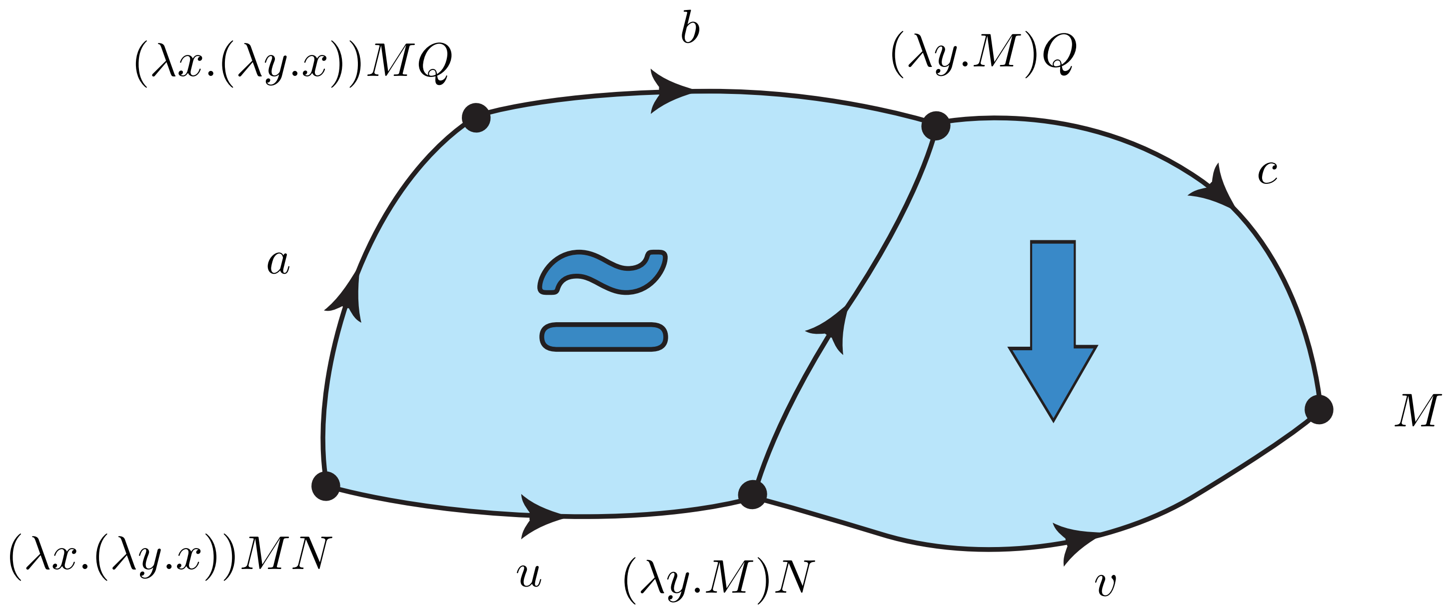
Corollary [**Finite Developments**]

All the developments of a finite set V of β-redexes are finite.

Third part

Standardisation theorem

Geometry of rewriting



Key idea: let us rewrite the rewriting paths !

Left-to-right vs. right-to-left strategy

Consider the λ-term

$$(\lambda x.\lambda y.x)\ a\ (\Delta\Delta)$$

constructed by applying the first projection to a and to $\Delta\Delta$.

The right-to-left strategy

$$(\lambda x.\lambda y.x)\ a\ \underline{(\Delta\Delta)} \quad \longrightarrow_\beta \quad (\lambda x.\lambda y.x)\ a\ \underline{(\Delta\Delta)} \quad \longrightarrow_\beta \quad \cdots$$

computes for ever while the left-to-right strategy

$$\underline{(\lambda x.\lambda y.x)\ a}\ (\Delta\Delta) \quad \longrightarrow_\beta \quad \underline{(\lambda y.a)\ (\Delta\Delta)} \quad \longrightarrow_\beta \quad a$$

terminates.

Irreversible tiles

We write

$$f \quad \blacktriangleright \quad g$$

when

$$f \quad = \quad v \cdot u' \qquad\qquad g \quad = \quad u \cdot h_v$$

are two developments of the pair of β-redexes $\{u, v\}$ where

$$u \ : \ M \longrightarrow P \qquad\qquad v \ : \ M \longrightarrow Q$$

and moreover

$$u \prec v$$

Reversible tiles

We write

$$f \quad \diamond \quad g$$

when

$$f \quad = \quad v \cdot u' \qquad\qquad g \quad = \quad u \cdot v'$$

are two developments of the pair of β-redexes $\{u, v\}$ where

$$u \;:\; M \longrightarrow P \qquad\qquad v \;:\; M \longrightarrow Q$$

and moreover

$$u \,\|\, v$$

where $u \,\|\, v$ means that the two β-redexes are disjoint:

$$\neg\,(\,u \prec v) \qquad \text{and} \qquad \neg\,(\,v \prec u)$$

Standardisation tiles

We write

$$f \quad \triangleright \quad g$$

when

$$f \quad = \quad v \cdot u' \qquad\qquad g \quad = \quad u \cdot h_v$$

are two developments of the pair of β-redexes $\{u, v\}$ where

$$u \ : \ M \longrightarrow P \qquad\qquad v \ : \ M \longrightarrow Q$$

and moreover

$$u \prec v \qquad \text{or} \qquad u \parallel v$$

Standardisation path

Given two rewriting paths

$$d\,,\,e \quad : \quad P \quad \xrightarrow{\ *\ } \quad Q$$

we write

$$d \ \stackrel{1}{\Rightarrow}\ e$$

when d and e factor as

$$d \ = \ P \xrightarrow{d_1} M \xrightarrow{f} N \xrightarrow{d_2} Q \qquad\qquad e \ = \ P \xrightarrow{d_1} M \xrightarrow{g} N \xrightarrow{d_2} Q$$

where f and g are related by a standardization tile:

$$f = v \cdot u' \qquad \triangleright \qquad g = u \cdot h_v$$

Standardisation paths

Given two rewriting paths

$$d \,,\, e \quad : \quad P \quad \xrightarrow{*} \quad Q$$

we write

$$d \quad \Rightarrow \; e$$

when there exists a sequence of standardization steps

$$d \quad \overset{1}{\Rightarrow} \quad f_1 \quad \overset{1}{\Rightarrow} \quad \cdots \quad \overset{1}{\Rightarrow} \quad f_n \quad \overset{1}{\Rightarrow} \quad e$$

transforming the rewriting path d into the rewriting path e.

In that case, one says that the path e is **more standard** than d.

Reversible standardization paths

A standardization path

$$\theta \quad : \quad d \quad \Rightarrow \quad e$$

is called **reversible** when all the standardization steps

$$d \quad \overset{1}{\Rightarrow} \quad f_1 \quad \overset{1}{\Rightarrow} \quad \cdots \quad \overset{1}{\Rightarrow} \quad f_n \quad \overset{1}{\Rightarrow} \quad e$$

are reversible. In that case, we write

$$\theta \quad : \quad d \quad \simeq \quad e$$

Standard path

A rewriting path

$$f \quad : \quad M \quad \xrightarrow{*} \quad N$$

is called **standard** when every standardization path

$$\theta \quad : \quad f \quad \Rightarrow \quad g$$

starting from the rewriting path f is reversible:

$$\theta \quad : \quad f \quad \simeq \quad g$$

Standardization theorem

Existence. From every rewriting path

$$f \quad : \quad M \quad \stackrel{*}{\longrightarrow} \quad N$$

there exists a standardization path

$$\theta \quad : \quad f \quad \Rightarrow \quad g \quad : \quad M \quad \stackrel{*}{\longrightarrow} \quad N$$

which transforms the path f into a standard path g.

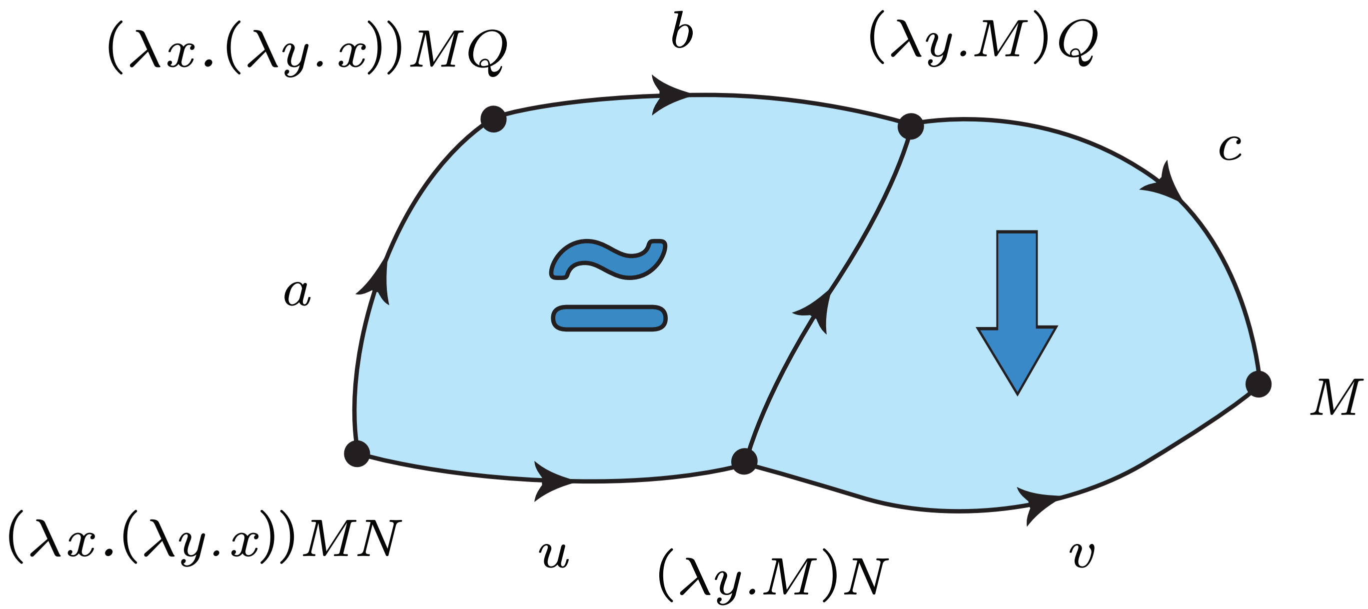
Uniqueness. Every two standard paths

$$f\,,\,g \quad : \quad M \quad \stackrel{*}{\longrightarrow} \quad N$$

equivalent modulo general β-redex permutation are equivalent modulo **reversible** permutation

$$f \sim g \quad \Rightarrow \quad f \simeq g.$$

Illustration



Here, the path $a \cdot b \cdot c$ is transformed into the standard path $u \cdot v$.

$$a \cdot b \cdot c \quad \Rightarrow \quad u \cdot w \cdot c \quad \Rightarrow \quad u \cdot v$$

Idea of the proof

To every non-empty rewriting path

$$f \quad : \quad M \quad \xrightarrow{*} \quad N$$

one associates a β-redex

$$\mathbf{outermost}\,(f) \quad : \quad M \quad \longrightarrow \quad P$$

defined by induction on the length of the rewriting path:

$\triangleright$ $\quad \mathbf{outermost}\,(u) \quad = \quad u$

$\triangleright$ $\quad \mathbf{outermost}\,(u \cdot g) \quad = \quad \begin{cases} v & \text{when} \quad v \prec_{left-outer} u \\ & \text{and} \quad v \,[\![\, u \,]\!]\, \mathbf{outermost}\,(g) \\ u & \text{otherwise} \end{cases}$

Leftmost-outermost ordering

One writes

$$u \prec_{left-outer} v$$

when the occurrence

$$o_u$$

of the β-redex u is smaller than the occurrence

$$o_v$$

of the β-redex v in the lexicographic order:

$$o_u \quad \leq_{lex} \quad o_v.$$

This defines a **total** ordering on the β-redex of a λ-term.

Leftmost-outermost ordering

Typically, the β-redex

$$u \quad : \quad \underline{(\lambda x.(\lambda y.x))\,a}\ (\Delta\Delta) \quad \longrightarrow \quad (\lambda y.a)\,(\Delta\Delta)$$

is left-outer to the β-redex

$$v \quad : \quad (\lambda x.(\lambda y.x))\ a\ \underline{(\Delta\Delta)} \quad \longrightarrow \quad (\lambda x.(\lambda y.x))\,a\ (\Delta\Delta)$$

in the λ-term

$$(\lambda x.(\lambda y.x))\ a\ (\Delta\Delta)$$

Key lemma

Lemma.

Suppose that the two non empty rewriting paths

$$f\,,\,g \quad : \quad M \quad \xrightarrow{*} \quad N$$

are related by a standardisation path

$$\theta \quad : \quad f \quad \Rightarrow \quad g$$

In that case,

$$\mathbf{outermost}\,(f) \quad = \quad \mathbf{outermost}\,(g).$$

Proof. The property holds in the case of a standardisation step

$$f \quad \triangleright \quad g$$

and this particular case induces the general case.

Corollary

Corollary.

Suppose that the rewriting path

$$f \quad : \quad P \quad \xrightarrow{*} \quad N$$

is standard and that

$$u \quad = \quad \textbf{outermost}\,(u \cdot f)$$

for a β-redex

$$u \quad : \quad M \quad \longrightarrow \quad P$$

In that case, the rewriting path

$$u \cdot f \quad : \quad M \quad \xrightarrow{*} \quad N$$

is standard.

Standardisation algorithm

Given a rewriting path

$$f \quad : \quad M \quad \xrightarrow{*} \quad N$$

▷ extract the β-redex

$$u = \mathbf{outermost}\,(f) \quad : \quad M \quad \longrightarrow \quad P$$

from the path f by applying a series of standardisation steps

▷ apply the standardisation algorithm on any rewriting path

$$g \quad : \quad P \quad \xrightarrow{*} \quad N$$

obtained as a residual of f after $\mathbf{outermost}\,(f)$.

Theorem. The algorithm terminates and produces a standard path.

Termination of the algorithm

Imagine that the standardization algorithm applied to the path

$$f \quad : \quad M \quad \xrightarrow{*} \quad N$$

produces an infinite sequence $\quad u_1 \cdots u_n \cdots \quad$ of β-redexes.

Suppose moreover that f is **minimal in length** among such paths.

In that case, the rewriting path f factors as

$$f \quad = \quad v \cdot g \quad : \quad M \xrightarrow{v} P \xrightarrow{*} N$$

where the standardisation algorithm **terminates** on the path g.

By definition, there exists a natural number N such that

$$u_{N+1} \quad \cdots \; u_{N+p} \quad \cdots$$

is a development of the set of residuals of v after the path $u_1 \cdots u_N$.

This contradicts the finite development theorem.

Uniqueness

Notation. Given a rewriting path

$$f \quad : \quad M \quad \stackrel{*}{\longrightarrow} \quad N$$

we write

$$\mathbf{std}\,(f) \quad : \quad M \quad \stackrel{*}{\longrightarrow} \quad N$$

for the standard path obtained as result of the algorithm.

Property. For every two rewriting paths

$$f, g \quad : \quad M \quad \stackrel{*}{\longrightarrow} \quad N$$

one has:

$$f \sim g \quad \Rightarrow \quad \mathbf{std}\,(f) = \mathbf{std}\,(g)$$

$$f \text{ standard} \quad \Rightarrow \quad f \simeq \mathbf{std}\,(f)$$

Standardization theorem

Existence. From every rewriting path

$$f \quad : \quad M \quad \overset{*}{\longrightarrow} \quad N$$

there exists a standardization path

$$\theta \quad : \quad f \quad \Rightarrow \quad g \quad : \quad M \quad \overset{*}{\longrightarrow} \quad N$$

which transforms the path f into a standard path g.

Uniqueness. Every two standard paths

$$f\,,\,g \quad : \quad M \quad \overset{*}{\longrightarrow} \quad N$$

equivalent modulo **general** β-redex permutation are equivalent modulo **reversible** permutation

$$f \sim g \quad \Rightarrow \quad f \simeq g.$$

A sesqui-category of β-rewriting

The sesqui-category $\mathscr{L}_{sesqui}$ has

- the λ-terms as objects,
- the β-rewriting paths $f : M \xrightarrow{*} N$ as morphisms,
- the standardization paths $\theta : f \Rightarrow g$ as 2-cells.

Ancestors along a standardisation path

Every standardisation path

$$\theta \quad : \quad f = u_1 \ldots u_m \quad \Rightarrow \quad g = v_1 \ldots v_n$$

induces a function

$$[\theta] \quad : \quad [1, \ldots, n] \quad \longrightarrow \quad [1, \ldots, m]$$

which relates every β-redex of the rewriting path f

$$v_j$$

to a β-redex of the rewriting path g

$$u_{[\theta](j)}$$

called its **ancestor** along the standardisation path θ.

A 2-category of β-rewriting

The 2-category $\mathscr{L}$ has

- ▷ the λ-terms as objects,
- ▷ the β-rewriting paths $f : M \stackrel{*}{\longrightarrow} N$ as morphisms,
- ▷ the standardization paths $\theta : f \Rightarrow g$ modulo $\cong$ as 2-cells.

Here, two standardization paths

$$\theta_1 \, , \, \theta_2 \quad : \quad f \quad \Rightarrow \quad g \quad : \quad M \quad \stackrel{*}{\longrightarrow} \quad N$$

are related by $\cong$ precisely when they define the same function

$$[\theta_1] \quad = \quad [\theta_2]$$

Standardization theorem revisited

Existence. From every rewriting path

$$f \;:\; M \xrightarrow{*} N$$

there exists a **unique** standardization 2-cell

$$\theta \;:\; f \;\Rightarrow\; g \;:\; M \xrightarrow{*} N$$

which transforms the path f into a standard path g.

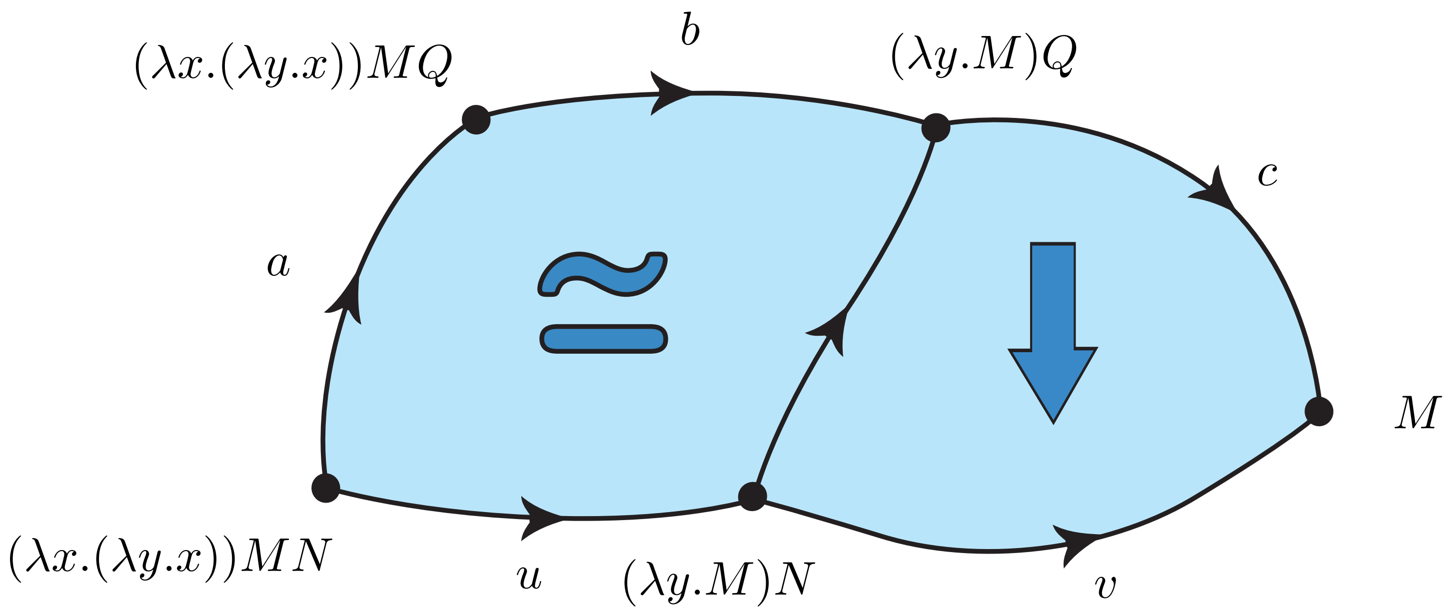
Uniqueness. Every two standard paths

$$f\,,\,g \;:\; M \xrightarrow{*} N$$

equivalent modulo general β-redex permutation are equivalent modulo a **reversible** 2-cell

$$f \sim g \;\Rightarrow\; f \simeq g.$$

Illustration

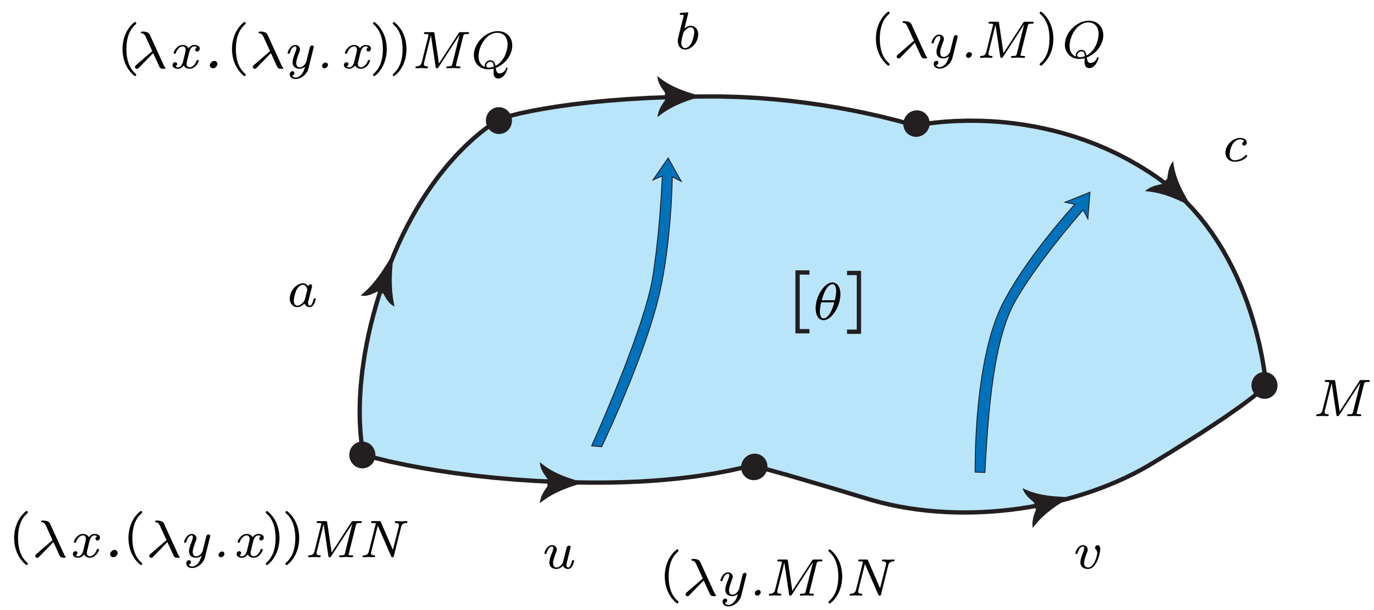


Here, the standardisation cell

$$\theta \quad : \quad a \cdot b \cdot c \quad \Rightarrow \quad u \cdot w \cdot c \quad \Rightarrow \quad u \cdot v$$

transforms the path $a \cdot b \cdot c$ into the standard path $u \cdot v$.

Illustration



Note that the standardisation cell θ « erases » the β-redex a in the sense that the β-redex a is not in the image of $[\theta]$.