Lambda calculs et catégories

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Synopsis of the lecture

1 – General presentation of the course

2 – Categories and functors

3 – Cartesian categories

General introduction

Programming language semantics

A **mathematical** study of programming languages
and of their compilation schemes

High-level languages built on the λ-calculus as kernel:

PCF	Algol	ML	OCAML	JAVA
λ-calculis	states	exceptions	modules	concurrency
higher order		references	objects	synchronisation
typing				threads
recursion				

Formalisation and certification of **low level languages**

The logical origins

A proof of the formula

$$A \quad \wedge \quad B$$

is a pair

$$(\varphi\,,\,\psi)$$

consisting of a proof

$$\varphi$$

of the formula A and of a proof

$$\psi$$

of the formula B.

The logical origins

A proof of the formula

$$A \quad \Rightarrow \quad B$$

is an algorithm

$$\psi$$

which transforms every proof

$$\varphi$$

of the formula A into a proof

$$\psi(\varphi)$$

of the formula B.

The simply-typed λ-calculus

Variable

$$\frac{}{x : A \ \vdash \ x : A}$$

Abstraction

$$\frac{\Gamma , x : A \ \vdash \ P : B}{\Gamma \ \vdash \ \lambda x.P : A \Rightarrow B}$$

Application

$$\frac{\Gamma \ \vdash \ P : A \Rightarrow B \qquad \Delta \ \vdash \ Q : A}{\Gamma, \Delta \ \vdash \ PQ : B}$$

Weakening

$$\frac{\Gamma \ \vdash \ P : B}{\Gamma , x : A \ \vdash \ P : B}$$

Contraction

$$\frac{\Gamma , x : A , y : A \ \vdash \ P : B}{\Gamma , z : A \ \vdash \ P[x, y \leftarrow z] : B}$$

Permutation

$$\frac{\Gamma , x : A , y : B , \Delta \ \vdash \ P : C}{\Gamma , y : B , x : A , \Delta \ \vdash \ P : C}$$

Curry-Howard correspondance

Variable

$$\frac{}{A \vdash A}$$

Abstraction

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B}$$

Application

$$\frac{\Gamma \vdash A \Rightarrow B \qquad \Delta \vdash A}{\Gamma, \Delta \vdash B}$$

Weakening

$$\frac{\Gamma \vdash B}{\Gamma, A \vdash B}$$

Contraction

$$\frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}$$

Permutation

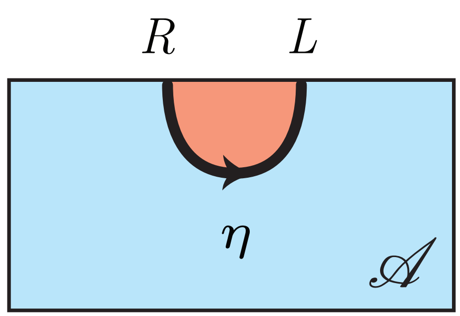
$$\frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C}$$

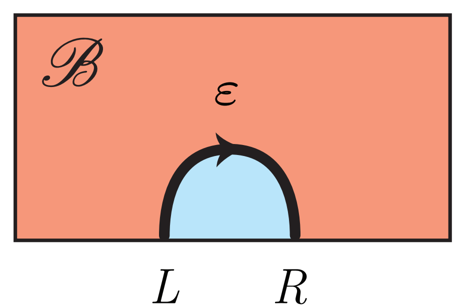
The algebraic nature of proofs

Illustration:

An adjunction is given by two graphical combinators

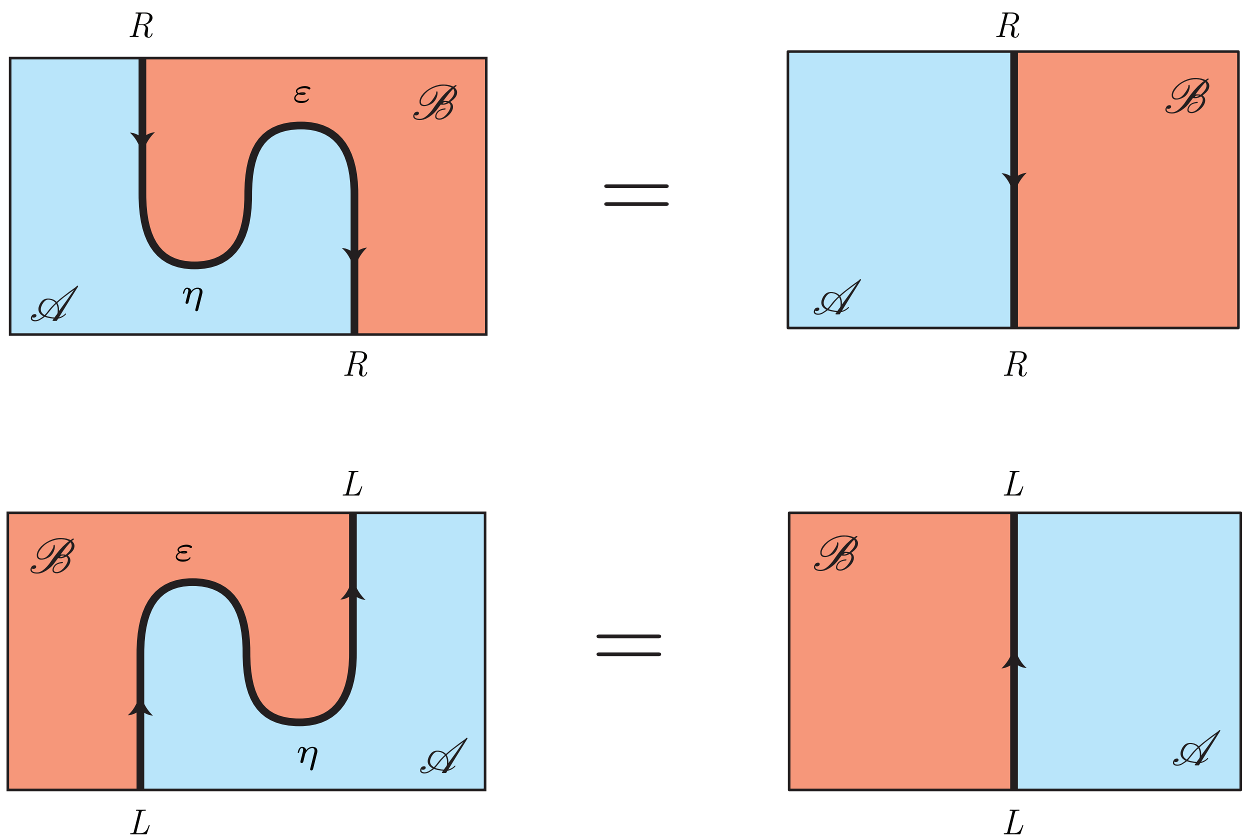
$$\eta : Id \Rightarrow R \circ L \qquad\qquad \varepsilon : L \circ R \Rightarrow Id$$



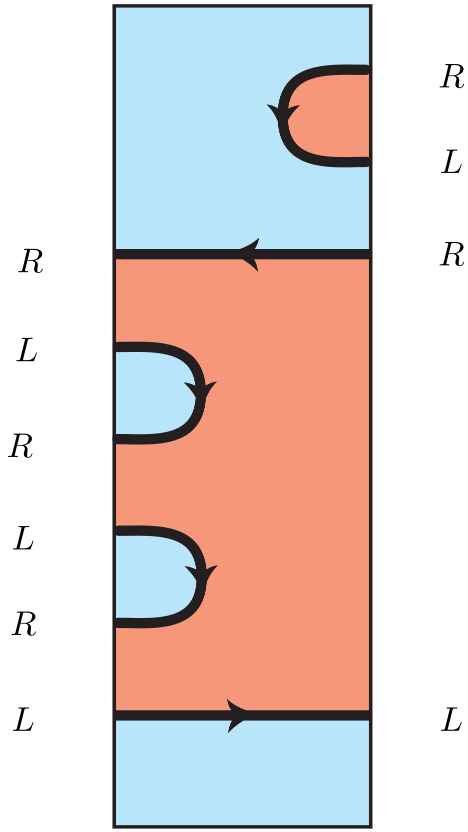


The algebraic nature of proofs

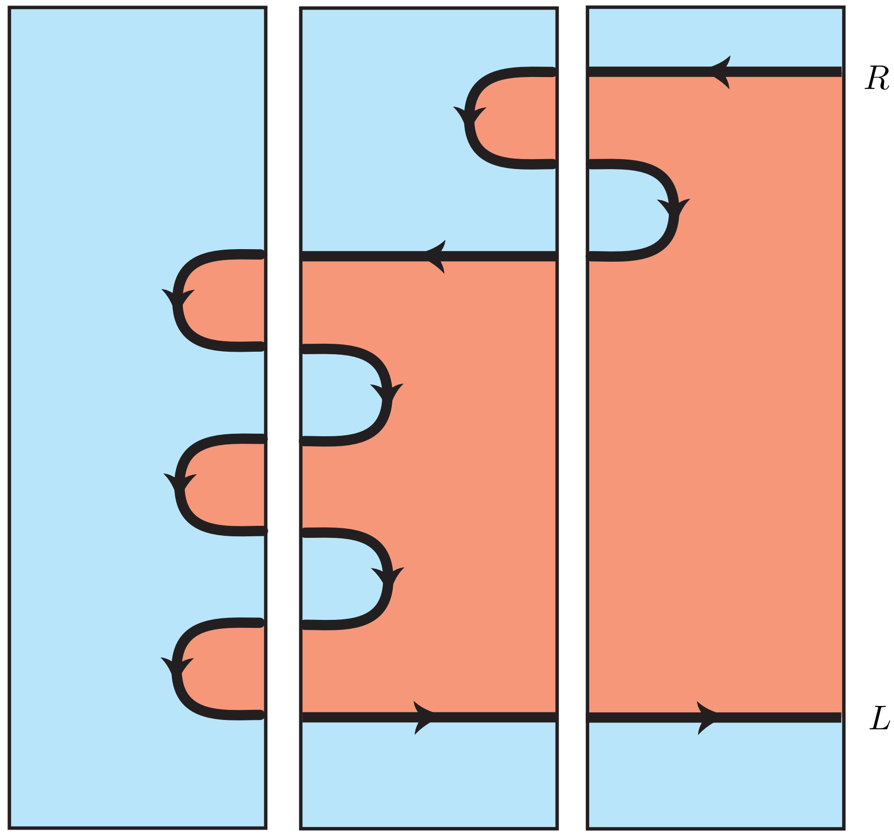
which satisfy the two equalities below:



String diagrams



A diagrammatic composition law

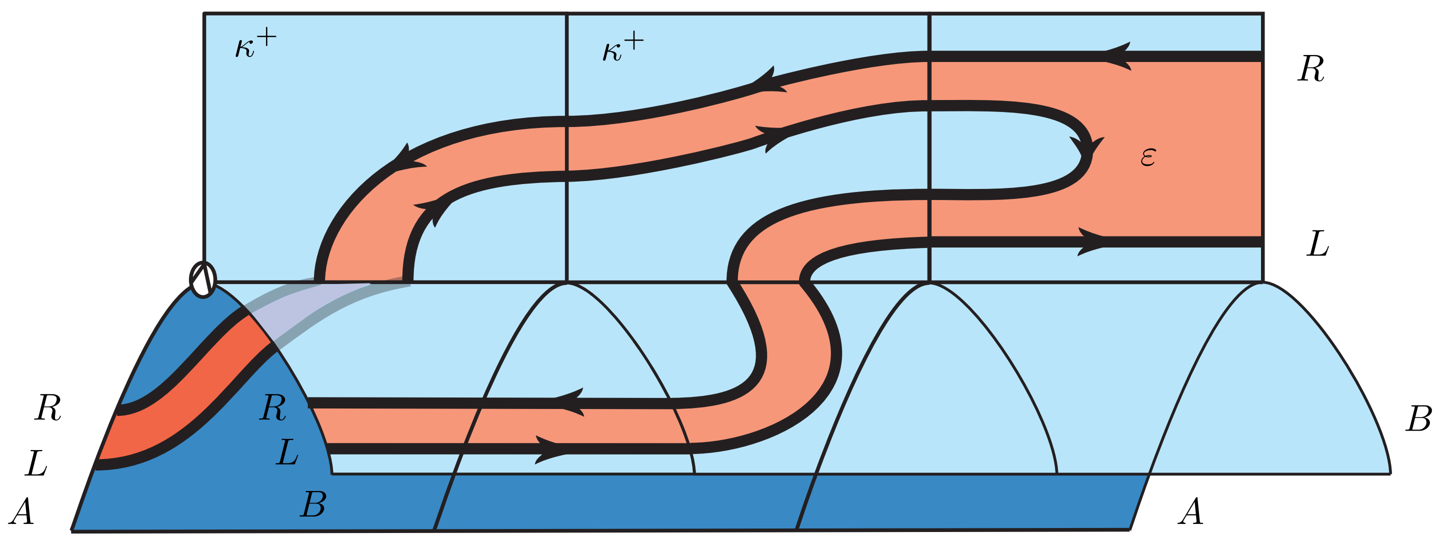


String diagrams

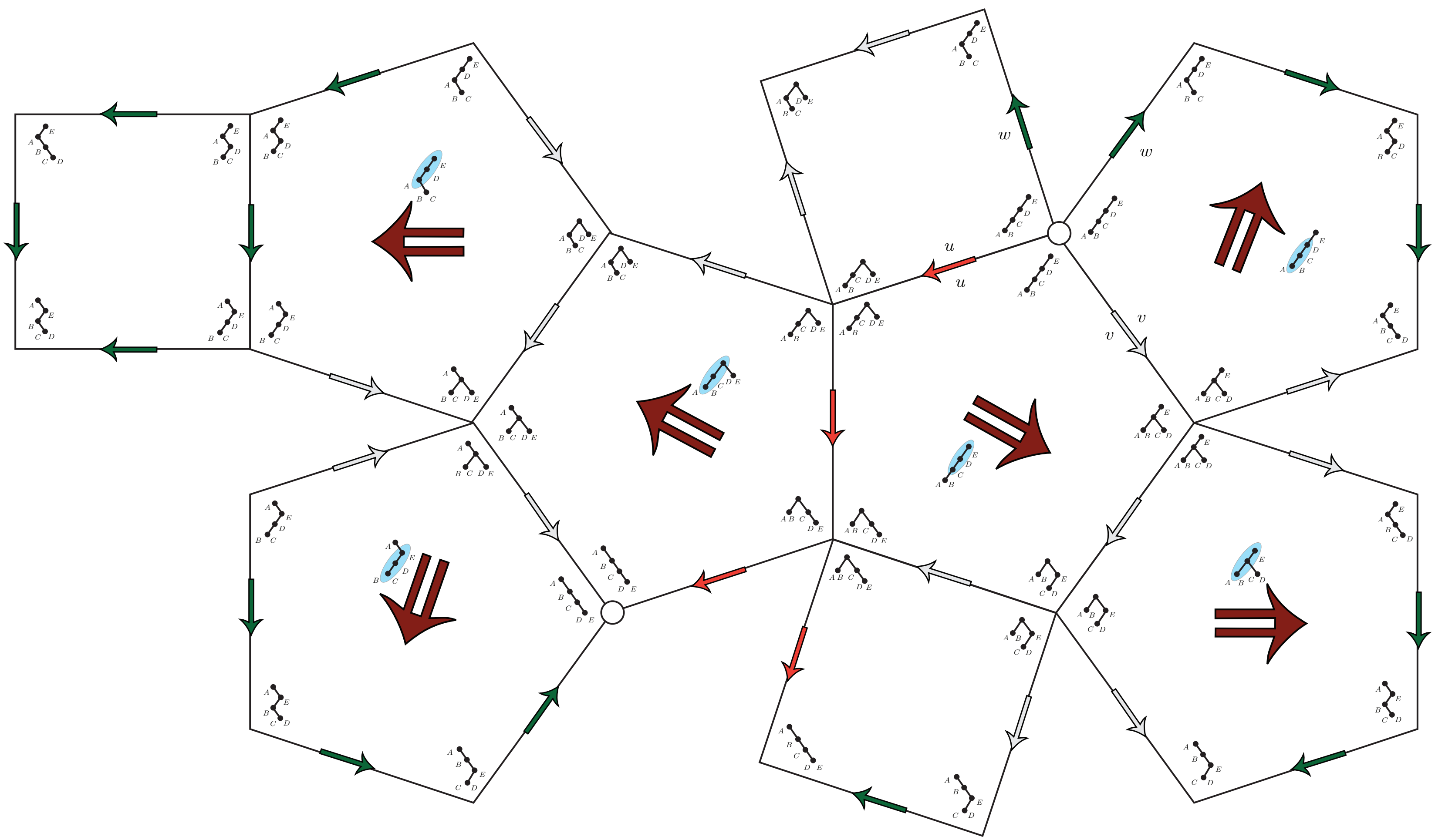
The λ-term

$$\varphi : \neg\neg A , \psi : \neg\neg B \quad \vdash \quad \lambda k.\, \varphi(\lambda a.\, \psi(\lambda b.\, k(a,b))) : \neg\neg (A \otimes B)$$

has the following diagram as control flow



The Stassheff associahedron



Computational effects

Typically described by a **monad** in Haskell

$$T \quad : \quad \mathscr{C} \quad \longrightarrow \quad \mathscr{C}$$

$$\boxed{\begin{array}{ccccc} \eta & : & Id_{\mathscr{C}} & \Rightarrow & T \\ \mu & : & T \circ T & \Rightarrow & T \end{array}}$$

The global state monad

A program accessing **one register** with a set of values $Val = \{true, false\}$

$$A \xrightarrow{\text{impure}} B$$

is interpreted as a function

$$Val \times A \xrightarrow{\text{pure}} Val \times B$$

thus as a function

$$A \xrightarrow{\text{pure}} Val \Rightarrow (Val \times B)$$

Hence, the global state monad

$$\boxed{T \quad : \quad A \quad \mapsto \quad Val \Rightarrow (Val \times A)}$$

The local state monad

The slightly intimidating monad

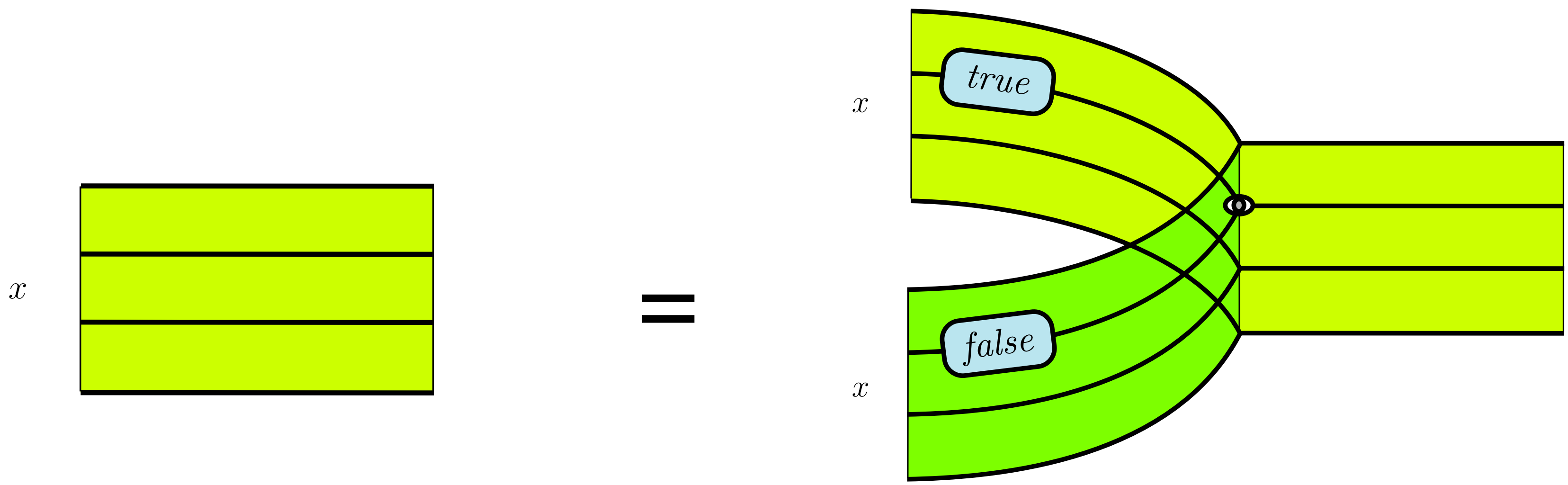
$$\boxed{\quad L\,A \quad : \quad n \quad \mapsto \quad S^{\,n} \quad \Rightarrow \quad \left(\int^{p \in Inj} \quad S^{\,p} \times A_p \times Inj\,(n, p) \quad \right)\quad}$$

on the presheaf category $[Inj, Set]$ where the contravariant presheaf

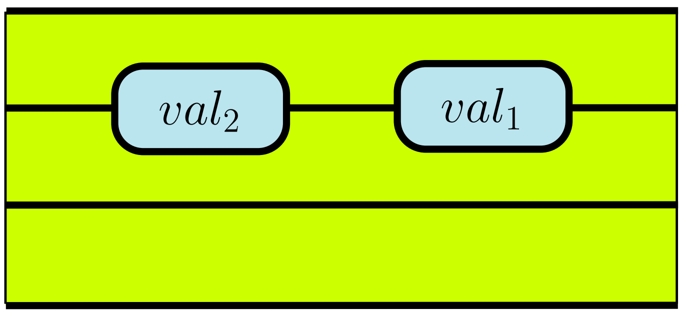
$$p \mapsto Val^{p} \quad : \quad Inj \quad \longrightarrow \quad Set$$

describes the states available at degree p.

An equational reformulation



An equational reformulation



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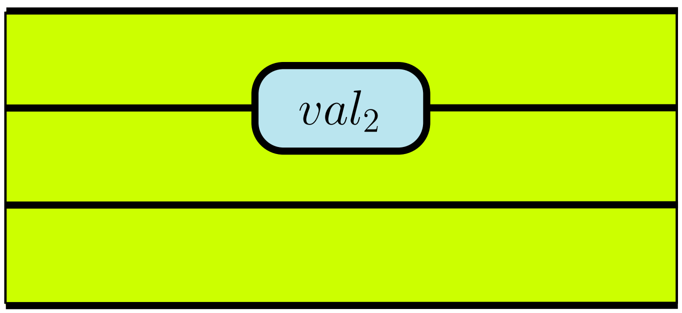
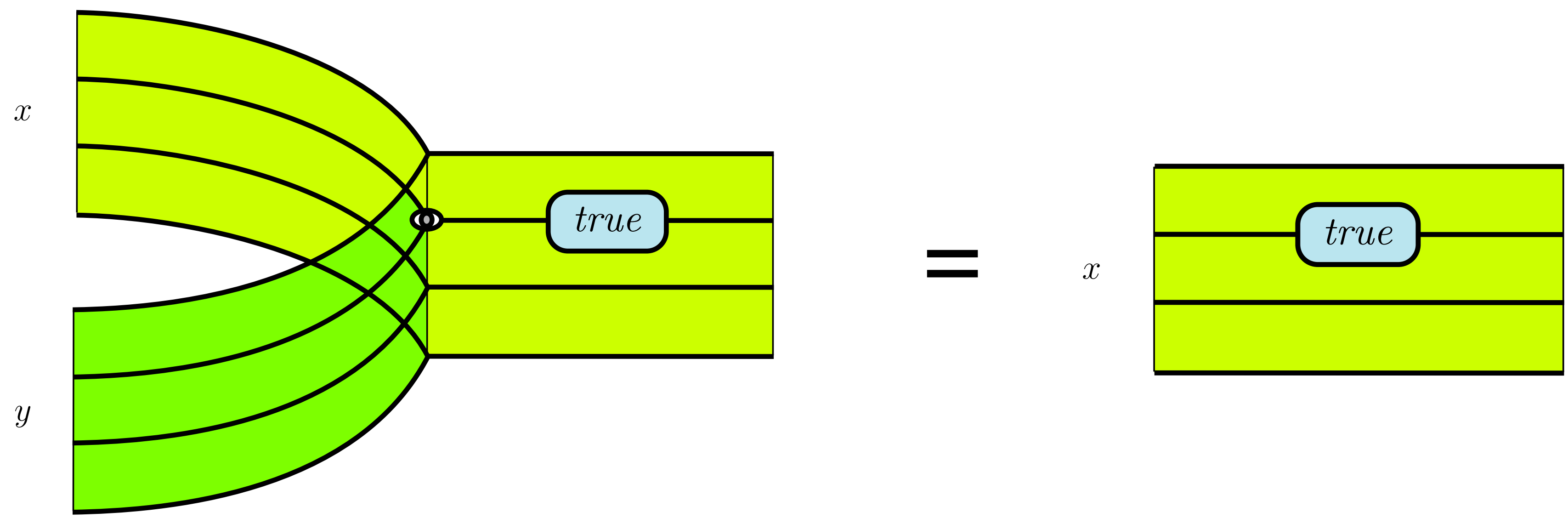
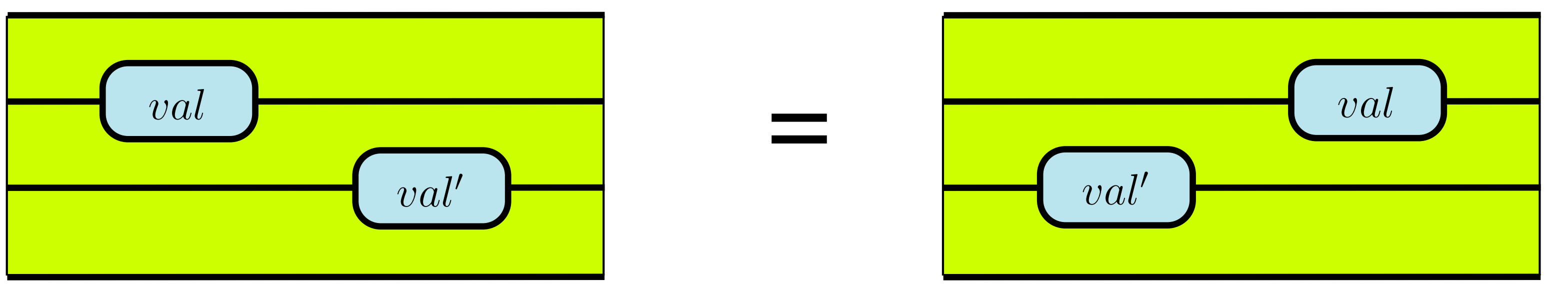


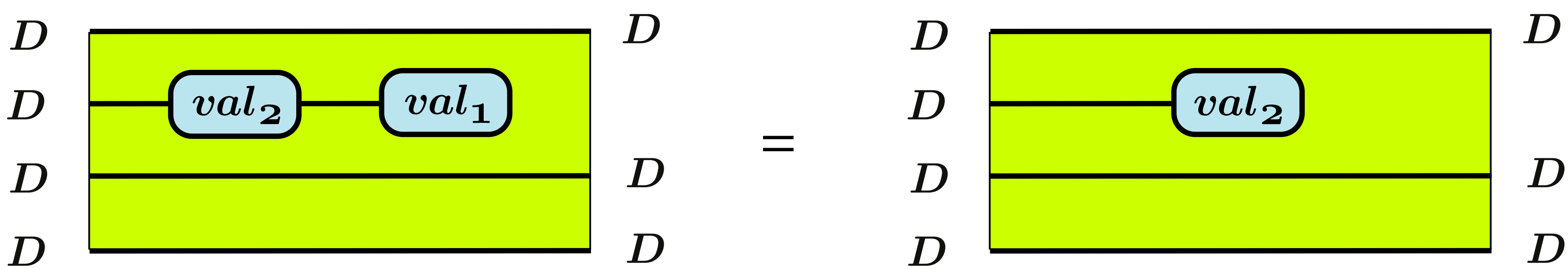
An equational reformulation



An equational reformulation



An equational reformulation



Categories and functors

Categories

A category $\mathscr{C}$ is given by

[0] a class of **objects**

[1] a set $\mathbf{Hom}(A,B)$ of **morphisms**

$$f \quad : \quad A \quad \longrightarrow \quad B$$

for every pair of objects (A,B)

[2] a **composition law**

$$\circ \quad : \quad \mathbf{Hom}(B,C) \times \mathbf{Hom}(A,B) \quad \longrightarrow \quad \mathbf{Hom}(A,C)$$

[2] an **identity** morphism

$$id_A \quad : \quad A \quad \longrightarrow \quad A$$

for every object A,

Categories

satisfying the following properties:

[3] the composition law $\circ$ is associative:

$$\begin{array}{ll} \forall f \in \mathbf{Hom}(A,B) & \\ \forall g \in \mathbf{Hom}(B,C) & \qquad h \circ (g \circ f) = (h \circ g) \circ f \\ \forall h \in \mathbf{Hom}(C,D) & \end{array}$$

[3] the morphisms id are neutral elements

$$\forall f \in \mathbf{Hom}(A,B) \qquad f \circ id_A \; = \; f \; = \; id_B \circ f$$

A hint of higher-dimensional wisdom

$$
\begin{tikzcd}
 & B \arrow[dr, "f"] & \\
A \arrow[ur, "g"] \arrow[rr, "g \circ f"'] & & C
\end{tikzcd}
$$

The composition law hides a 2-dimensional simplex

A hint of higher-dimensional wisdom

$$
\begin{tikzcd}
& B \arrow[r, "g"] & C \arrow[dr, "h"] & \\
A \arrow[ur, "f"] \arrow[urr, "g \circ f"'] \arrow[rrr, "h \circ (g \circ f)"'] & & & D
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& B \arrow[r, "g"] \arrow[drr, "h \circ g"'] & C \arrow[dr, "h"] & \\
A \arrow[ur, "f"] \arrow[rrr, "(h \circ g) \circ f"'] & & & D
\end{tikzcd}
$$

The associativity rule hides a 3-dimensional simplex

Functors

A **functor** between categories

$$F \quad : \quad \mathscr{C} \quad \longrightarrow \quad \mathscr{D}$$

is defined as the following data:

$[0]$ an object FA of $\mathscr{D}$ for every object A of $\mathscr{C}$,

$[1]$ a function

$$F_{A,B} \quad : \quad \mathbf{Hom}_{\mathscr{C}}(A,B) \quad \longrightarrow \quad \mathbf{Hom}_{\mathscr{D}}(FA,FB)$$

for every pair of objects (A,B) of the category $\mathscr{C}$.

Functors

One requires moreover

[2] that F preserves composition

$$FA \xrightarrow{Ff} FB \xrightarrow{Fg} FC \qquad = \qquad FA \xrightarrow{F(g \circ f)} FC$$

[2] that F preserves the identities

$$FA \xrightarrow{Fid_A} FA \qquad = \qquad FA \xrightarrow{id_{FA}} FA$$

Illustration [*orders*]

Every ordered set

$$(X, \leq)$$

defines a category

$$[X, \leq]$$

- ▷ whose objects are the elements of X
- ▷ whose hom-sets are defined as

$$\mathbf{Hom}(x, y) \quad = \quad \begin{cases} \{*\} & \text{if } \ x \leq y \\ \varnothing & \text{otherwise} \end{cases}$$

In this category, there exists at most one map between two objects

Illustration [*orders*]

Exercise: given two ordered sets

$$(X, \leq) \qquad (Y, \leq)$$

a functor

$$F \quad : \quad [X, \leq] \quad \longrightarrow \quad [Y, \leq]$$

is the same thing as a monotonic function

$$F \quad : \quad (X, \leq) \quad \longrightarrow \quad (Y, \leq)$$

between the underlying ordered sets.

Illustration [*monoids*]

A monoid $(M, \cdot, e)$ is a set M equipped with a binary operation

$$\cdot \quad : \quad M \times M \quad \longrightarrow \quad M$$

and a neutral element

$$e \quad : \quad \{*\} \quad \longrightarrow M$$

satisfying the two properties below:

Associativity law	$\forall x, y, z \in M,$	$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
Unit law	$\forall x \in M,$	$x \cdot e = x = e \cdot x.$

Illustration [*monoids*]

Key observation: there is a one-to-one relationship

$$(M, \cdot, e) \quad \mapsto \quad \Sigma\,(M, \cdot, e)$$

between

- ▷ monoids
- ▷ categories with one object $*$

obtained by defining $\Sigma(M, \cdot, e)$ as the category with unique hom-set

$$\Sigma(M, \cdot, e)\ (*, *) \quad = \quad M$$

and composition law and unit defined as

$$g \circ f \quad = \quad g \cdot f \qquad\qquad\qquad id_* = e$$

Illustration [*monoids*]

Key observation: given two monoids

$$(M, \cdot, e) \qquad\qquad (N, \bullet, u)$$

a functor

$$F \quad : \quad \Sigma\,(M, \cdot, e) \quad \longrightarrow \quad \Sigma\,(N, \bullet, u)$$

is the same thing as a homomorphism

$$f \quad : \quad (M, \cdot, e) \quad \longrightarrow \quad (N, \bullet, u)$$

between the underlying monoids.

Recall that a homomorphism is a function f such that

$$\forall x, y \in M, \quad f(x \cdot y) \;=\; f(x) \bullet f(y) \qquad\qquad f(e) \;=\; u$$

Illustration [*actions*]

The action of a monoid

$$(M, \cdot, e)$$

on a set

$$X$$

is the same thing as a functor

$$\Sigma\,(M, \cdot, e) \longrightarrow \mathbf{Set}$$

Illustration [*representations*]

The action of a monoid

$$(M, \cdot, e)$$

on a vector space

$$V$$

is the same thing as a functor

$$\Sigma\,(M, \cdot, e) \quad \longrightarrow \quad \mathbf{Vect}$$

Cartesian categories

Isomorphism

In a category $\mathscr{C}$, a morphism

$$f \quad : \quad A \quad \longrightarrow \quad B$$

is called an **isomorphism** when there exists a morphism

$$g \quad : \quad B \quad \longrightarrow \quad A$$

satisfying

$$g \circ f = id_A \qquad \text{et} \qquad f \circ g = id_B.$$

Exercise.

- Show that $g \circ f : A \longrightarrow C$ is an isomorphism when $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are isomorphisms.
- Show that every functor $F : \mathscr{C} \longrightarrow \mathscr{D}$ transports an isomorphism of $\mathscr{C}$ into an isomorphism of $\mathscr{D}$.

Products

The **product** of two objects A and B in a category $\mathscr{C}$ is an object $A \times B$ equipped with two morphisms

$$\pi_1 : A \times B \longrightarrow A \qquad\qquad \pi_2 : A \times B \longrightarrow B$$

such that for every diagram

$$\begin{tikzcd}
 & & A \\
X \arrow[urr, bend left, "f"] \arrow[drr, bend right, "g"'] & A \times B \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] & \\
 & & B
\end{tikzcd}$$

there exists a unique morphism $h : X \longrightarrow A \times B$ making the diagram

$$\begin{tikzcd}
 & & A \\
X \arrow[r, "h"] \arrow[urr, bend left, "f"] \arrow[drr, bend right, "g"'] & A \times B \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] & \\
 & & B
\end{tikzcd}$$

commute.

Illustrations

1. The cartesian product in the category $\mathbf{Set}$,

2. The cartesian product in the category $\mathbf{Ord}$ of ordered sets,

3. The lub $a \wedge b$ of two elements a and b in an ordered set $(X, \leq)$.

Terminal object

An object $\mathbf{1}$ is **terminal** in a category $\mathscr{C}$ when $\mathbf{Hom}(A, \mathbf{1})$ is a singleton for all objects A.

One may consider $\mathbf{1}$ as the nullary product in $\mathscr{C}$.

Example 1. the singleton $\{*\}$ in the categories $\mathbf{Set}$ and $\mathbf{Ord}$,

Example 2. the maximum of an ordered set $(X, \leq)$

Cartesian category

A **cartesian category** is a category $\mathscr{C}$ equipped with a product

$$A \times B$$

for all pairs A, B of objects, and of a terminal object

$$\mathbf{1}$$

The category of small categories

Definition

A category is small when its class of objects is a set.

Definition

The category **Cat** of small categories has

- small categories as objects
- functors as morphisms.

Exercise. Show that the category **Cat** is cartesian.

Bifunctors

A **bifunctor** between categories

$$F \quad : \quad \mathscr{C}, \mathscr{D} \quad \longrightarrow \quad \mathscr{E}$$

is given by:

- ▷ a functor $\quad F(A, -) \quad : \quad \mathscr{D} \longrightarrow \quad \mathscr{E}$

  for every object A of the category $\mathscr{C}$

- ▷ a functor $\quad F(-, B) \quad : \quad \mathscr{C} \longrightarrow \quad \mathscr{E}$

  for every object B of the category $\mathscr{D}$

Bifunctors

such that the diagram

$$\begin{tikzcd}
F(A,B) \arrow[r, "{F(A,g)}"] \arrow[d, "{F(f,B)}"'] & F(A,B') \arrow[d, "{F(f,B')}"] \\
F(A',B) \arrow[r, "{F(A',g)}"] & F(A',B')
\end{tikzcd}$$

commutes for all maps $f : A \longrightarrow A'$ in $\mathscr{C}$ and $g : B \longrightarrow B'$ in $\mathscr{D}$.

Reformulation

Proposition.

A bifunctor

$$F \quad : \quad \mathscr{C}, \mathscr{D} \quad \longrightarrow \quad \mathscr{E}$$

is the same thing as a functor

$$F \quad : \quad \mathscr{C} \times \mathscr{D} \quad \longrightarrow \quad \mathscr{E}$$

where $\mathscr{C} \times \mathscr{D}$ is the product of the categories $\mathscr{C}$ and $\mathscr{D}$.

Exercise

Show that every cartesian category $\mathscr{C}$ is equipped with a bifunctor

$$A, B \mapsto A \times B \quad : \quad \mathscr{C}, \mathscr{C} \quad \longrightarrow \quad \mathscr{C}$$

as well as with a functor

$$* \mapsto 1 \quad : \quad \mathbb{1} \quad \longrightarrow \quad \mathscr{C}$$

from the terminal category $\mathbb{1}$ with one object $*$ and one morphism id_*.