

# Lambda calculs et catégories

Paul-André Melliès

Master Parisien de Recherche en Informatique

Ecole Normale Supérieure

# Synopsis of the lecture

- 1 – Natural transformations
- 2 – The 2-category of categories
- 3 – String diagrams

# Natural transformations

A notion of morphism between functors

# Transformations

A transformation

$$\theta : F \longrightarrow G$$

between two functors

$$F, G : \mathcal{A} \longrightarrow \mathcal{B}$$

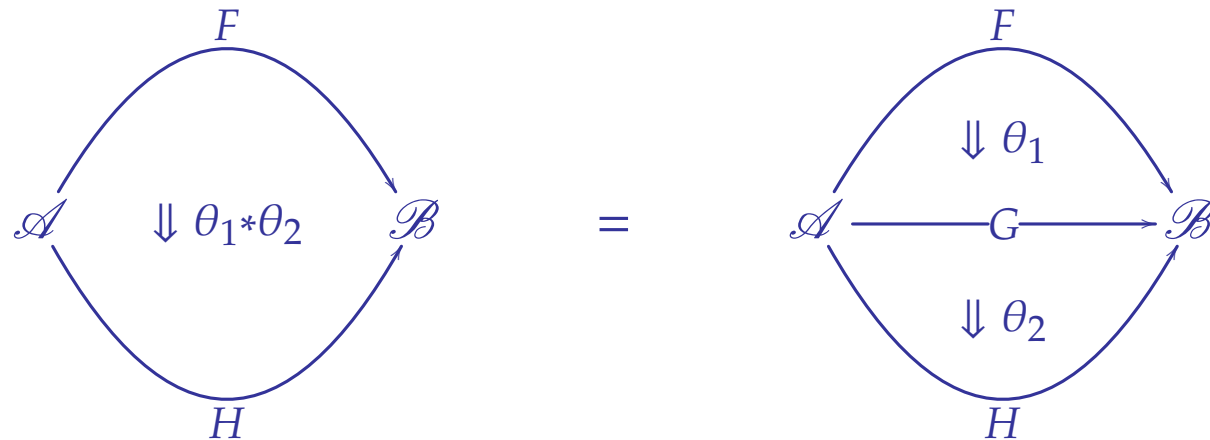
is a family of morphisms

$$(\theta_A : FA \longrightarrow GA)_{A \in \text{Obj}(\mathcal{A})}$$

of the category  $\mathcal{B}$  indexed by the objects of the category  $\mathcal{A}$ .

# Vertical composition of transformations

The transformations compose vertically



and thus define a category

$$\mathbf{Trans} ( \mathcal{A} , \mathcal{B} )$$

for all categories  $\mathcal{A}$  and  $\mathcal{B}$ .

## Left action

In the following situation:

$$\begin{array}{ccccc} & & F & & \\ & \curvearrowright & & \curvearrowleft & \\ \mathcal{A} & & \Downarrow \theta & & \mathcal{B} \xrightarrow{H} \mathcal{C} \\ & \curvearrowleft & & \curvearrowright & \\ & & G & & \end{array}$$

the **left action** of the functor  $H$  on the transformation

$$\theta : F \longrightarrow G : \mathcal{A} \longrightarrow \mathcal{B}$$

is defined as the transformation

$$H \circ_L \theta : H \circ F \longrightarrow H \circ G : \mathcal{A} \longrightarrow \mathcal{C}$$

whose instance at object  $A$  is defined as the morphism

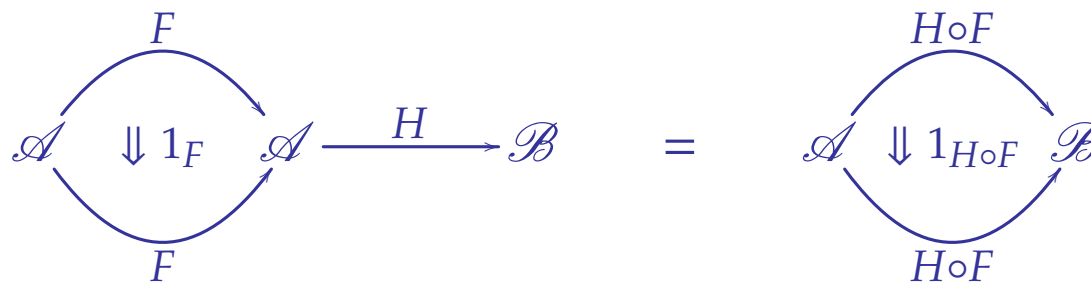
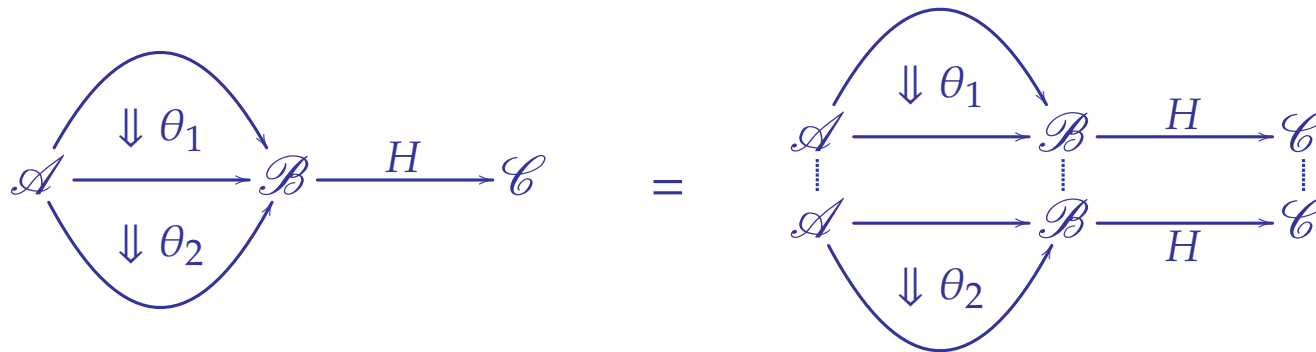
$$H \circ F(A) \xrightarrow{H(\theta_A)} H \circ G(A).$$

# Properties of the left action [1]

From a diagrammatic point of view, the two equations

$$H \circ_L (\theta_2 * \theta_1) = (H \circ_L \theta_2) * (H \circ_L \theta_1) \quad H \circ_L 1_F = 1_{H \circ F}$$

mean that



## Properties of the left action (2)

These two equations mean that

$$\begin{array}{lcl} H \circ_L - & : & \mathbf{Trans}(\mathcal{A}, \mathcal{B}) \longrightarrow \mathbf{Trans}(\mathcal{A}, \mathcal{C}) \\ & & \theta \mapsto H \circ_L \theta \end{array}$$

defines a functor, while the two equations

$$(H_1 \circ H_2) \circ_L F = H_1 \circ_L (H_2 \circ_L F) \qquad id_{\mathcal{B}} \circ_L \theta = \theta$$

mean that  $\circ_L$  defines an action.



## Right action

In the following situation:

$$\mathcal{A} \xrightarrow{H} \mathcal{B} \begin{array}{c} \xrightarrow{F} \mathcal{C} \\ \Downarrow \theta \\ \xrightarrow{G} \mathcal{C} \end{array}$$

the functor  $H$  acts on the transformation

$$\theta : F \longrightarrow G : \mathcal{B} \longrightarrow \mathcal{C}$$

and transports it into the transformation:

$$\theta \circ_R H : F \circ H \longrightarrow G \circ H : \mathcal{A} \longrightarrow \mathcal{C}$$

whose instance at  $A$  is defined as the morphism

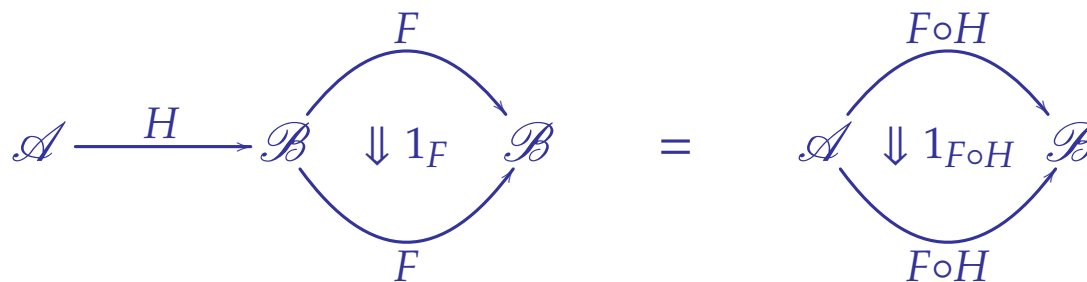
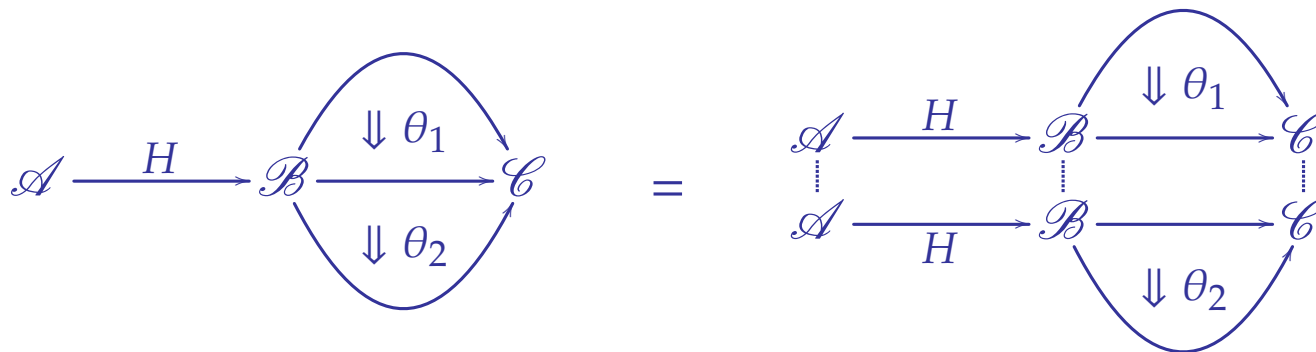
$$F \circ H(A) \xrightarrow{\theta_{H(A)}} G \circ H(A).$$

# Properties of the right action (1)

From a diagrammatic point of view, the two equations

$$(\theta_2 * \theta_1) \circ_R H = (\theta_2 \circ_R H) * (\theta_1 \circ_R H) \quad 1_F \circ_R H = 1_{F \circ H}$$

mean that



## Properties of the right action (2)

The two equations mean that

$$\begin{array}{ccc} - \circ_R H & : & \mathbf{Trans}(\mathcal{B}, \mathcal{C}) \longrightarrow \mathbf{Trans}(\mathcal{A}, \mathcal{C}) \\ & & \theta \mapsto \theta \circ_R H \end{array}$$

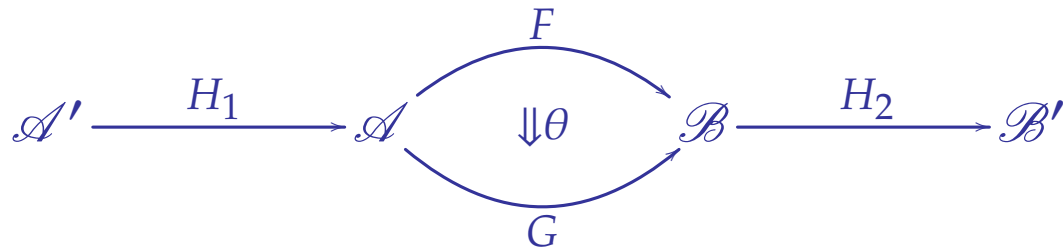
defines a functor, while the two equations

$$\theta \circ_R (H_2 \circ H_1) = (\theta \circ_R H_2) \circ_R H_1 \qquad \theta \circ_R id_{\mathcal{A}} = \theta$$

mean that  $\circ_R$  defines an action.

# Compatibility of the left and right actions

Last equation: in the situation



the order in which one makes the functors

$$H_1 : \mathcal{A}' \longrightarrow \mathcal{A} \qquad H_2 : \mathcal{B} \longrightarrow \mathcal{B}'$$

act on the transformation  $\theta$  does not matter:

$$(H_2 \circ_L \theta) \circ_R H_1 = H_2 \circ_L (\theta \circ_R H_1)$$

# Sesqui-category

A sesqui-category  $\mathcal{D}$  is

[0] a class of objects

[1,2] equipped with a category

$$\mathcal{D}(A, B)$$

for every pair of objects  $(A, B)$  of the sesqui-category, where

the objects of  $\mathcal{D}(A, B)$  = the morphisms from  $A$  to  $B$

equipped with a pair of actions  $\circ_L$  and  $\circ_R$  satisfying...

# Sesqui-categories

equipped with a pair of actions  $\circ_L$  and  $\circ_R$  satisfying the equations

$$\begin{array}{ll}
 h \circ_L (\theta_2 * \theta_1) & = (h \circ_L \theta_2) * (h \circ_L \theta_1) & h \circ_L 1_f & = 1_{h \circ f} \\
 (h_1 \circ h_2) \circ_L f & = h_1 \circ_L (h_2 \circ_L f) & id_{\mathcal{B}} \circ_L \theta & = \theta \\
 (\theta_2 * \theta_1) \circ_R h & = (\theta_2 \circ_R h) * (\theta_1 \circ_R h) & 1_f \circ_R h & = 1_{f \circ h} \\
 \theta \circ_R (h_2 \circ h_1) & = (\theta \circ_R h_2) \circ_R h_1 & \theta \circ_R id_{\mathcal{A}} & = \theta
 \end{array}$$

$$(h_2 \circ_L \theta) \circ_R h_1 = h_2 \circ_L (\theta \circ_R h_1)$$

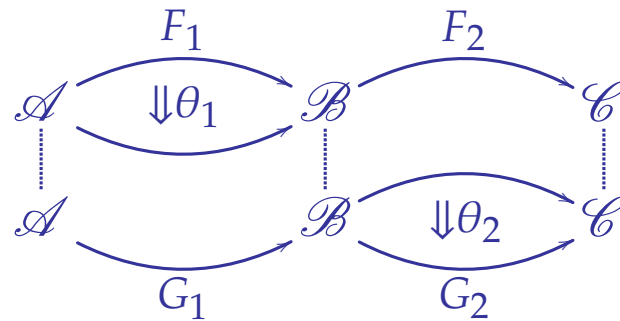
## Theorem.

Categories, functors and transformations define a sesqui-category.

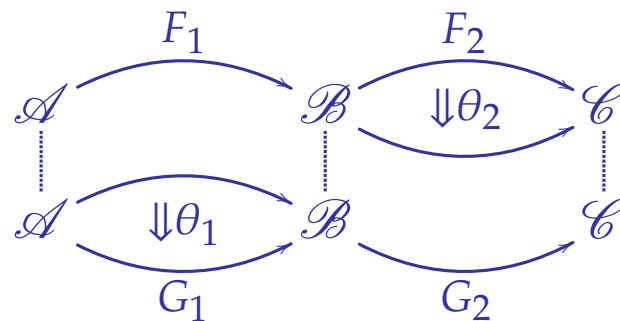
# The sesqui-category of categories and transformations

Let  $\theta_1$  and  $\theta_2$  be two transformations in  $\mathcal{A} \begin{matrix} \xrightarrow{F_1} \\ \Downarrow \theta_1 \\ \xrightarrow{G_1} \end{matrix} \mathcal{B} \begin{matrix} \xrightarrow{F_2} \\ \Downarrow \theta_2 \\ \xrightarrow{G_2} \end{matrix} \mathcal{C}$

In general, the transformation obtained by applying  $\theta_1$  then  $\theta_2$



is not the same as the transformation obtained by applying  $\theta_1$  then  $\theta_2$ :



# Natural transformations

A transformation  $\theta : F \Rightarrow G : \mathcal{A} \longrightarrow \mathcal{B}$   
is **natural** when the diagram

$$\begin{array}{ccc} FA & \xrightarrow{\theta_A} & GA \\ Ff \downarrow & & \downarrow Gf \\ FB & \xrightarrow{\theta_B} & GB \end{array}$$

commutes for every morphism  $f : A \longrightarrow B$ .

**Notation.** we write

$$\mathbf{Nat}(\mathcal{A}, \mathcal{B})$$

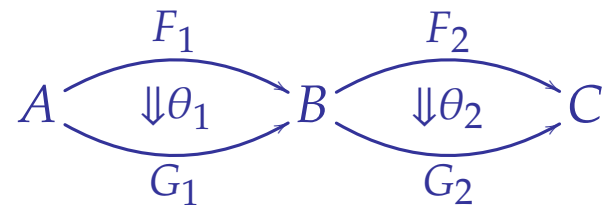
for the category of functors and natural transformations

$$\theta : F \Rightarrow G : \mathcal{A} \longrightarrow \mathcal{B}$$

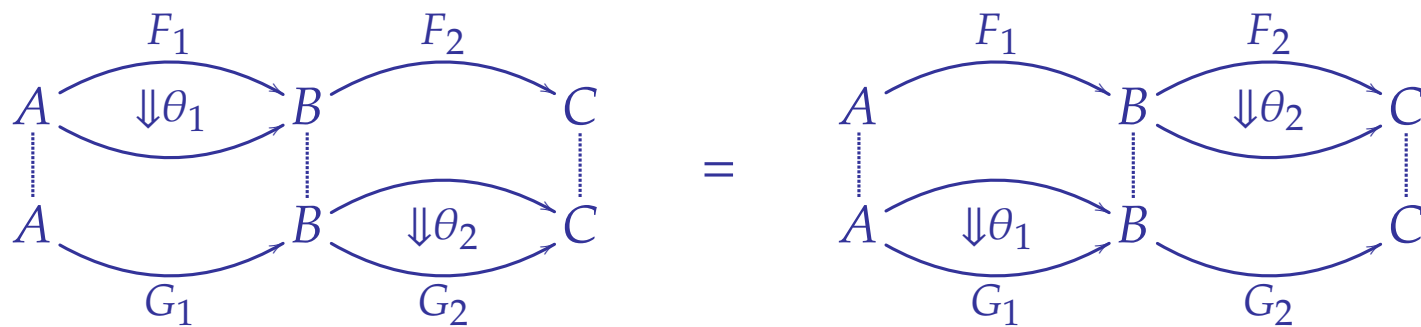


# Exchange law

A pair of 2-cells  $\theta_1$  and  $\theta_2$  in a sesqui-categorie  $\mathcal{D}$



satisfy the exchange law when the equality

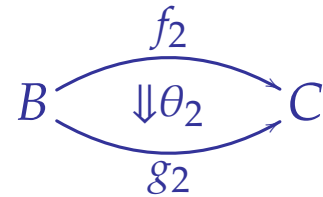


holds.

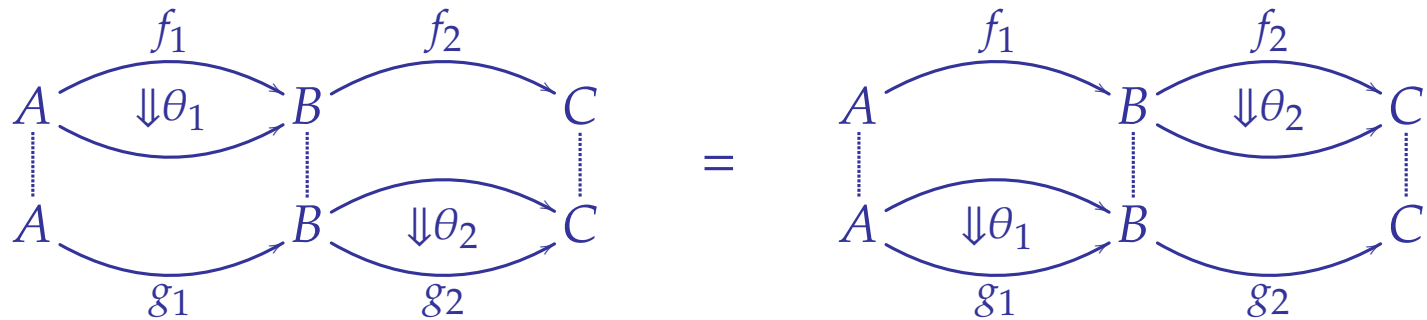
The order in which one applies  $\theta_1$  and  $\theta_2$  does not matter.

# Definition

A 2-cell



is called **central on the left** when the exchange law



is satisfied for every 2-cell  $\theta_1$  of the sesqui-category  $\mathcal{D}$ .

## Exercise

Show that in the sesqui-category with

- ▷ categories as objects
- ▷ functors as 1-cells
- ▷ transformations as 2-cells

the natural transformations are the 2-cells central on the left.

Deduce the existence of a functor

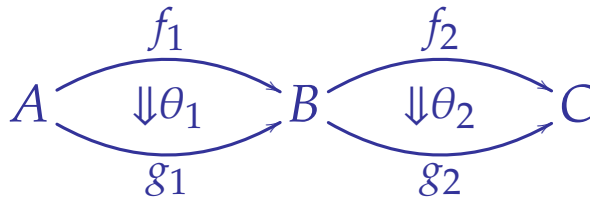
$$\mathbf{Nat}(\mathcal{B}, \mathcal{C}) \times \mathbf{Nat}(\mathcal{A}, \mathcal{B}) \longrightarrow \mathbf{Nat}(\mathcal{A}, \mathcal{C})$$

# The 2-category of categories

Categories, functors, natural transformations

## 2-categories

A 2-category  $\mathcal{D}$  is a sesqui-category such that the **exchange law** is satisfied for every pair of 2-cells



## 2-categories (alternative definition)

A 2-category  $\mathcal{D}$  is given by

[0] a class of **objects**

[1,2] a category  $\mathcal{D}(A, B)$  for every pair of objects  $(A, B)$

[2,3,4] a **composition law** defined as a functor

$$\circ : \mathcal{D}(B, C) \times \mathcal{D}(A, B) \longrightarrow \mathcal{D}(A, C)$$

[2,3,4] an **identity** defined as a functor

$$id_A : \mathbb{1} \longrightarrow \mathcal{D}(A, A)$$

this for all objects  $A, B, C$  of the 2-category,

## 2-categories (alternative definition)

1— such that the composition law  $\circ$  is associative in the sense that

$$\begin{array}{ccc}
 \mathcal{D}(C, D) \times \mathcal{D}(B, C) \times \mathcal{D}(A, B) & \xrightarrow{\circ \times \mathcal{D}(A, B)} & \mathcal{D}(B, D) \times \mathcal{D}(A, B) \\
 \mathcal{D}(C, D) \times \circ \downarrow & & \downarrow \circ \\
 \mathcal{D}(C, D) \times \mathcal{D}(A, C) & \xrightarrow{\circ} & \mathcal{D}(A, D)
 \end{array}$$

commutes.

## 2-categories (alternative definition)

2— such that  $id$  is a neutral element of  $\circ$  in the sense that

$$\begin{array}{ccc}
 \mathcal{D}(A, B) & \xlongequal{\quad} & \mathcal{D}(A, B) \\
 \cong \downarrow & & \uparrow \circ \\
 \mathcal{D}(A, B) \times \mathbb{1} & \xrightarrow{\mathcal{D}(A, B) \times id_A} & \mathcal{D}(A, B) \times \mathcal{D}(A, A)
 \end{array}$$

and

$$\begin{array}{ccc}
 \mathcal{D}(A, B) & \xlongequal{\quad} & \mathcal{D}(A, B) \\
 \cong \downarrow & & \uparrow \circ \\
 \mathbb{1} \times \mathcal{D}(A, B) & \xrightarrow{id_B \times \mathcal{D}(A, B)} & \mathcal{D}(B, B) \times \mathcal{D}(A, B)
 \end{array}$$

commute for all  $A$  and  $B$ .



## Notation

One writes

$$\theta : f \Rightarrow g : A \longrightarrow B$$

when

$$\theta : f \longrightarrow g$$

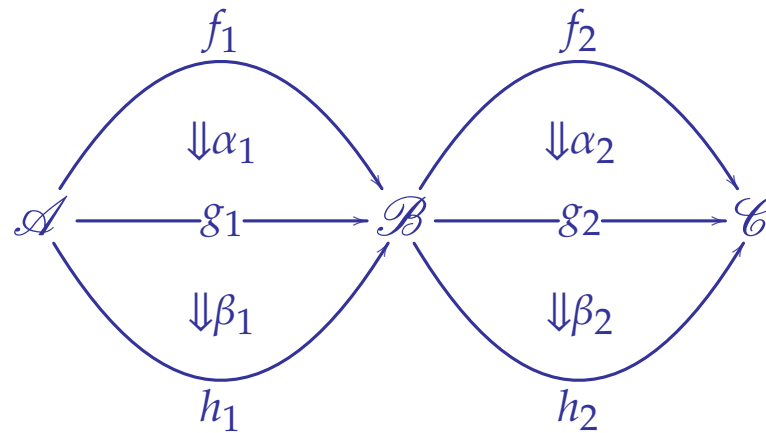
is a morphism of the category  $\mathcal{D}(A, B)$ .

# Godement law

In a 2-category

$$\mathcal{D}(A, B)$$

the two canonical ways to compose the 2-cells



coincide:

$$(\beta_2 * \alpha_2) \circ (\beta_1 * \alpha_1) = (\beta_2 \circ \beta_1) * (\alpha_2 \circ \alpha_1)$$

# Suspension

The notion of monoidal category will be defined very soon.

Every strict monoidal category  $\mathcal{C}$  may be seen as the 2-category  $\Sigma(\mathcal{C})$

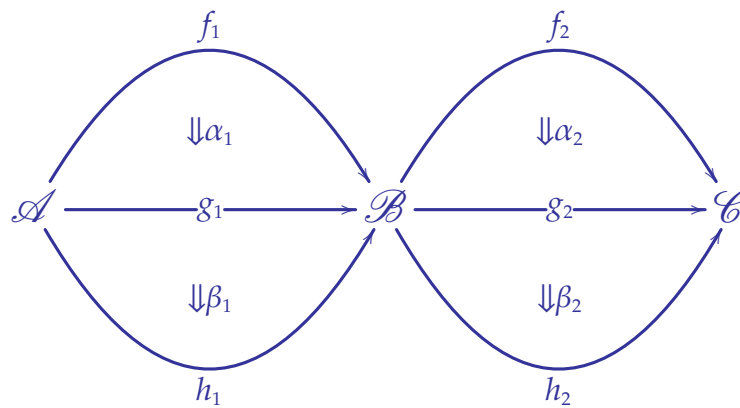
- ▷ which contains only one 0-cell,
- ▷ whose 1-cells are the 0-cells of  $\mathcal{C}$
- ▷ whose 2-cells are the 1-cells of  $\mathcal{C}$

equipped with the induced composition laws.

A sesqui-category  $\Sigma(\mathcal{C})$  with one object is the same thing as a premonoidal category  $(\mathcal{C}, \otimes, I)$ .

## Useful equality

In a 2-category  $\mathcal{D}(\mathcal{A}, \mathcal{B})$ , the two canonical ways to compose the 2-cells



commute:

$$(\beta_2 * \alpha_2) \circ (\beta_1 * \alpha_1) = (\beta_2 \circ \beta_1) * (\alpha_2 \circ \alpha_1)$$

# The 2-category of sets and relations

The 2-category  $\mathcal{Rel}$  is defined as follows:

- ▷ its 0-cells are the sets,
- ▷ its 1-cells are the relations between sets,

$$A \xrightarrow{f \cdot g} B = A \xrightarrow{f} B \xrightarrow{g} C$$

relationally composed:

$$a [f \cdot g] c \iff \exists b \in B, \quad a [f] b \text{ et } b [g] c.$$

- ▷ its 2-cells are inclusions:

$$A \begin{array}{c} \xrightarrow{f} \\ \Downarrow \\ \xrightarrow{g} \end{array} B \iff f \subseteq g$$

In particular, the categories  $\mathcal{Rel}(A, B)$  are order categories.

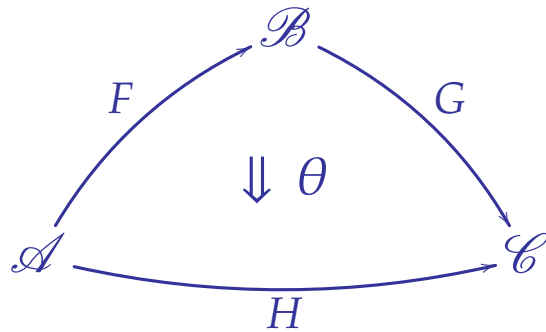
# String diagrams

A notation introduced by Roger Penrose

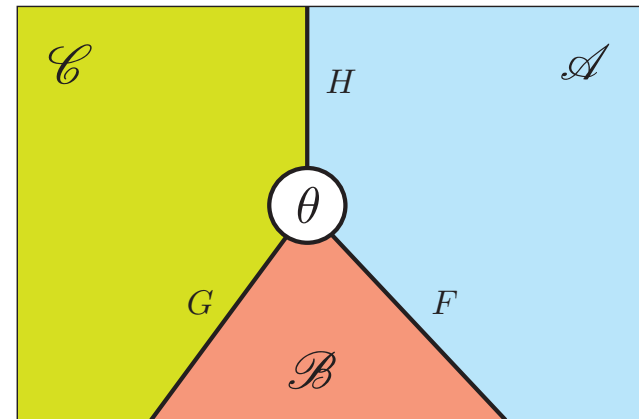
# String diagrams

## Two key ideas

1. apply the Poincaré duality on the original pasting diagrams:



is depicted as

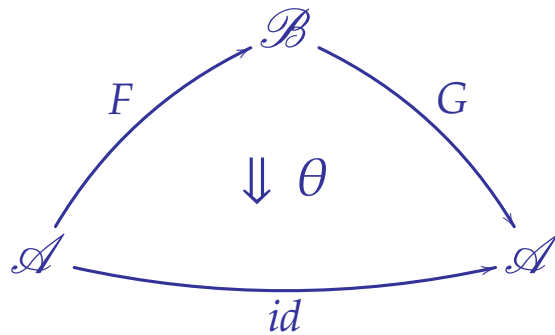


$$\theta : G \circ F \Rightarrow H$$

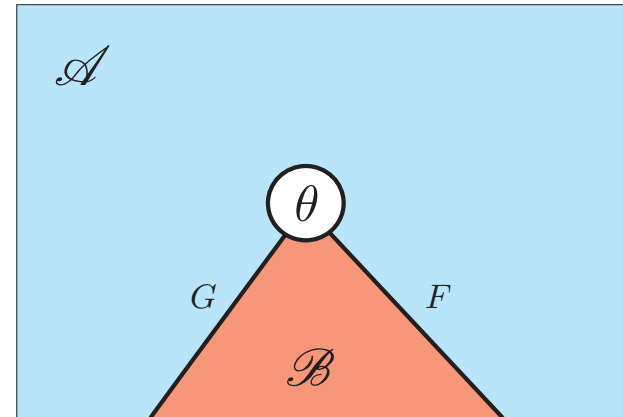
# String diagrams

## Two key ideas

2. hide the identity 1-cells in the picture:



is depicted as



$$\theta : G \circ F \Rightarrow id$$

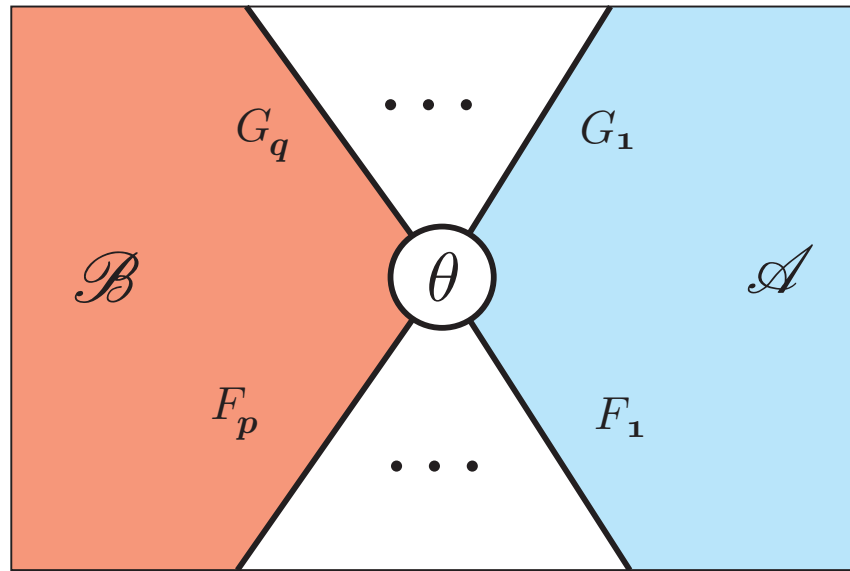


# String diagrams

More generally, a 2-dimensional cell

$$\theta : F_1 \circ \dots \circ F_p \Rightarrow G_1 \circ \dots \circ G_q : \mathcal{A} \longrightarrow \mathcal{B}$$

is depicted as



## **Exercise**

Draw the exchange law and explain the connection to concurrency

## Short bibliography of the course

On categorical semantics of linear logic and 2-categories:

### **Categorical semantics of linear logic.**

Survey published in

« Interactive models of computation and program behaviour ».

Pierre-Louis Curien, Hugo Herbelin, Jean-Louis Krivine, Paul-André Melliès.

Panoramas et Synthèses 27, Société Mathématique de France, 2009.

On string diagrams:

Christian Kassel

### **Quantum groups**

Graduate Texts in Mathematics 155

Springer Verlag 1995.

Peter Selinger

### **A survey of graphical languages for monoidal categories.**

New Structures for Physics

Springer Lecture Notes in Physics 813, pp. 289-355, 2011.

### **Functorial boxes in string diagrams**

Proceedings of CSL 2006.

Lecture Notes in Computer Science 4207.