Lambda calculs et catégories

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Synopsis of the lecture

1 – Adjunctions

2 – Monads

Adjunctions

A notion of duality between functors

Adjunction

An **adjunction** is a triple (L, R, ϕ) where L and R are two functors

$$L : \mathscr{A} \longrightarrow \mathscr{B} \qquad\qquad R : \mathscr{B} \longrightarrow \mathscr{A}$$

and ϕ is a family of bijections, for all objects A in $\mathscr{A}$ and B in $\mathscr{B}$,

$$\phi_{A,B} : \mathscr{B}(LA, B) \;\cong\; \mathscr{A}(A, RB)$$

natural in A et B. One also writes

$$\frac{LA \longrightarrow_{\mathscr{B}} B}{A \longrightarrow_{\mathscr{A}} RB} \quad \phi_{A,B}$$

One says that L **is left adjoint to** R, noted $L \dashv R$.

The 2-dimensional version of isomorphism

The naturality of the bijection ϕ

Natural in A and B means that the family of bijections

$$\phi_{A,B} \quad : \quad \mathscr{B}(LA, B) \;\cong\; \mathscr{A}(A, RB)$$

transforms every commutative diagram

$$\begin{array}{ccc} LA & \xrightarrow{\;g\;} & B \\ {\scriptstyle Lh_A}\uparrow & & \downarrow{\scriptstyle h_B} \\ LA' & \xrightarrow[\;f\;]{} & B' \end{array}$$

into a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\;\phi_{A,B}(g)\;} & RB \\ {\scriptstyle h_A}\uparrow & & \downarrow{\scriptstyle Rh_B} \\ A' & \xrightarrow[\;\phi_{A',B'}(f)\;]{} & RB' \end{array}$$

Example: the free vector space

$$\mathbf{Set} \underset{R}{\overset{L}{\rightleftarrows}} \mathbf{Vect} \qquad (\perp)$$

where

$\mathscr{A} = \mathbf{Set}$	:	the category of sets and functions
$\mathscr{B} = \mathbf{Vect}$	:	the category of vector spaces on a field k
R	:	the « forgetful » functor $V \mapsto U(V)$
L	:	the « free vector space » functor $X \mapsto kX$

$$kX := \left\{ \sum_{x \in X} \lambda_x \, x \;\middle|\; \lambda_x \in k \text{ null almost everywhere.} \right\}$$

Illustration: the tensor algebra

$$\mathbf{Vect} \underset{R}{\overset{L}{\rightleftarrows}} \mathbf{Alg} \qquad (L \dashv R,\ \perp)$$

where

$\mathscr{A} = \mathbf{Vect}$: the category of vector spaces
$\mathscr{B} = \mathbf{Alg}$: the category of algebras and homomorphisms,

R : the « forgetful » functor $A \mapsto U(A)$.
L : the « free algebra » functor $V \mapsto TV$.

$$TV \;:=\; \bigoplus_{n\in\mathbb{N}} V^{\otimes n}$$

Definition of a Lie algebra

Vector space $\mathfrak{g}$ equipped with a Lie bracket

Anti-symmetry:

$$[x, y] \quad = \quad -[y, x]$$

Jacobi identity:

$$[x, [y, z]] \quad + \quad [y, [z, x]] \quad + \quad [z, [x, y]] \quad = \quad 0$$

Example: the vector space of vector fields on a smooth manifold.

Illustration: the enveloping algebra of a Lie algebra

$$\mathbf{Lie} \underset{R}{\overset{L}{\rightleftarrows}} \mathbf{Alg} \qquad (L \dashv R)$$

where

$\mathscr{A} = \mathbf{Lie}$: the category of Lie algebras,
$\mathscr{B} = \mathbf{Alg}$: the category of algebras,

R : equips A with the canonical Lie bracket $[a, b] = ab - ba$,
L : « enveloping algebra » functor $\mathfrak{g} \mapsto U(\mathfrak{g})$.

$$U(\mathfrak{g}) \quad := \quad T\mathfrak{g} \quad / \quad I(\mathfrak{g})$$

where $I(\mathfrak{g})$ is the ideal generated by $ab - ba - [a, b]$.

Illustration: the free category

$$\mathbf{Graph} \underset{R}{\overset{L}{\rightleftarrows}} \mathbf{Cat} \qquad (L \dashv R)$$

where

$\mathscr{A} = \mathbf{Graph}$: the category of graphs,
$\mathscr{B} = \mathbf{Cat}$: the category of categories and functors,

R : the « forgetful » functor
L : the « free category » functor

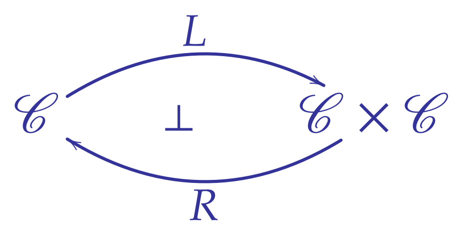
Illustration: the terminal object

$$\mathscr{C} \underset{R}{\overset{L}{\underset{\perp}{\rightleftarrows}}} \mathbb{1}$$

where

$\mathscr{A} = \mathscr{C}$	:	any category equipped with a terminal object 1
$\mathscr{B} = \mathbb{1}$	:	the singleton category
L	:	the canonical (and unique) functor
R	:	the functor whose image is the terminal object 1

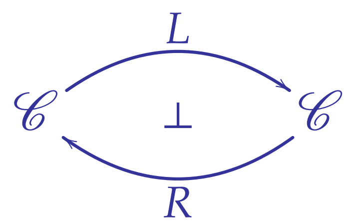
Illustration: cartesian categories



where

$\mathscr{A} = \mathscr{C}$	:	any cartesian category
$\mathscr{B} = \mathscr{C} \times \mathscr{C}$	:	the product category
L	:	the diagonal functor $A \mapsto (A, A)$
R	:	the functor $(A, B) \mapsto A \times B$

Illustration: cartesian closed categories

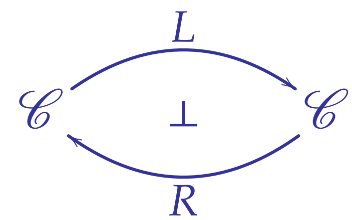


where

$\mathscr{A} = \mathscr{B} = \mathscr{C}$	:	any cartesian closed category $\mathscr{C}$
L	:	the functor $B \mapsto A \times B$
R	:	the functor $B \mapsto A \Rightarrow B$

for a given object A of the cartesian closed category $\mathscr{C}$.

Illustration: negation



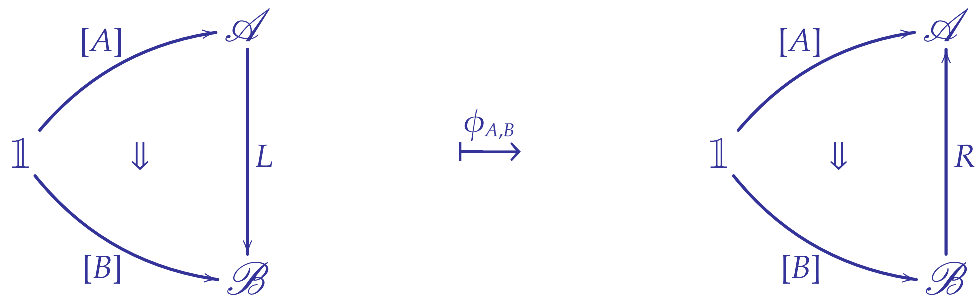
where

$\mathscr{A} = \mathscr{C}$	:	any cartesian closed category $\mathscr{C}$
$\mathscr{B} = \mathscr{C}^{\,op}$	:	the opposite category $\mathscr{C}^{\,op}$
L	:	the negation functor $A \mapsto A \Rightarrow \perp$
R	:	the negation functor $A \mapsto A \Rightarrow \perp$

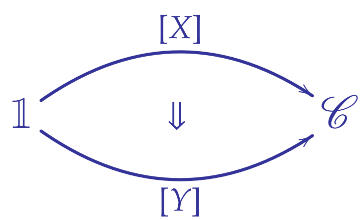
for a given object $\perp$ of the cartesian closed category $\mathscr{C}$.

Adjunction in the 2-category Cat

A bijection ϕ between the natural transformations



Here, a morphism $X \longrightarrow Y$ in the category $\mathscr{C}$ is seen as a natural transformation $[X] \longrightarrow [Y]$.



Adjunction in the 2-category Cat

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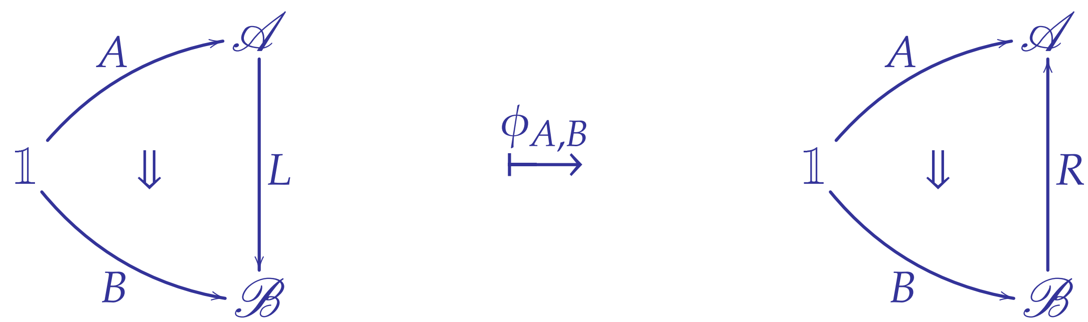

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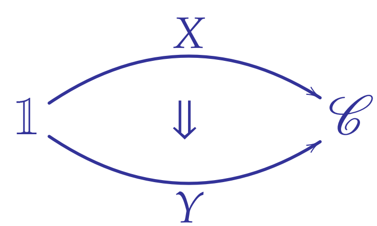
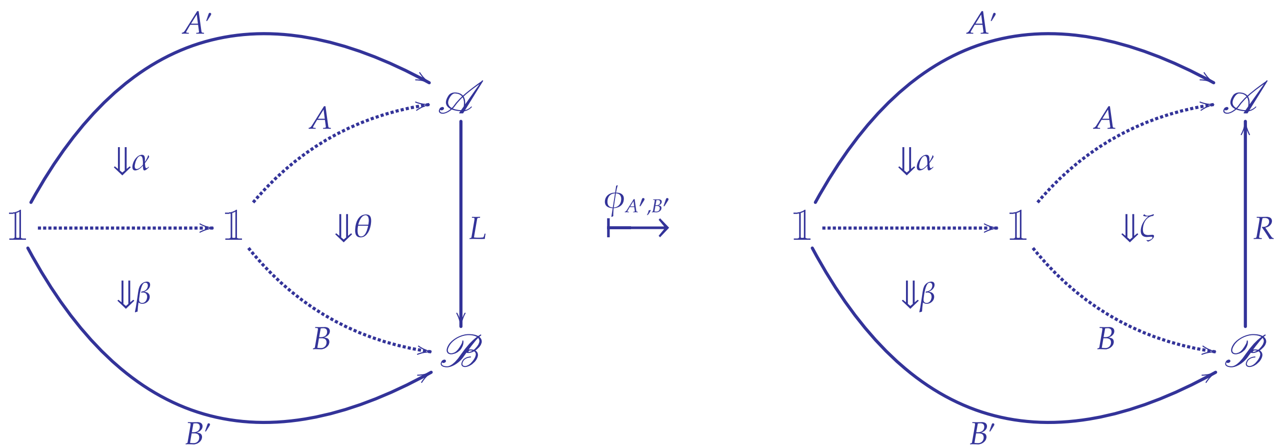


A 2-dimensional naturality condition

One reformulates the naturality conditionin that way:

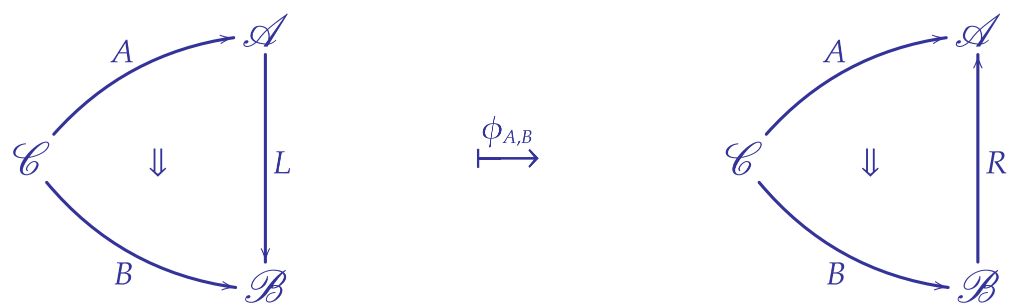
The bijection ϕ is natural with respect to the natural transformations α and β.



Adjunction in the 2-category Cat

This point of view leads to a more satisfactory definition of adjunction:

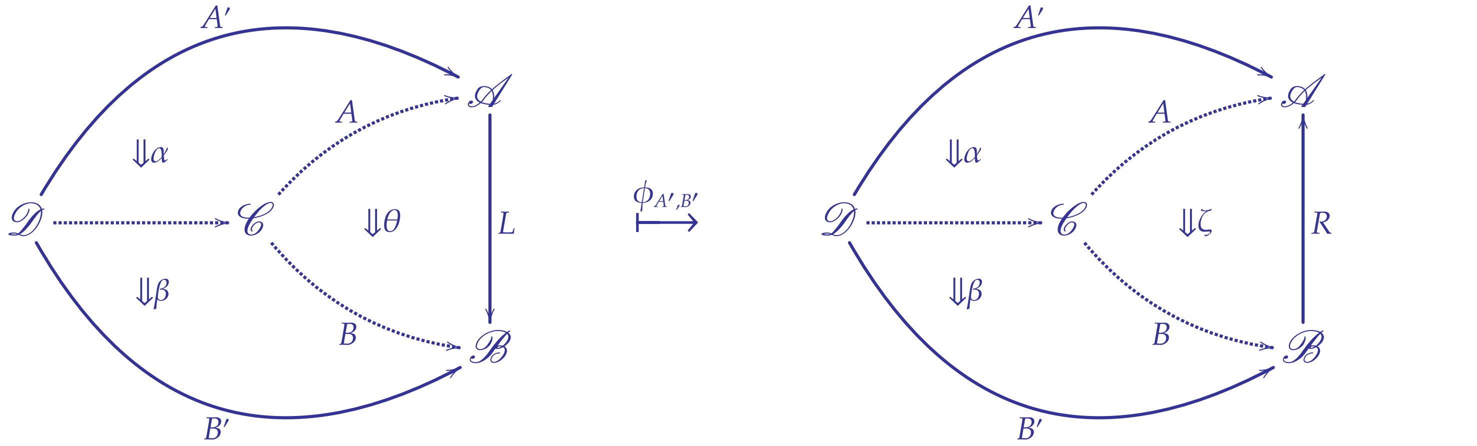
A bijection ϕ between the natural transformations



Adjunction in the 2-category Cat

One reformulates the naturality condition as follows:

The bijection ϕ is natural with respect to the natural transformations α et β.



Algebraic presentation of the adjunction

An **adjunction** is a quadruple $(L, R, \eta, \varepsilon)$ where L and R are functors

$$L \;:\; \mathscr{A} \longrightarrow \mathscr{B} \qquad\qquad R \;:\; \mathscr{B} \longrightarrow \mathscr{A}$$

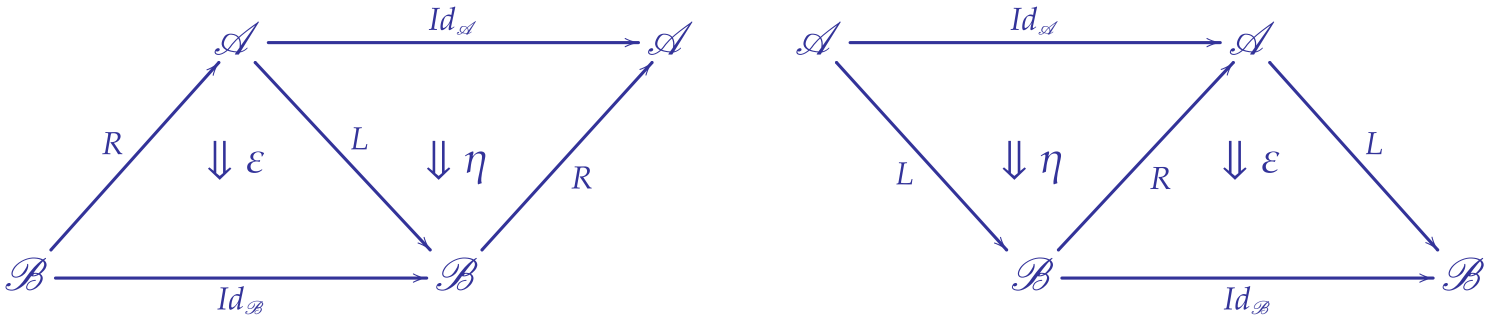
and η and ε are natural transformations:

$$\eta \;:\; Id_{\mathscr{A}} \stackrel{.}{\longrightarrow} RL \qquad\qquad \varepsilon \;:\; LR \stackrel{.}{\longrightarrow} Id_{\mathscr{B}}$$

such that the composite

$$R \xrightarrow{\ \eta R\ } RLR \xrightarrow{\ R\varepsilon\ } R \qquad\qquad L \xrightarrow{\ L\eta\ } LRL \xrightarrow{\ \varepsilon F\ } L$$

depicted as



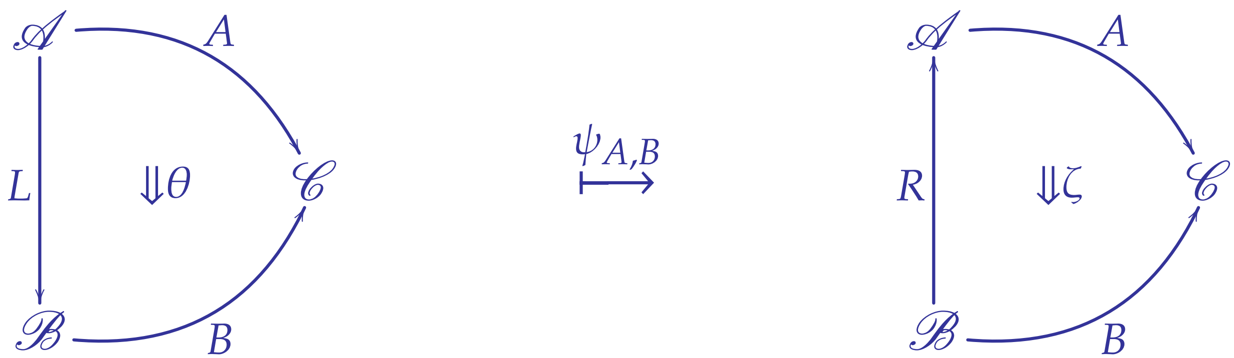
are the natural identities

$$Id_R : R \Rightarrow R \qquad\qquad Id_L : L \Rightarrow L$$

of the functors R and L.

Dual definition (but equivalent) of adjunction

By duality, an adjunction is given by a family of bijections ψ between the sets of 2-cells

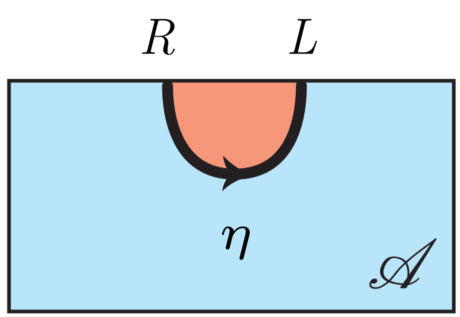


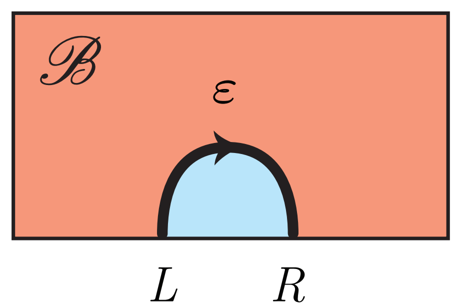
natural in A and B.

The 2-dimensional topology of adjunctions

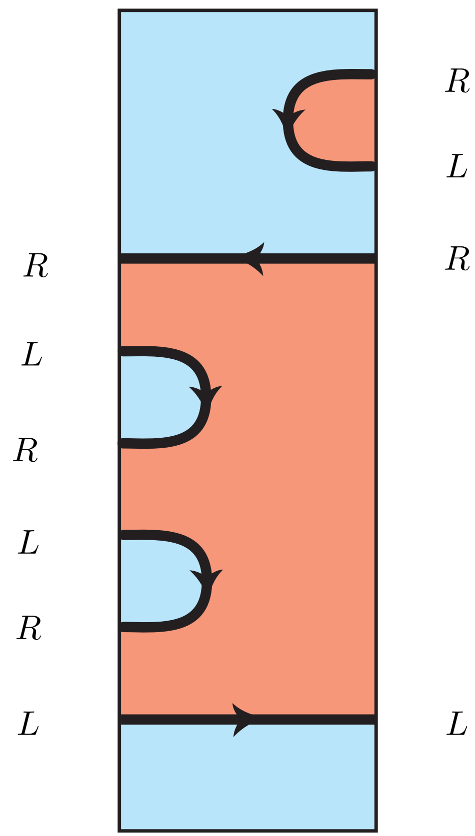
The **unit** and **counit** of the adjunction $L \dashv R$ are depicted as

$\eta : Id \Rightarrow R \circ L$ $\qquad$ $\varepsilon : L \circ R \Rightarrow Id$

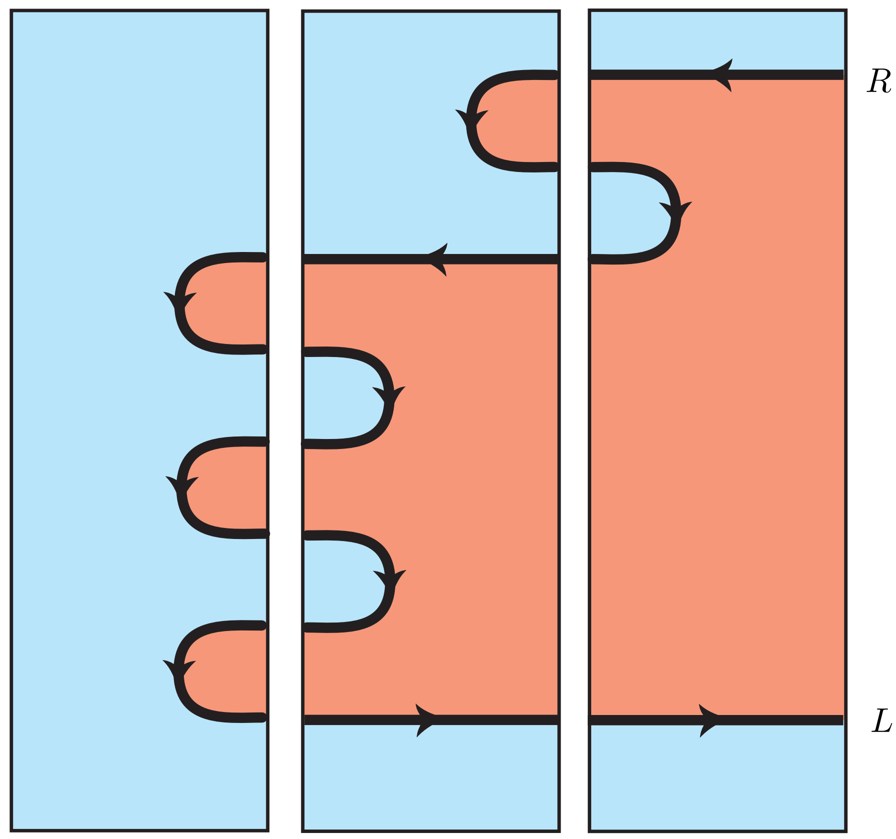




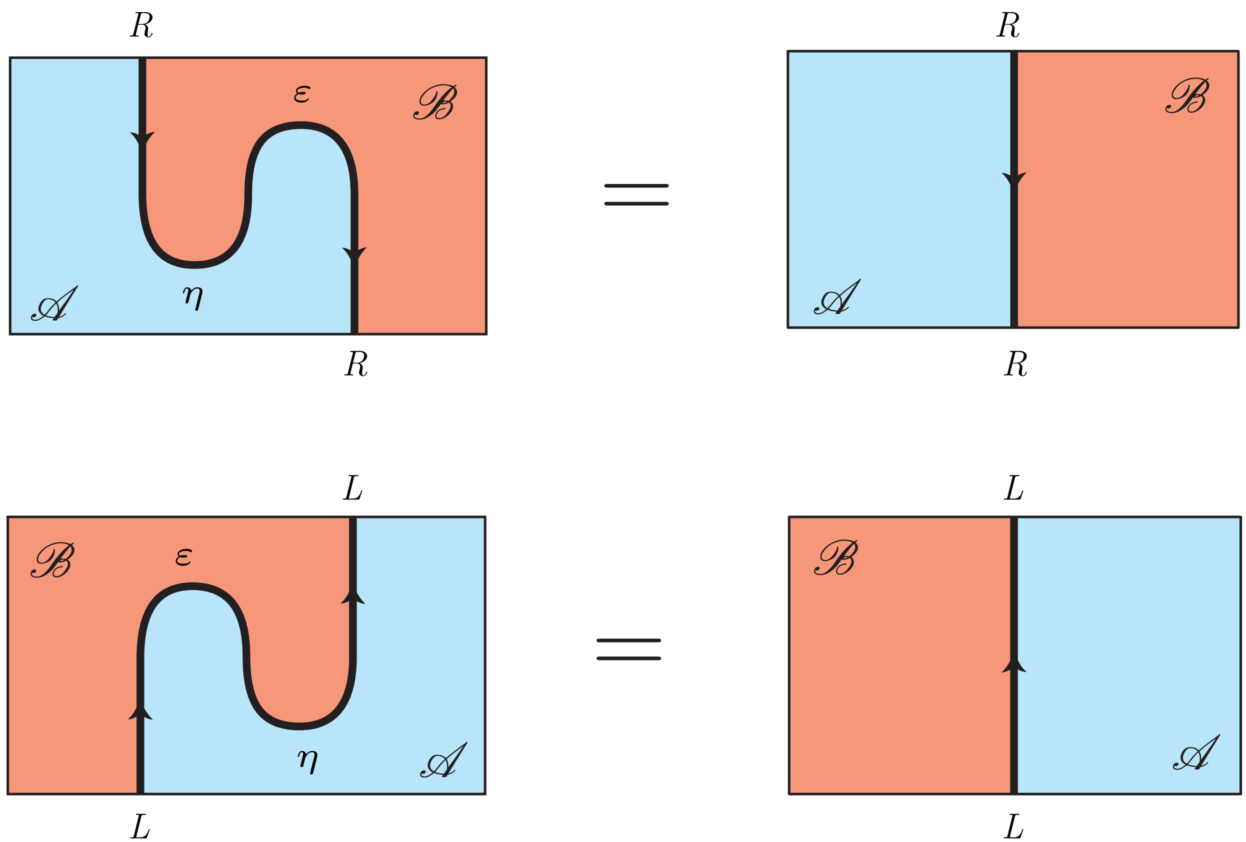
A typical 2-cell generated by an adjunction



A purely diagrammatic composition



The 2-dimensional dynamics of adjunctions



String diagrams

The λ-term

$$\varphi : \neg\neg A , \psi : \neg\neg B \quad \vdash \quad \lambda k.\, \varphi(\lambda a.\psi(\lambda b.k(a,b))) : \neg\neg(A \otimes B)$$

has the following control flow diagram

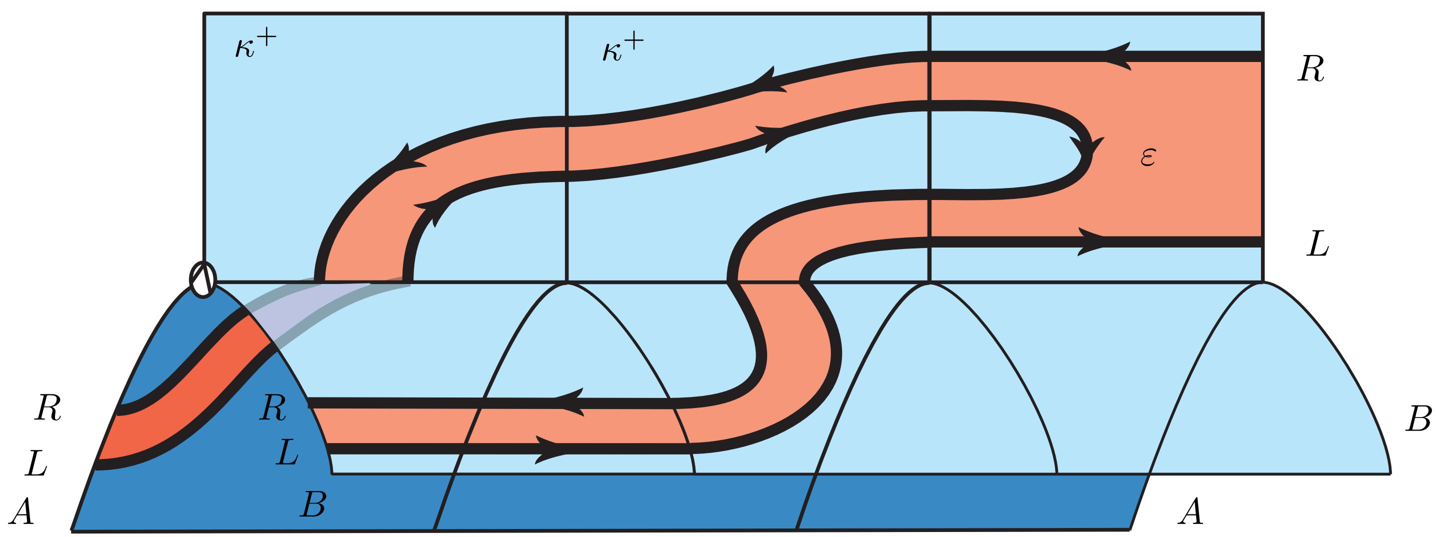


Illustration: the 2-category of sets and relations

Show that a relation

$$f \quad : \quad A \quad \longrightarrow \quad B$$

is left adjoint if and only if it is functional:

$$\forall a \in A. \quad \exists!\, b \in B. \qquad a\,[f]\,b$$

Show that its right adjoint g is the relation defined as

$$\forall a \in A. \quad \forall b \in B. \qquad a\,[f]\,b \quad \Longleftrightarrow \quad b\,[g]\,a.$$

Monads

Kleisli category, Eilenberg-Moore category

Monads

Suppose given a 0-cell $\mathscr{C}$ in a 2-category $\mathcal{W}$.

A monad T on a 0-cell $\mathscr{C}$ is a 1-cell

$$T \quad : \quad \mathscr{C} \quad \longrightarrow \quad \mathscr{C}$$

equipped with a multiplication

$$\mu \quad : \quad T \circ T \quad \Rightarrow \quad T \quad : \quad \mathscr{C} \quad \longrightarrow \quad \mathscr{C}$$

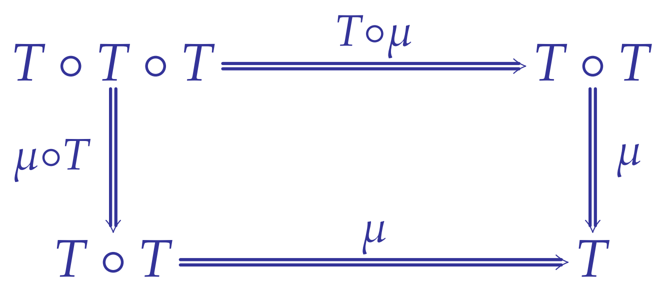
and with a unit

$$\eta \quad : \quad Id_{\mathscr{C}} \quad \Rightarrow \quad T \quad : \quad \mathscr{C} \quad \longrightarrow \quad \mathscr{C}$$

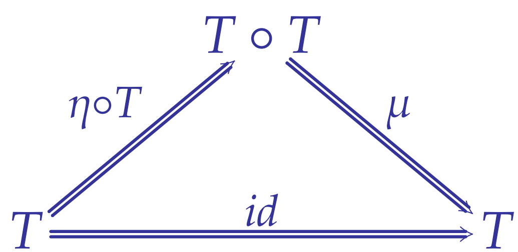
satisfying the expected associativity and unit laws.

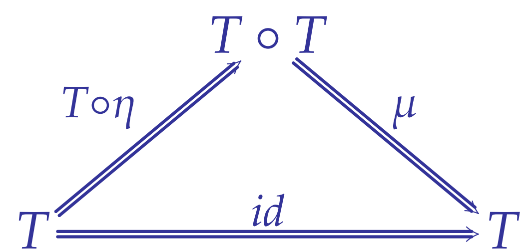
Monads

- Associativity law:



- Left and right unit laws:





Every adjunction defines a monad

(with a graphical proof)

Illustration: the state monad

Every set S induces a monad

$$X \quad \mapsto \quad S \Rightarrow (S \times X) \quad : \quad \mathbf{Set} \quad \longrightarrow \quad \mathbf{Set}$$

called **the state monad**. This monad is induced by the adjunction

$$\mathbf{Set} \underset{R}{\overset{L}{\rightleftarrows}} \mathbf{Set} \qquad (L \perp R)$$

where

$$\begin{array}{lcl} L & : & X \mapsto S \times X \\ R & : & X \mapsto S \Rightarrow X. \end{array}$$

Algebra

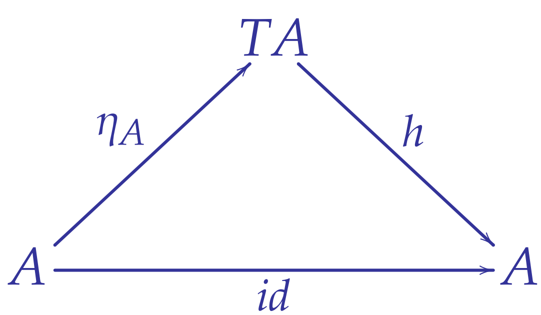
Suppose given a monad T on a category $\mathscr{C}$.

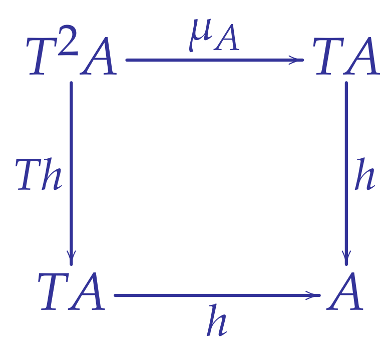
An algebra of the monad (T, μ, η) is a pair (A, h) consisting of

- ▷ an object A of the category $\mathscr{C}$
- ▷ a morphism

$$h \quad : \quad TA \longrightarrow A$$

making the diagrams





commute.

Algebra homomorphism

An algebra homomorphism

$$f \quad : \quad (A, h_A) \quad \longrightarrow \quad (B, h_B)$$

is a morphism

$$f \quad : \quad A \quad \longrightarrow \quad B$$

making the diagram

$$\begin{array}{ccc} TA & \xrightarrow{Tf} & TB \\ {\scriptstyle h_A}\downarrow & & \downarrow{\scriptstyle h_B} \\ A & \xrightarrow[f]{} & B \end{array}$$

commute in the category $\mathscr{C}$.

Kleisli category

The Kleisli category $\mathscr{C}_T$ of a monad (T, μ, η) is the category $\mathscr{C}$

▷ with the same objects as the category $\mathscr{C}$,

▷ with the morphisms

$$A \longrightarrow TB$$

in the category $\mathscr{C}$ as morphisms

$$A \longrightarrow\!\!\!\!\rightarrow B$$

in the Kleisli category.

Kleisli category

The identities

$$id_A \quad : \quad A \longrightarrow\!\!\!\!\rightarrow A$$

are given by the morphisms

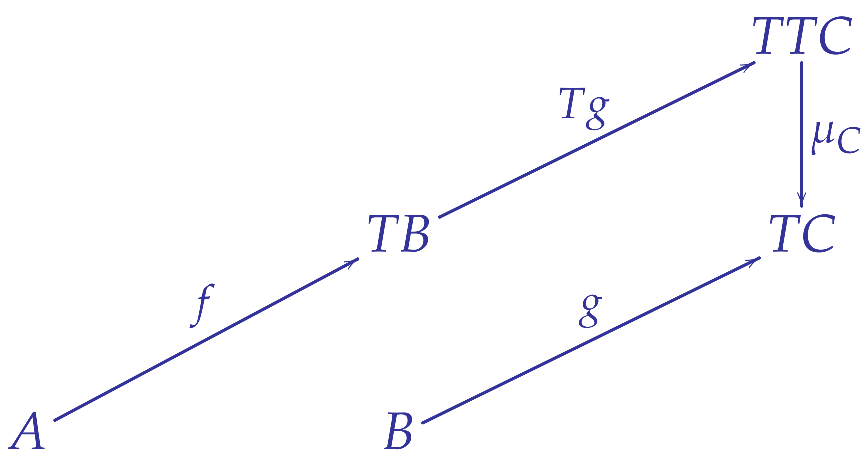
$$\eta_A \quad : \quad A \longrightarrow TA.$$

The two morphisms

$$f : A \longrightarrow\!\!\!\!\rightarrow B \qquad\qquad g : B \longrightarrow\!\!\!\!\rightarrow C$$

are composed as follows

$$g \circ_K f \quad :=$$

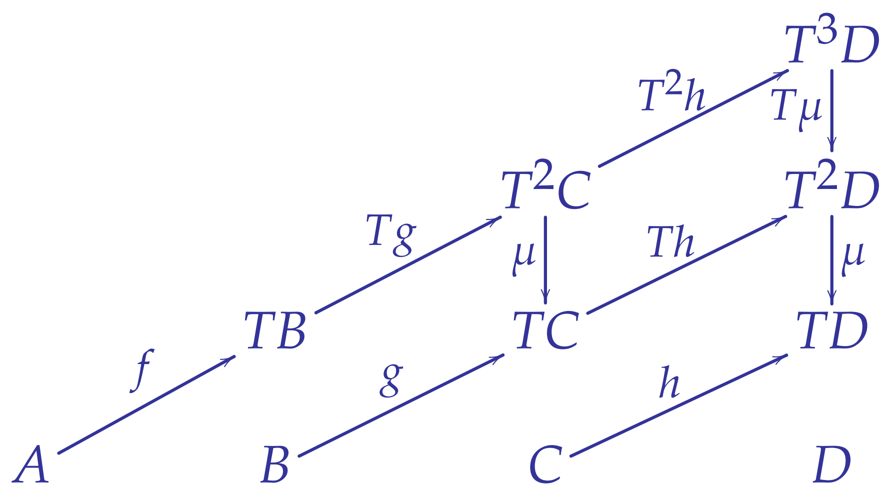


Exercise

Show that:

- ▷ that the identities of the Kleisli category are identities
- ▷ that its composition is associative.

Remark: checking associativity requires to consider the diagram



and to show that the two maps from A to TD coincide.

Short bibliography of the course

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