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Multiset ordering, battle of the hydra
Well-quasi-orderings, Higman and Kruskal theorems

λ-calculs et catégories (31 octobre 2016)

1 Multiset ordering

Recall that a multiset M on a set S is defined as a function

$$M : S \to \mathbb{N}.$$

The intuition is that the function M describes how many times each element $s \in S$ appears in the multiset. A multiset M is called finite when its domain

$$\mathrm{dom}(M) = \{ s \in S \mid M(s) \neq 0 \}$$

is finite. Notation: we write $M = [\, a, c, c, c, d, d, d, d \,]$ for the finite multiset

$$M \quad = \quad a \mapsto 1 \quad , \quad b \mapsto 0 \quad , \quad c \mapsto 3 \quad , \quad d \mapsto 4$$

on the set $S = \{a, b, c, d\}$. We also write $s \in M$ when $M(s) \neq 0$ or equivalently, when $s \in \mathrm{dom}(M)$. Given two multisets M, N, we define the multiset $M \uplus N$ as follows:

$$M \uplus N = s \mapsto M(s) + N(s).$$

We write $M \subseteq N$ when $\forall s \in S, M(s) \leq N(s)$ and define in that case the multiset $M \setminus N$ as follows:

$$M \setminus N = s \mapsto M(s) - N(s).$$

Notation: the set of finite multisets on S is denoted $\mathscr{M}(S)$.

We suppose from now on that $(S, \succeq)$ is a partially ordered set. As explained during the course, we write

$$M \succ_{\mathsf{mset}} N$$

for two finite multisets M, N on S, when there exists two finite multisets X, Y such that

$$N = (M \setminus X) \uplus Y \qquad \text{where} \quad \forall y \in Y, \exists x \in X, x \succ y.$$

§1. Given two finite multisets M, N, we write $M \rhd_{\mathsf{mset}} N$ when

$$N = (M \setminus [\, x \,]) \uplus Y$$

for an element $x \in M$ and a finite multiset Y such that $\forall y \in Y, x \succ y$. Show that the relation $\succeq_{\mathsf{mset}}$ is the smallest partial ordering on finite multisets containing $\rhd_{\mathsf{mset}}$.

§2. Notation: we find convenient to write $M \rhd_x^Y N$ or simply $M \rhd_x N$ in the situation described in §1 where $M \rhd_{\mathsf{mset}} N$. We want to show that the partial ordering $(\mathscr{M}(S), \succeq_{\mathsf{mset}})$ is well-founded when $(S, \succeq)$ is well-founded. To that purpose, we proceed by contradiction, and suppose the existence of an infinite decreasing sequence of finite multisets:

$$M_1 \rhd_{\mathsf{mset}} M_2 \rhd_{\mathsf{mset}} M_3 \rhd_{\mathsf{mset}} M_4 \rhd_{\mathsf{mset}} \dots \rhd_{\mathsf{mset}} M_n \rhd_{\mathsf{mset}} \dots \tag{1}$$

where for the sake of notations:

$$M_n \rhd_{x_n}^{Y_n} M_{n+1}.$$

We construct a family T_n of rooted trees in the following way:

- the rooted tree T_1 has a leaf attached to its root $*$ for each element $x \in M_1$ with repetition,
- each rooted tree T_{n+1} is obtained from the rooted tree T_n by picking a leaf x_n in the tree T_n and by attaching a leaf to x_n for each element $y \in Y_n$ with repetition.

Illustrate the construction of the trees T_1, T_2, T_3, T_4, T_5 for the sequence of multisets:

$$[\, a, a, b \,] \rhd_a [\, a, b, b, c \,] \rhd_b [\, a, b, c, c, c, d \,] \rhd_b [\, a, c, c, c, d, d, d, d \,]$$

on the ordered set $S = \{a, b, c, d\}$ where $a \succ b \succ c \succ d$. Are there different choices of construction of that specific sequence of rooted trees?

§3. Deduce from the construction in §2. and König's lemma that there exists no infinite sequence (1) when the partial ordering $(S, \succeq)$ is well-founded.

§4. Deduce that the partial ordering $(\mathscr{M}(S), \succeq_{\mathsf{mset}})$ is well-founded if and only if the partial ordering $(S, \succeq)$ is well-founded.

2 Nested multiset ordering

Informally speaking, a nested multiset $M \in \mathscr{M}^*(S)$ over a base set S is a multiset whose elements may belong to the base set S, or may be multisets containing both elements of S and multisets of elements of S, and so on. For example,

$$[\,[\, a, a \,], [\,[\, a \,], c, d \,], a, b \,]$$

is a nested multiset over $S = \{a, b, c, d\}$. We are looking for a more formal definition. Given two disjoint sets S and T, a pointed multiset over S, T is a finite multiset on their disjoint union $S \uplus T$ which contains at least an element of S. The set of pointed multisets over S, T is noted

$$\mathscr{M}^\bullet(S, T) \subseteq \mathscr{M}(S \uplus T).$$

We define a hierarchy of sets

$$\mathscr{M}^{<k+1}(S) \qquad\qquad \mathscr{M}^{k+1}(S)$$

of nested multisets over the base set S of height strictly less than, and strictly equal to, $k + 1 \in \mathbb{N}$. The hierarchies are defined as

$$\mathscr{M}^{<k+1}(S) = \mathscr{M}^k(S) \uplus \mathscr{M}^{<k}(S) \qquad\qquad \mathscr{M}^{k+1}(S) = \mathscr{M}^\bullet(\mathscr{M}^k(S), \mathscr{M}^{<k}(S))$$

with base case

$$\mathscr{M}^{<0}(S) = \emptyset \qquad\qquad \mathscr{M}^{0}(S) = S.$$

§1. Suppose that the base set S contains only atoms, and show by induction on $k \in \mathbb{N}$ that

$$\mathscr{M}^{<k}(S) \quad = \quad \biguplus_{p<k} \quad \mathscr{M}^{p}(S)$$

and that $\mathscr{M}^{<k}(S)$ and $\mathscr{M}^{k}(S)$ are disjoint sets.

§2. From this, deduce that

$$\forall p, q \in \mathbb{N}, \qquad\qquad p \neq q \quad \Rightarrow \quad \mathscr{M}^{p}(S) \cap \mathscr{M}^{q}(S) = \emptyset$$

The set $\mathscr{M}^{*}(S)$ of nested multisets over the base set S is defined as

$$\mathscr{M}^{*}(S) \quad = \quad \biguplus_{k\in\mathbb{N}} \quad \mathscr{M}^{k}(S)$$

§3. We suppose from now on that the base set $(S, \succeq)$ is partially ordered. We define an ordering on $\mathscr{M}^{<k}(S)$ and on $\mathscr{M}^{k}(S)$ by induction:

- every element of $\mathscr{M}^{k+1}(S)$ is strictly larger than every element of $\mathscr{M}^{<k+1}(S)$,
- two nested multisets $M, N \in \mathscr{M}^{<k+1}(S)$ of height $k+1$ are ordered by the multiset ordering of
$$\mathscr{M}^{k+1}(S) \quad \subseteq \quad \mathscr{M}(\mathscr{M}^{k}(S) \uplus \mathscr{M}^{<k}(S))$$

Show by induction on k that this defines an ordering relation

$$M \succ^{k}_{\mathsf{mset}} N$$

between nested multisets $M, N \in \mathscr{M}^{k}(S)$ of height k. Deduce the existence of an ordering relation

$$M \succ^{*}_{\mathsf{mset}} N$$

between nested multisets $M, N \in \mathscr{M}^{*}(S)$

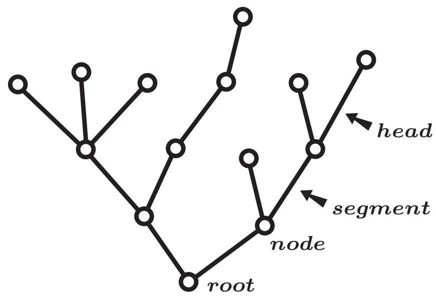
§4. Repeat the argument by induction of §3. to establish that the partial order

$$(\mathscr{M}^{*}(S), \succ^{*}_{\mathsf{mset}})$$

on nested multisets is well-founded when the base set $(S, \succeq)$ is well-founded.

3 Battle of Hercules against the hydra

A hydra is a finite rooted tree which may be considered as a finite collection of straight line segments, each joining two nodes, such that every node is connected by a unique path of segments to the root. For example:

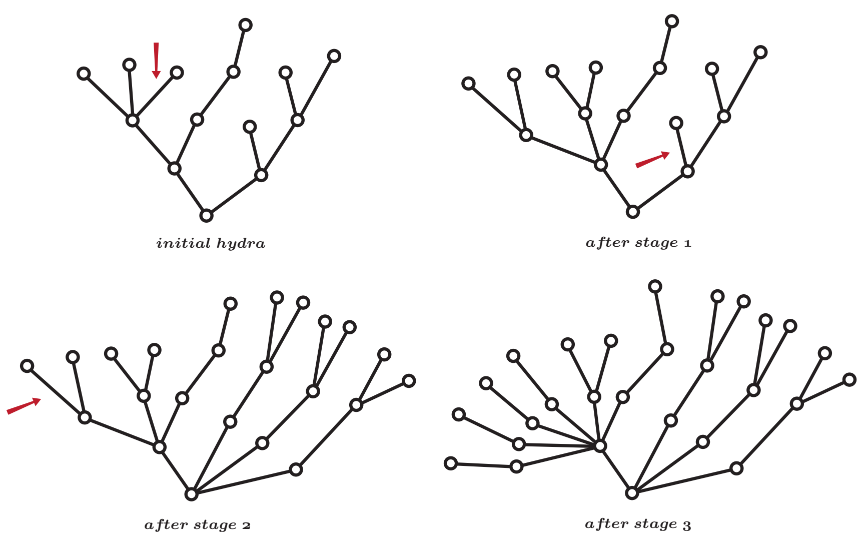


A top node of a hydra is one which is a node of only one segment, and is not the root. A *head* of the hydra is a top node together with its attached segment.

A battle between Hercules and a given hydra proceeds as follows: at stage $n \geq 1$, Hercules chops off one head from the hydra. The hydra then grows n new heads in the following manner:

From the node that used to be attached to the head which was just chopped off, traverse one segment towards the root until the next node is reached. From this node sprout n replicas of the part of the hydra (after decapitation) which is "above" the segment just traversed, i.e., those nodes and segments from which, in order to reach the root, this segment would have to be traversed. If the head just chopped off had the root as one of its nodes, no new head is grown.

Thus the battle might for instance commence like this, assuming that at each stage Hercules decides to chop off the head marked with an arrow:



Hercules wins if after some finite number of stages, nothing is left of the hydra but its root. A strategy is a function which determines for Hercules which head to chop off at each stage of any battle.

§1. Establish a one-to-one relationship between hydra and nested sequences of finite multisets, generated by an empty base set $S = \emptyset$.

§2. Deduce that every strategy of Hercules is a winning strategy.

4 Well-quasi-orderings and Kruskal theorem

A well-quasi-ordering (wqo) is a partial order $(A, \preceq)$ such that for every infinite sequence

$$(a_n)_{n\in\mathbb{N}} \quad = \quad a_0, a_1, \ldots, a_n, \ldots \tag{2}$$

of elements of A, there exists a pair of indices $i, j \in \mathbb{N}$ such that

$$i < j \qquad \text{and} \qquad a_i \preceq a_j.$$

A sequence is called *good* in that case, and *bad* otherwise. Hence, a wqo is a partial order where every infinite sequence is good.

§1. Show that for every sequence (2) of a wqo, there exists an index $i \in \mathbb{N}$ an an infinite number of indices $j \geq i$ such that

$$a_i \preceq a_j.$$

§2. Deduce that for every sequence (2) of a wqo, there exists an infinite subsequence $(a_{\varphi(i)})_{i\in\mathbb{N}}$ such that

$$\forall i, j, \qquad i < j \quad \Rightarrow \quad a_{\varphi(i)} \preceq a_{\varphi(j)}.$$

§3. Apply Ramsey theorem to establish that for every sequence (2) of elements of a partial order $(A, \preceq)$, one of the following three cases occurs:

- the sequence contains an infinite antichain,
- the sequence contains an infinite increasing subsequence,
- the sequence contains an infinite strictly decreasing subsequence.

§4. Show that a partial order is a wqo if and only if it is well-founded and it does not contain any infinite antichain.

Given two words $u = a_1 \cdots a_m$ and $v = b_1 \cdots b_n$, one writes

$$u \quad \preceq_{\mathbf{emb}} \quad v$$

when there exists an injective and monotone function

$$\varphi \quad : \quad \{1, \ldots, m\} \quad \longrightarrow \quad \{1, \ldots, n\}$$

such that

$$\forall i \in \{1, \ldots, m\} \qquad\qquad a_i \quad \preceq \quad b_{\varphi(i)}.$$

§5. Check that $\preceq_{\mathbf{emb}}$ defines a partial order on finite words.

§6. Higman Theorem. Show that $\preceq_{\mathbf{emb}}$ defines a wqo on finite words, using the "smallest counterexample" argument below.

§6a. Proceed by contradiction, and suppose that there exists an infinite bad sequence. Construct a sequence v in the following way. Start from the index $i = 0$ and at each index i, take the shortest word v_i such that there exists a bad sequence

$$v_0, \ldots, v_i, u_{i+1}, u_{i+2}, \ldots$$

Show that the resulting sequence

$$S = v_0, \ldots, v_i, v_{i+1}, v_{i+2}, \ldots$$

is bad.

§6b. Show that the sequence S does not contain the empty word.

§6c. We write

$$v_i = a_i \cdot w_i$$

where a_i is the first letter of the word v_i, and w_i its suffix. There exists an infinite sequence

$$\forall i, j \qquad i < j \quad \Rightarrow \quad a_{\varphi(i)} \quad \preceq \quad a_{\varphi(j)}$$

Consider the corresponding sequence S' of suffixes

$$S' \quad = \quad w_{\varphi(0)}, \cdots, w_{\varphi(n)}, \cdots$$

Show that the sequence S' is bad.

§6d. Deduce a contradiction from the construction of S, and conclude that the embedding order $\preceq_{\mathbf{emb}}$ is a wqo.

Given a partial order on A, the homeomorphic embedding ordering of trees labelled by A is defined as

$$a(s_1, \ldots, s_m) \preceq_{\mathbf{hom}} b(t_1, \ldots, t_n)$$

precisely when

$$a \preceq b$$

and moreover there exists an injective and monotone function

$$\varphi \quad : \quad \{1, ..., m\} \quad \longrightarrow \quad \{1, ..., n\}$$

such that

$$\forall i \in \{1, ..., m\}, \qquad s_i \preceq_{\mathbf{hom}} t_j.$$

§7. Kruskal tree theorem. Adapt the previous argument and establish that $\preceq_{\mathbf{hom}}$ is a wqo on finite trees labelled by A when $\preceq$ is a wqo on A.