

Travaux Dirigés

Distributivity laws between functors and monads Grothendieck construction and set-theoretic colimits

λ-calculs et catégories (12 décembre 2016)

1 Distributivity laws between functors and monads

§1. Suppose given two categories \mathcal{A} and \mathcal{B} , each of them equipped with a monad

$$(S, \mu_S, \eta_S) : \mathcal{A} \longrightarrow \mathcal{A} \qquad (T, \mu_T, \eta_T) : \mathcal{B} \longrightarrow \mathcal{B}$$

A homomorphism

$$(F, \lambda) : (\mathcal{A}, S) \longrightarrow (\mathcal{B}, T) \tag{1}$$

is defined as a functor $F : \mathcal{A} \rightarrow \mathcal{B}$ equipped with distributivity law

$$\lambda : T \circ F \Rightarrow F \circ S$$

making the diagrams of natural transformations below commute:

$$\begin{array}{ccc}
 T \circ T \circ F & \xrightarrow{T \circ \lambda} & T \circ F \circ S & \xrightarrow{\lambda \circ S} & F \circ S \circ S \\
 \mu_{T \circ F} \downarrow & & & & \downarrow F \circ \mu_S \\
 T \circ F & \xrightarrow{T \circ \lambda} & F \circ S & & \\
 & & & &
 \end{array}
 \tag{a}$$

$$\begin{array}{ccc}
 & F & \\
 \eta_T \swarrow & & \searrow \eta_S \\
 T \circ F & \xrightarrow{\lambda} & F \circ S
 \end{array}
 \tag{b}$$

§1. Formulate the two commutative diagrams (a) and (b) as families of commutative diagrams between maps living in the category \mathcal{B} .

§2. Depict the commutative diagrams (a) and (b) in the language of string diagrams.

§3. Show that every homomorphism (F, λ) as in (1) induces a functor

$$\tilde{F} : \mathbf{Alg}(S) \longrightarrow \mathbf{Alg}(T)$$

making the diagram below commute:

$$\begin{array}{ccc}
 \mathbf{Alg}(S) & \xrightarrow{\tilde{F}} & \mathbf{Alg}(T) \\
 U_S \downarrow & & \downarrow U_T \\
 \mathcal{A} & \xrightarrow{F} & \mathcal{B}
 \end{array}
 \tag{*}$$

where U_S and U_T are the forgetful functors associated to the monads S and T , respectively.

§4. Conversely, show that every functor

$$\tilde{F} : \mathbf{Alg}(S) \longrightarrow \mathbf{Alg}(T)$$

making the diagram (*) commute induces a distributivity law $\lambda : T \circ F \Rightarrow F \circ S$ making the two diagrams (a) and (b) commute.

§5. Conclude that a homomorphism $(F, \lambda) : (\mathcal{A}, S) \rightarrow (\mathcal{B}, T)$ between two monads may be equivalently defined as a pair (F, \tilde{F}) of functors

$$F : \mathcal{A} \longrightarrow \mathcal{B} \qquad \tilde{F} : \mathbf{Alg}(S) \longrightarrow \mathbf{Alg}(T)$$

making the diagram (*) commute.

§6. Deduce that there is a category **Mon** of monads and homomorphisms between them.

§7. Describe the free abelian group functor $F : \mathbf{Sets} \rightarrow \mathbf{Sets}$ which transports every set A to the free abelian group FA generated by the set A .

§8. Construct a family of functions

$$\lambda_A : TF(A) \longrightarrow FT(A)$$

parametrized by an object $A \in \mathcal{A}$ and check that the family λ is natural in A and makes the diagrams (a) and (b) commute.

§9. From this, deduce the existence of a functor

$$\tilde{F} : \mathbf{Monoid} \longrightarrow \mathbf{Monoid}$$

from the category of monoids and homomorphisms, making the diagram below commute:

$$\begin{array}{ccc} \mathbf{Monoid} & \xrightarrow{\tilde{F}} & \mathbf{Monoid} \\ U \downarrow & (*) & \downarrow U \\ \mathbf{Sets} & \xrightarrow{F} & \mathbf{Sets} \end{array}$$

§10. Describe the natural transformations μ_F and η_F equipping the functor F as a monad (F, μ_F, η_F) .

§11. A distributivity law

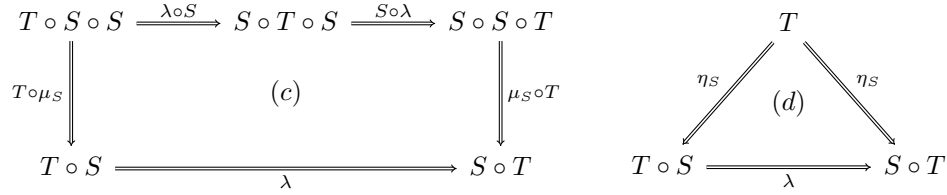
$$\lambda : T \circ S \Rightarrow S \circ T$$

between two monads on the same category

$$(S, \mu_S, \eta_S) : \mathcal{A} \longrightarrow \mathcal{A} \qquad (T, \mu_T, \eta_T) : \mathcal{A} \longrightarrow \mathcal{A}$$

is a natural transformation making the diagrams below commute

$$\begin{array}{ccc} T \circ T \circ S & \xrightarrow{T \circ \lambda} & T \circ S \circ T & \xrightarrow{\lambda \circ T} & S \circ T \circ T \\ \mu_{T \circ S} \downarrow & & & & \downarrow S \circ \mu_T \\ T \circ S & \xrightarrow{\lambda} & S \circ T & & \end{array} \qquad \begin{array}{ccc} & S & \\ \eta_T \swarrow & & \searrow \eta_T \\ T \circ S & \xrightarrow{\lambda} & S \circ T \end{array}$$



Depict the commutative diagrams (c) and (d) in the language of string diagrams.

§12. Show that every distributive law $\lambda : T \circ S \Rightarrow S \circ T$ between two monads S and T on the same category \mathcal{A} induces a monad structure on the composite functor $S \circ T : \mathcal{A} \rightarrow \mathcal{A}$.

§13. Show that the natural transformation λ defined in §7. defines a distributivity law between the monads $S = F$ and T .

§14. Show that the monad $S \circ T : \mathbf{Sets} \rightarrow \mathbf{Sets}$ associated to the distributivity law $\lambda : T \circ S \Rightarrow S \circ T$ coincides with the free algebra monad (here, by algebra, we mean \mathbb{Z} -algebra).

2 Grothendieck construction and colimits computed in the category of sets and functions

We recall that a contravariant presheaf on a small category \mathcal{C} is a functor

$$\varphi : \mathcal{C}^{op} \rightarrow \mathbf{Sets}.$$

The purpose of this exercise is to compute the colimit of this functor, seen as a diagram in the category \mathbf{Sets} . Every contravariant presheaf φ induces a category $\mathbf{Groth}[\varphi]$ together with a projection functor

$$\pi[\varphi] : \mathbf{Groth}[\varphi] \rightarrow \mathcal{C}. \quad (2)$$

The objects of the category are the pairs (c, x) with c an object of \mathcal{C} and x an element of $\varphi(c)$; the maps

$$(c, x) \rightarrow (d, y)$$

of the category are maps $f : c \rightarrow d$ of the underlying category \mathcal{C} such that

$$\varphi(f)(y) = x.$$

§1. Show that these data define a category $\mathbf{Groth}[\varphi]$ together with a functor (2).

§2. Show that every natural transformation

$$\theta : \varphi \Rightarrow \psi : \mathcal{C}^{op} \rightarrow \mathbf{Sets}$$

induces a functor

$$\mathbf{Groth}[\theta] : \mathbf{Groth}[\varphi] \rightarrow \mathbf{Groth}[\psi]$$

making the diagram below commute

$$\begin{array}{ccc}
\mathbf{Groth}[\varphi] & \xrightarrow{\mathbf{Groth}[\theta]} & \mathbf{Groth}[\psi] \\
\searrow \pi[\varphi] & & \swarrow \pi[\psi] \\
& \mathcal{C} &
\end{array}$$

§3. Conversely, show that every functor

$$F : \mathbf{Groth}[\varphi] \longrightarrow \mathbf{Groth}[\psi]$$

making the diagram below commute

$$\begin{array}{ccc} \mathbf{Groth}[\varphi] & \xrightarrow{F} & \mathbf{Groth}[\psi] \\ & \searrow \pi[\varphi] & \swarrow \pi[\psi] \\ & \mathcal{C} & \end{array}$$

is of the form $F = \mathbf{Groth}[\theta]$ for a unique natural transformation

$$\theta : \varphi \Rightarrow \psi : \mathcal{C}^{op} \longrightarrow \mathbf{Sets}$$

§4. For every object c of the category \mathcal{C} , construct a function

$$\theta_c : \varphi(c) \longrightarrow \pi_0(\mathbf{Groth}[\varphi])$$

where the set

$$\pi_0(\mathbf{Groth}[\varphi])$$

denotes the set of connected components of the category $\mathbf{Groth}[\varphi]$, defined as the connected components of the underlying graph.

§5. Show that the diagram below commutes

$$\begin{array}{ccc} \varphi(c) & \xrightarrow{\varphi(f)} & \varphi(d) \\ & \searrow \theta_c & \swarrow \theta_d \\ & \pi_0(\mathbf{Groth}[\varphi]) & \end{array}$$

for every map $f : c \rightarrow d$ in the category \mathcal{C} . Deduce from this that the family θ defines a natural transformation

$$\theta : \varphi \Rightarrow \pi_0(\mathbf{Groth}[\varphi])$$

and thus a cone.

§5. Show that the cone is a colimiting cone, and thus that the colimit of the diagram

$$\varphi : \mathcal{C}^{op} \longrightarrow \mathbf{Sets}$$

coincides with the set

$$\pi_0(\mathbf{Groth}[\varphi])$$

of connected components of the Grothendieck category $\mathbf{Groth}[\varphi]$. From this, deduce that the category \mathbf{Sets} has all small colimits.