

§2. Given two pullback diagrams

$$\begin{array}{ccc}
 Y'' & \xrightarrow{p'} & Y' \\
 g'' \downarrow & (a) & \downarrow g' \\
 X'' & \xrightarrow{f'} & X'
 \end{array}
 \qquad
 \begin{array}{ccc}
 Y' & \xrightarrow{p} & Y \\
 g' \downarrow & (b) & \downarrow g \\
 X' & \xrightarrow{f} & X
 \end{array}$$

in a category \mathcal{C} , show that the commutative diagram

$$\begin{array}{ccccc}
 Y'' & \xrightarrow{p'} & Y' & \xrightarrow{p} & Y \\
 g'' \downarrow & & & & \downarrow g \\
 X'' & \xrightarrow{f'} & X' & \xrightarrow{f} & X
 \end{array}
 \quad (c)$$

obtained by “glueing” the two diagrams (a) and (b) defines a pullback diagram in the category \mathcal{C} .

§3. Suppose given three commutative diagrams (a)(b)(c) in a category \mathcal{C} . We have seen in the previous question that when (b) is a pullback diagram,

$$(a) \text{ is a pullback diagram} \quad \Rightarrow \quad (c) \text{ is a pullback diagram}$$

Establish the converse property that

$$(c) \text{ is a pullback diagram} \quad \Rightarrow \quad (a) \text{ is a pullback diagram}$$

when (b) is a pullback diagram.

§4. Exhibit an example of three commutative diagrams (a)(b)(c) such that

$$(a) \text{ and } (c) \text{ are pullback diagrams... but } (b) \text{ is not a pullback diagram!}$$

Hint: one can take $X = \{x\}$ et $X'' = \{x''\}$ singleton sets and $X' = \{x_1, x_2\}$ a two-element set in the category $\mathcal{C} = \text{Sets}$.

2 Monomorphisms and epimorphisms

§1. An arrow $m : A \rightarrow B$ of a category \mathcal{C} is called a monomorphism (mono for short) when m is left-simplifiable in the sense that

$$m \circ f = m \circ g \quad \Rightarrow \quad f = g$$

for every pair of arrows $f, g : X \rightarrow A$. Show that a function $m : A \rightarrow B$ is a mono in the category Sets precisely when it is an injective function.

§2. An arrow $e : A \rightarrow B$ of a category \mathcal{C} is called an epimorphism (epi for short) when e is right-simplifiable in the sense that

$$f \circ e = g \circ e \quad \Rightarrow \quad f = g$$

for every pair of arrows $f, g : B \rightarrow Y$. Show that a function $e : A \rightarrow B$ is an epi in the category Sets precisely when it is a surjective function.

§3. Show that in any category \mathcal{C} , the composite $g \circ f : A \rightarrow C$ of two monos $f : A \rightarrow B$ and $g : B \rightarrow C$ is a mono, and that the composite of two epis is an epi.

§4. Show that an arrow $m : A \rightarrow B$ is a mono precisely when the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{id} & A \\ id \downarrow & & \downarrow m \\ A & \xrightarrow{m} & B \end{array}$$

is a pullback diagram in the category \mathcal{C} . Explain what the property means in the specific case of a function $m : A \rightarrow B$ in the category Sets.

§5. Show that every pullback diagram

$$\begin{array}{ccc} V & \xrightarrow{p} & U \\ m' \downarrow & (\otimes) & \downarrow m \\ B & \xrightarrow{f} & A \end{array}$$

in a category \mathcal{C} satisfies the following property:

$$m : U \rightarrow A \text{ is a mono} \quad \Rightarrow \quad m' : V \rightarrow B \text{ is a mono.}$$

Show that the converse property does not hold by constructing a counter-example in the category Sets.

3 Comma categories and subobject categories

§1. Every object A in a category \mathcal{C} induces a category \mathcal{C}/A called the comma category on the object A , and defined in the following way. The objects of \mathcal{C}/A are the pairs (X, f) consisting of an object $X \in \mathcal{C}$ and of an arrow

$$f : X \rightarrow A$$

with target A . The arrows of the category \mathcal{C}/A

$$h : (X, f) \longrightarrow (Y, g)$$

are the morphisms

$$h : X \longrightarrow Y$$

of the underlying category \mathcal{C} , making the diagram below commute:

$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ f \searrow & & \swarrow g \\ & A & \end{array}$$

Establish our claim above that \mathcal{C}/A defines a category.

§2. Show that a commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{p_2} & Y \\ p_1 \downarrow & (*) & \downarrow g \\ X & \xrightarrow{f} & Z \end{array}$$

in the category \mathcal{C} is the same thing as a diagram

$$\begin{array}{ccc} & (P, u) & \\ p_1 \swarrow & & \searrow p_2 \\ (X, f) & (**) & (Y, g) \end{array}$$

in the category \mathcal{C}/Z . Show moreover that the commutative diagram $(*)$ is a pullback in the category \mathcal{C} precisely when the span diagram $(**)$ defines a cartesian product of (X, f) and (Y, g) in the comma category \mathcal{C}/Z . Deduce from this that the pullback diagram $(*)$ associated to a pair of morphisms $f : X \rightarrow Z$ and $g : Y \rightarrow Z$ is unique up to isomorphism.

§3. Every object A in a category \mathcal{C} induces a category $\text{Sub}(A)$ called the category of subobjects of A , and defined in the following way. Its objects (U, m) are the pairs consisting of an object $U \in \mathcal{C}$ and of a mono $m : U \rightarrow A$ with target A . Its morphisms $h : (U, m) \rightarrow (V, n)$ are the morphisms $h : U \rightarrow V$ of the underlying category \mathcal{C} making the diagram

$$\begin{array}{ccc} U & \xrightarrow{h} & V \\ m \searrow & & \swarrow n \\ & A & \end{array}$$

commute in the category \mathcal{C} . The category $\text{Sub}(A)$ is thus the full subcategory of monos in the comma category \mathcal{C}/A . Show that the category $\text{Sub}(A)$ is a preorder category, in the sense there exists at most one arrow $h : (U, m) \rightarrow (V, n)$ between two objects (U, m) and (V, n) .

§4. Show that in the case $\mathcal{C} = \text{Sets}$, one recovers the powerset $(\mathcal{P}(A), \subseteq)$ with subsets $U, V \subseteq A$ ordered by inclusion $U \subseteq V$, as the ordered set of equivalence classes associated to the preorder $\text{Sub}(A)$. A useful convention in category theory is to identify the preorder category $\text{Sub}(A)$ with the ordered set $(\mathcal{P}(A), \subseteq)$ in that case.

§5. A category \mathcal{C} has pullbacks when there exists a pullback diagram $(*)$ for every pair of arrows $f : X \rightarrow Z$ and $g : Y \rightarrow Z$. Show that in a category \mathcal{C} with pullbacks, every arrow $f : B \rightarrow A$ induces a monotone function

$$f^* : \text{Sub}(A) \longrightarrow \text{Sub}(B)$$

defined by transporting every mono $m : U \rightarrow A$ to the mono $m' : V \rightarrow B$ using the pullback diagram (\otimes) in Exercise 2.5. Give an explicit description of the resulting monotone function

$$f^* : \mathcal{P}(A) \longrightarrow \mathcal{P}(B)$$

in the case when $\mathcal{C} = \text{Sets}$ and when $f : A \rightarrow B$ is a function between two sets A and B .