Examen partiel 2016-2017

Modèles des langages de programmation
Master Parisien de Recherche en Informatique

Le vendredi 2 décembre 2016 de 16h15 à 19h15

**On pourra répondre aux questions en français ou en anglais.**
**Questions may be answered in French or in English.**

**Exercise 1.** Consider the typing derivation of the simply-typed $\lambda$-calculus

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x : A \vdash x : A}}{f : A \Rightarrow A, x : A \vdash x : A}\;\text{Weakening}}{f : A \Rightarrow A \vdash \lambda x.x : A \Rightarrow A}\;\text{Abstraction}}{\vdash \lambda f.\lambda x.x : (A \Rightarrow A) \Rightarrow (A \Rightarrow A)}$$

in the type system studied in the course.
**(1a.)** Describe another derivation tree leading to the same typing judgement

$$\vdash \quad \lambda f.\lambda x.x \quad : \quad (A \Rightarrow A) \Rightarrow (A \Rightarrow A)$$

**(1b.)** Explain why the interpretation of the two typing derivations are the same in any cartesian closed category.
**(1c.)** Redo the same exercise for the typing derivation below :

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{g : A \Rightarrow A \vdash g : A \Rightarrow A} \qquad \dfrac{\dfrac{}{h : A \Rightarrow A \vdash h : A \Rightarrow A} \qquad \dfrac{}{x : A \vdash x : A}}{h : A \Rightarrow A, x : A \vdash h(x) : A}}{g : A \Rightarrow A, h : A \Rightarrow A, x : A \vdash g(h(x)) : A}}{f : A \Rightarrow A, x : A \vdash f(f(x)) : A}\;\text{Contraction}}{f : A \Rightarrow A \vdash \lambda x.f(f(x)) : A \Rightarrow A}\;\text{Abstraction}}{\vdash \lambda f.\lambda x.f(f(x)) : (A \Rightarrow A) \Rightarrow (A \Rightarrow A)}$$

**Exercise 2.** Every category $\mathcal{C}$ comes equipped with a unique functor

$$L \quad : \quad \mathcal{C} \quad \longrightarrow \quad \mathbf{1}$$

to the category $\mathbf{1}$ which contains a unique object $*$ and a unique morphism : the identity morphism on $*$. Show that the category $\mathcal{C}$ has a terminal object if and only if the functor $L$ has a right adjoint

$$R \quad : \quad \mathbf{1} \quad \longrightarrow \quad \mathcal{C}.$$

**Problème.** The purpose of this problem is to describe the coherence spaces as the objects $X$ isomorphic to their double negation $\sim\sim X$ in a larger category **Config** of arguably more primitive spaces, called configuration spaces. In particular, we will see how the constructions on coherence spaces studied in the course (classical duality, tensorial product, sum, exponentials) can be deduced from more primitive constructions on configuration spaces. A configuration space is defined here as a pair

$$X \quad = \quad (|X|, \mathrm{Conf}(X))$$

consisting of a countable set $|X|$ called the web of $X$ and of a set

$$\mathrm{Conf}(X) \quad \subseteq \quad \wp(|X|)$$

of subsets (finite or infinite) of the web $|X|$. The elements of $\mathrm{Conf}(X)$ are called the configurations of $X$. One requires moreover that every element $x \in |X|$ of the web is an element of some configuration $u \in \mathrm{Conf}(X)$ :

$$\forall x \in |X|, \quad \exists u \in \mathrm{Conf}(X), \qquad x \in u.$$

One remarks immediately that every coherence space $A$ defines a configuration space $U(A)$ with the same web :

$$|U(A)| \quad = \quad |A|$$

and whose configurations are the cliques of $A$.

The negation of a configuration space $X$ is defined as the configuration space $\sim X$ with same web

$$|\sim X| \quad = \quad |X|$$

and whose configurations are defined in the following way :

$$\mathrm{Conf}(\sim X) \quad = \quad \{ \quad u \subseteq |X| \quad | \quad \forall v \in \mathrm{Conf}(X), \quad u \perp v \quad \}$$

where

$$u \perp v$$

means that the intersection $u \cap v$ contains at most an element.

**(3a.)** Show that

$$\mathrm{Conf}(X) \quad \subseteq \quad \mathrm{Conf}(\sim\sim X)$$

for every configuration space $X$.

**(3b.)** Show that

$$\sim\sim\sim X \quad = \quad \sim X$$

for every configuration space $X$.

**(3c.)** Show that for every configuration space $X$, the configuration space

$$\sim X$$

is of the form

$$\sim X \quad = \quad U(A)$$

for some coherence space $A$ which will be described.

**(3d.)** Show in particular that

$$\sim U(A) \quad = \quad U(A^{\perp})$$

for every coherence space $A$, where the coherence space $A^{\perp}$ is the dual of $A$ defined in the course.

**(3e.)** Deduce from this the equality

$$U(A) \quad = \quad \sim\sim U(A)$$

for every coherence space $A$.

**(3f.)** Deduce from the previous questions that there exists a one-to-one relationship between the coherence spaces $A$ and the configuration spaces $X$ such that

$$X \quad = \quad \sim\sim X.$$

**(3g.)** The category **Config** is defined in the following way : its objects are the configuration spaces, and the morphisms

$$f \quad : \quad X \quad \longrightarrow \quad Y$$

are the binary relations

$$f \quad \subseteq \quad |X| \times |Y|$$

such that :

— the relation $f$ transports the configurations forward :

$$\forall u \in \mathrm{Conf}(X), \quad f(u) \quad \in \quad \mathrm{Conf}(Y)$$

where one writes

$$f(u) \quad = \quad \{ \quad y \in |Y| \quad | \quad \exists x \in u \text{ such that } (x, y) \in f \quad \}$$

— the relation $f$ is locally injective in the sense that for every configuration $u \in \mathrm{Conf}(X)$, one has :

$$\forall x_1, x_2 \in u, \qquad ( \; \exists y \in |Y|, \quad (x_1, y) \in f \text{ and } (x_2, y) \in f \; ) \quad \Rightarrow \quad x_1 = x_2.$$

The identity morphism on the configuration space $X$ is the morphism defined as follows :

$$\mathrm{id}_X \quad = \quad \{ \quad (x, x) \quad | \quad x \in |X| \quad \}.$$

Show that these data define indeed a category, where morphisms are composed as relations [ note : one can use the fact that the binary relations between sets define a category noted **Rel** ].

**(3h.)** Show that the construction $U(-)$ defines a fully faithful functor

$$U(-) \quad : \quad \mathbf{Coh} \quad \longrightarrow \quad \mathbf{Config}$$

Recall that the functor $U$ is fully faithful when the function

$$\begin{array}{ccc} \mathbf{Coh}(A, B) & \to & \mathbf{Config}(U(A), U(B)) \\ f & \mapsto & U(f) \end{array}$$

is bijective for every pair of coherence spaces $A$ and $B$.

This result enables us to see the category **Coh** as the full subcategory of configuration spaces $X$ such that $X = \sim\sim X$ in the category **Config**. Recall that a category $\mathcal{B}$ is a *full* subcategory of a category $\mathcal{C}$ when the class of objects of $\mathcal{B}$ is included in the class of objects of $\mathcal{C}$, and when the sets of morphisms $\mathcal{B}(B_1, B_2)$ and $\mathcal{C}(B_1, B_2)$ between two objects $B_1$ and $B_2$ of $\mathcal{B}$ (and thus of $\mathcal{C}$) are the same in the categories $\mathcal{B}$ and $\mathcal{C}$ :

$$\mathcal{B}(B_1, B_2) = \mathcal{C}(B_1, B_2)$$

with same composition law and identity morphisms in $\mathcal{B}$ and in $\mathcal{C}$.

**(3i.)** Show that the construction $\sim$ defines a functor

$$\sim \quad : \quad \mathbf{Config} \quad \longrightarrow \quad \mathbf{Config}^{\,op}$$

which transports a morphism $f : X \to Y$ in the morphism

$$\sim f \quad : \quad \sim Y \quad \longrightarrow \quad \sim X$$

defined as

$$(y, x) \in \sim f \qquad \Longleftrightarrow \qquad (x, y) \in f$$

for every $x \in |\,X\,|$ and $y \in |\,Y\,|$.

**(3j.)** Deduce from the two previous questions that $\sim\sim$ defines a functor

$$\sim\sim \quad : \quad \mathbf{Config} \quad \longrightarrow \quad \mathbf{Coh}.$$

From now on, one writes this double negation functor as

$$F \quad : \quad \mathbf{Config} \quad \longrightarrow \quad \mathbf{Coh}.$$

**(3k.)** Describe in a simple way the coherence space $F(X)$ associated to a configuration space $X$, and give the example of a configuration space $X$ such that the inclusion

$$\mathrm{Conf}(X) \subseteq \mathrm{Conf}(UF(X))$$

is a strict inclusion [note : it is possible to find such a configuration space $X$ with three points in its web $|\,X\,|$].

**(3l.)** Suppose given a configuration space $X$ and a coherence space $A$. Show that a binary relation

$$f \quad \subseteq \quad |\,X\,| \times |\,A\,|$$

is an element of

$$\mathbf{Config}(X, U(A))$$

if and only if the relation $f$ is an element of

$$\mathbf{Coh}(F(X), A).$$

**(3m.)** From this, deduce the existence of a bijection

$$\phi_{X,A} \quad : \quad \mathbf{Coh}(F(X), A) \quad \cong \quad \mathbf{Config}(X, U(A))$$

and show that it is natural in $X$ and in $A$.

From this, it follows that the functor

$$F \quad : \quad \mathbf{Config} \quad \longrightarrow \quad \mathbf{Coh}$$

is left adjoint to the functor

$$U \quad : \quad \mathbf{Coh} \quad \longrightarrow \quad \mathbf{Config}$$

**(3n.)** Describe the unit

$$\eta_X \quad : \quad X \quad \longrightarrow \quad UF(X)$$

and the counit

$$\varepsilon_A \quad : \quad FU(A) \quad \longrightarrow \quad A$$

of the adjunction $F \dashv U$ for a configuration space $X$ and for a coherence space $A$.

**(3o.)** Given two configuration spaces $X$ and $Y$, define the configuration space $X \bullet Y$ as follows :

$$|\, X \bullet Y \,| \quad = \quad |\, X \,| \times |\, Y \,|$$

$$\mathrm{Conf}(X \bullet Y) \quad = \quad \{ \;\; u \times v \quad | \quad u \in \mathrm{Conf}(X) \;\; \text{and} \;\; v \in \mathrm{Conf}(Y) \quad \}$$

We admit here that the tensor product defines a structure of symmetric monoidal category on the category **Config**. Show that

$$F(X) \; \otimes \; F(Y) \quad = \quad F(X \bullet Y)$$

where $\otimes$ is the tensor product on coherence space defined in the course. Explain in what sense this equality enables one to deduce the tensor product $\otimes$ on coherence spaces from the tensor product $\bullet$ on configuration spaces.

**(3p.)** Given two configuration spaces $X$ and $Y$, show that a binary relation

$$f \quad \subseteq \quad |\, X \,| \times |\, Y \,|$$

is an element of

$$\mathbf{Config}(X, \sim Y)$$

if and only if

$$\forall (u, v) \in \mathrm{Conf}(X) \times \mathrm{Conf}(Y), \qquad f \quad \perp \quad u \times v.$$

[Note : here, one uses the fact that every element $y$ of the web of $Y$ appears in one configuration $v$ of the configuration space $Y$.]

**(3q.)** Deduce a bijection

$$\mathbf{Config}(X, \sim Y) \quad \cong \quad \mathbf{Config}(X \bullet Y, \bot)$$

where $\perp$ is the configuration space whose web is the singleton $\{*\}$ and whose configurations are the empty set and the singleton $\{*\}$.

**(3r.)** Given two configuration spaces $X$ and $Y$, one defines the configuration space $X+Y$ as follows :

$$\begin{array}{rcl} |X+Y| & = & |X|+|Y| \\ \mathrm{Conf}(X+Y) & = & \{ \ \mathtt{inl}(u) \ \mid \ u \in \mathrm{Conf}(X) \ \} \\ & & \cup \ \{ \ \mathtt{inr}(v) \ \mid \ v \in \mathrm{Conf}(Y) \ \} \end{array}$$

Show that $X + Y$ defines a cartesian sum of $X$ and of $Y$ in the category **Config**. [Note : the cartesian sum is the dual of the cartesian product, in other words, a cartesian product in the opposite category $\mathbf{Config}^{op}$.]

**(3s.)** Show that

$$F(X) \oplus F(Y) \quad = \quad F(X+Y).$$

**(3t.)** For every configuration space $X$, one defines the configuration space $TX$ whose web is the set of finite sub-configurations

$$|TX| \quad = \quad \{ \quad u \text{ finite} \quad \mid \quad \exists v \in \mathrm{Conf}(X) \text{ tel que } u \subseteq v \quad \}$$

and whose configurations $u^\dagger$ are generated by the configurations of $X$ in the following sense :

$$\mathrm{Conf}(TX) \quad = \quad \{ \quad u^\dagger \quad \mid \quad u \in \mathrm{Conf}(X) \quad \}$$

where

$$u^\dagger \quad = \quad \text{the set of finite subsets of } u.$$

[Note : here, a sub-configuration $u$ of the configuration space $X$ is any set of elements of the web $|X|$ contained in some configuration $v$ of the configuration space $X$]. Show that for every coherence space $A$, one has

$$!\,A \quad = \quad F\ T\ U\ A.$$

**(3u.)** From this, deduce the existence of a morphism

$$m_X \quad : \quad F\ T\ X \quad \longrightarrow \quad !\ F\ X$$

for every configuration space $X$ [one admits here the fact that the construction $T(-)$ defines a functor $T : \mathbf{Config} \to \mathbf{Config}$.]

**(3v.)** Give an example of a configuration space $X$ such that the morphism

$$m_X \quad : \quad F\ T\,X \quad \longrightarrow \quad !\ F\ X$$

is not an isomorphism [note : it is sufficient to check that the two coherence spaces $F\,T\,X$ and $!\,FX$ do not have the same web.]