Examen partiel 2017-2018

Modèles des langages de programmation
Master Parisien de Recherche en Informatique

Le jeudi 30 novembre 2017 de 9h00 à 11h00

Les questions de l'examen sont rédigées en anglais mais les réponses pourront bien entendu être écrites en français.

Exercice 1. Consider the derivation tree of the simply-typed λ-calculus

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x : A \vdash x : A}}{f : A \Rightarrow A, x : A \vdash x : A}\text{Weakening}}{f : A \Rightarrow A \vdash \lambda x.x : A \Rightarrow A}\text{Abstraction}}{\vdash \lambda f.\lambda x.x : (A \Rightarrow A) \Rightarrow (A \Rightarrow A)}\text{Abstraction}$$

in the typing system studied in the course.

§1a. Describe another derivation tree leading to the same typing judgment:

$$\vdash \quad \lambda f.\lambda x.x \quad : \quad (A \Rightarrow A) \Rightarrow (A \Rightarrow A)$$

§1b. Explain with a categorical diagram why the interpretations of the two derivation trees are the same in any cartesian closed category.

Exercice 2. Every category $\mathcal{C}$ comes equipped with a unique functor

$$L \quad : \quad \mathcal{C} \quad \longrightarrow \quad \mathbf{1}$$

to the category $\mathbf{1}$ which contains a unique object $*$ and a unique morphism: the identity morphism on $*$.

§2a. Show that a category $\mathcal{C}$ admits a terminal object if and only if the functor L has a right adjoint.

$$R \quad : \quad \mathbf{1} \quad \longrightarrow \quad \mathcal{C}.$$

§2b. Similarly, every category $\mathcal{C}$ comes equipped with a diagonal functor

$$L \quad : \quad \mathcal{C} \quad \longrightarrow \quad \mathcal{C} \times \mathcal{C}$$

Show that a category $\mathcal{C}$ has cartesian products if and only if the functor L has a right adjoint:

$$R \quad : \quad \mathcal{C} \times \mathcal{C} \quad \longrightarrow \quad \mathcal{C}.$$

§2c. Describe when the category $\mathcal{C}$ has cartesian products, the unit η and counit ε of the adjunction considered in §2b.

$$\eta : Id \Rightarrow R \circ L \qquad \varepsilon : L \circ R \Rightarrow Id$$

Problem. We saw in the course that given two coherence spaces A and B, the coherence space $A\&B$ is the coherence space whose web

$$|A\&B| = |A| + |B|$$

is the disjoint sum of the webs of A and B, with the following coherence relation:

$$\begin{array}{lcl} \forall a, a' \in |A|, \quad \mathbf{inl}(a) \frown_{A\&B} \mathbf{inl}(a') & \iff & a \frown_A a' \\ \forall b, b' \in |B|, \quad \mathbf{inr}(b) \frown_{A\&B} \mathbf{inr}(b') & \iff & b \frown_B b' \end{array}$$

and

$$\forall a \in |A|, \forall b \in |B|, \qquad \mathbf{inl}(a) \frown_{A\&B} \mathbf{inr}(b).$$

§1. Every clique

$$f \quad : \quad A\&S \quad \multimap \quad B\&S$$

induces a family of unary (sorted) operations

$$\begin{array}{cccc} b & s & t & b \\ | & | & | & | \\ f & f & f & f \\ | & | & | & | \\ a & a & s & s \end{array}$$

one for each pair of elements $a \in |A|$, $b \in |B|$, and $s, t \in |S|$ such that

$$a[f]b \qquad a[f]s \qquad s[f]t \qquad s[f]b$$

respectively, where we use the notation $a[f]b$ to mean that $(a, b) \in f$. Let **trees** (f) denote the set of linear trees of the form:

$$\begin{array}{c} b \\ | \\ f \\ | \\ s_n \\ \vdots \\ s_1 \\ | \\ f \\ | \\ a \end{array}$$

containing at least one node f. Define the relation

$$\mathbf{trace}\,(f) \quad : \quad A \quad \longrightarrow \quad B$$

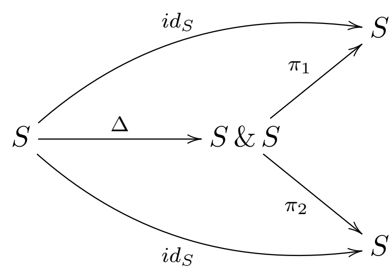
as the set of elements (a, b) (which may be also noted $a \multimap b$) such that there exists a linear tree in $\mathbf{trees}\,(f)$ with input sort a and output sort b. Show that the relation $\mathbf{trace}\,(f)$ defines a clique of the coherence space $A \multimap B$.

Remark: this construction defines a « trace operator » on the category of coherence spaces equipped with the cartesian product.

§2. Describe explicitly the diagonal clique

$$\Delta \quad : \quad S \quad \multimap \quad S \,\&\, S$$

characterized as the unique morphism making the diagram



commute in the category of coherence spaces.

§3. Given a clique

$$f \quad : \quad A \,\&\, S \quad \multimap \quad S$$

describe explicitly the clique $\Delta \circ f$ defined as the composite

$$A \,\&\, S \xrightarrow{\quad f \quad} S \xrightarrow{\quad \Delta \quad} S \,\&\, S.$$

§4. Every clique

$$f \quad : \quad A \,\&\, S \quad \multimap \quad S$$

induces a clique, noted

$$\mathbf{fixpoint}\,(f) = \mathbf{trace}\,(\Delta \circ f) \quad : \quad A \quad \multimap \quad S$$

and defined as the trace operator applied to the composite

$$A \,\&\, S \xrightarrow{\quad f \quad} S \xrightarrow{\quad \Delta \quad} S \,\&\, S.$$

considered in §3. Show that $\mathbf{fixpoint}\,(f)$ coincides with the set of all elements (a, s) (which may be also noted $a \multimap s$) with $a \in |A|$ and $s \in |S|$ such that there exists a linear tree in $\mathbf{trees}\,(f)$ with input sort a and output sort s.

§5. Show that the clique **fixpoint** (f) makes the diagram

$$\begin{array}{ccc} A \,\&\, A & \xrightarrow{A \,\&\, \mathbf{fixpoint}\,(f)} & A \,\&\, S \\ \uparrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle f} \\ A & \xrightarrow{\mathbf{fixpoint}\,(f)} & S \end{array}$$

commute in the category of coherence spaces.

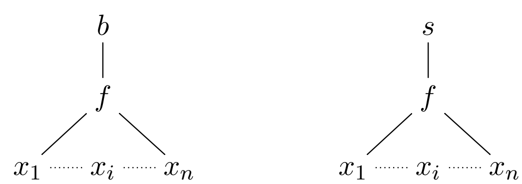
§6. At this stage, we are ready to lift to stable functions what we have just achieved for linear functions. We have seen during the course that every stable function

$$f \quad : \quad D_A \times D_S \quad \longrightarrow \quad D_B \times D_S$$

may be seen as a clique

$$f \quad : \quad !\,(\,A \,\&\, S\,) \quad \multimap \quad B \,\&\, S$$

This clique induces in turn an n-ary (sorted) operation



whenever

$$\{x_1, \cdots, x_n\} \quad [f] \quad b \qquad\qquad \{x_1, \cdots, x_n\} \quad [f] \quad s$$

where each x_i is an element of $|A \,\&\, S|$, that is, either an element of $|A|$ or an element of $|S|$. Note that the descendents of a node f are unordered. An element of **trees** (f) is defined as a tree constructed with these n-ary operations, and containing at least a node f. The relation

$$\mathbf{trace}\,(f) \quad : \quad !A \quad \longrightarrow \quad B$$

is then defined as the set of elements

$$\{a_1, \ldots, a_n\} \quad \multimap \quad b$$

such that there exists a tree in **trees** (f) with b as output sort and with $\{a_1, \cdots, a_n\}$ as set of input sorts. Show that **trace** (f) defines a clique of the coherence space $!A \multimap B$. [Hint: proceed as in question **§1.**]

§7. Every clique

$$f \quad : \quad !\,(\,A \,\&\, S\,) \quad \multimap \quad S$$

induces a clique

$$\mathbf{fixpoint}\,(f) = \mathbf{trace}\,(\Delta \circ f) \quad : \quad !\,A \quad \multimap \quad S$$

defined as the trace operator applied to the composite

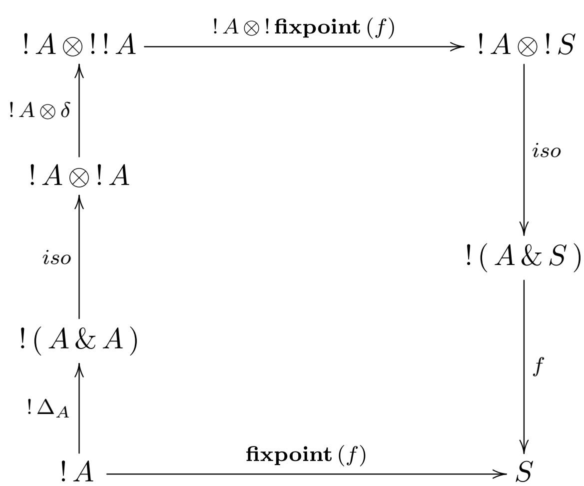
$$!\,(\,A \,\&\, S\,) \xrightarrow{\;f\;} S \xrightarrow{\;\Delta\;} S \,\&\, S.$$

Show that **fixpoint** (f) coincides with the set of all elements

$$\{a_1, \ldots, a_n\} \multimap s$$

such that there exists a tree in **trees** (f) with input sorts $\{a_1, \ldots, a_n\}$ and output sort s, where each a_i is an element of the web $|A|$.

§8. Show that the clique **fixpoint** (f) is a parametric fixpoint of f in the sense that the diagram



commutes in the category of coherence spaces.

§9. Explain how to compute the factorial function

$$\mathbf{factorial} \quad : \quad \mathbb{N} \quad \multimap \quad \mathbb{N}$$

as a fixpoint of the clique

$$\mathbf{fact} \quad : \quad !\,(\mathbb{N} \quad \multimap \quad \mathbb{N}) \quad \multimap \quad \mathbb{N} \quad \multimap \quad \mathbb{N}$$

defined as the set of elements

$$\begin{array}{|ccccc|} \hline !\,(\mathbb{N} \multimap \mathbb{N}) & \multimap & \mathbb{N} & \multimap & \mathbb{N} \\ \{\ p \mapsto q\ \} & \multimap & p+1 & \mapsto & (p+1)\times q \\ \emptyset & \multimap & 1 & \mapsto & 1 \\ \hline \end{array}$$

where $a \mapsto b$ is an alternative notation for (a, b) or for $a \multimap b$. Explain in particular why the tree

$$5 \multimap (5 \times 4 \times 3 \times 2 \times 1)$$
$$|$$
$$\textbf{fact}$$
$$|$$
$$4 \multimap (4 \times 3 \times 2 \times 1)$$
$$|$$
$$\textbf{fact}$$
$$|$$
$$3 \multimap (3 \times 2 \times 1)$$
$$|$$
$$\textbf{fact}$$
$$|$$
$$2 \multimap (2 \times 1)$$
$$|$$
$$\textbf{fact}$$
$$|$$
$$1 \multimap 1$$
$$|$$
$$\textbf{fact}$$

establishes that

$$5 \multimap (5 \times 4 \times 3 \times 2 \times 1)$$

is an element of the clique

$$\textbf{factorial} \quad : \quad \mathbb{N} \multimap \mathbb{N}.$$

§10. We have seen in the course how to see a pair of morphisms (= cliques) in the category of coherence spaces

$$f \quad : \quad !\,A \longrightarrow B \qquad\qquad g \quad : \quad !\,B \longrightarrow C$$

as a pair of morphisms

$$f \quad : \quad A \quad \stackrel{k}{\longrightarrow} \quad B \qquad\qquad\qquad g \quad : \quad B \quad \stackrel{k}{\longrightarrow} \quad C$$

in the Kleisli category associated to the comonad « ! ». Recall that the composite $g \bullet f$ of the two morphisms in the Kleisli category is defined as follows:

$$!\,A \xrightarrow{\ \delta_A\ } !\,!\,A \xrightarrow{\ !f\ } !B \xrightarrow{\ g\ } C$$

Express the set **trees** $(g \bullet f)$ directly from the sets **trees** (g) and **trees** (f).

§11. Explain (formally or informally) why the stable function

$$D_A \quad \longrightarrow \quad D_S$$

associated to the clique

$$\textbf{fixpoint}\,(f) \quad : \quad !\,A \quad \multimap \quad S$$

defines the least fixpoint of the stable function

$$D_A \times D_S \quad \longrightarrow \quad D_S$$

associated to the clique

$$f \quad : \quad !\,(\,A \,\&\, S\,) \quad \multimap \quad S.$$