Examen partiel 2018-2019

Modèles des langages de programmation
Master Parisien de Recherche en Informatique

Le mardi 27 novembre 2018 de 16h15 à 18h45

On pourra répondre aux questions en français ou en anglais.
Questions may be answered in French or in English.

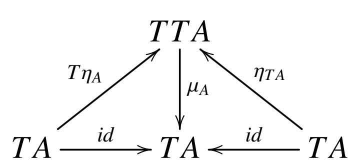
Exercise A. We recall that a monad (T, μ, η) on a category $\mathcal{A}$ is a functor

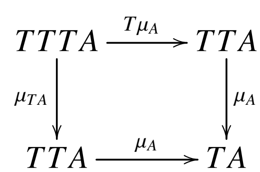
$$T : \mathcal{A} \longrightarrow \mathcal{A}$$

equipped with two natural transformations

$$\mu_A : TTA \longrightarrow TA \qquad\qquad \eta_A : A \longrightarrow TA$$

making the diagrams below commute :

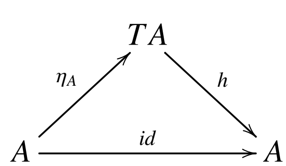


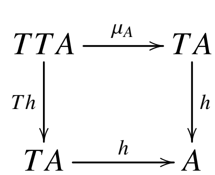


for every object A. An algebra (A, h) of such a monad T is then defined as a pair consisting of an object A of the category $\mathcal{A}$ and of a morphism

$$h : TA \longrightarrow A$$

making the two diagrams below commute :





for every object A. From now on, we make the assumption that the category $\mathcal{A}$ is monoidal, with tensor product noted $\otimes$ and tensor unit noted 1.

§1. Suppose given a monoid M in the monoidal category $\mathcal{A}$, with multiplication and unit noted

$$m : M \otimes M \to M \qquad\qquad e : 1 \to M.$$

Show that the induced functor

$$M \otimes - \quad : \quad A \quad \mapsto \quad M \otimes A$$

defines a monad (T, μ, η) on the category $\mathcal{A}$.

§2. Given a monoid M in the monoidal category $\mathcal{A} = \mathbf{Set}$ of sets and functions, with tensor product $\otimes$ defined as the set-theoretic cartesian product $\times$, show that an algebra (A, h) of the monad T associated to the monoid M in §1 is the same thing as an action (in the usual sense) of the monoid M on the set A.

§3. Suppose given a comonoid C in the monoidal category $(\mathcal{A}, \otimes, 1)$, with comultiplication and counit noted

$$d : C \to C \otimes C \qquad\qquad u : C \to 1.$$

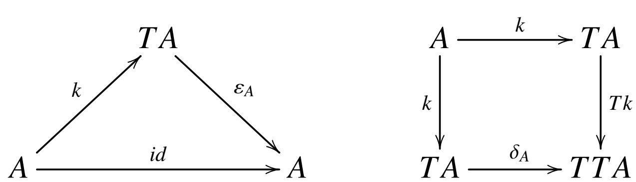
Show that the induced functor

$$C \otimes - \quad : \quad A \quad \mapsto \quad C \otimes A$$

defines a comonad (K, δ, ε) on the category $\mathcal{A}$.

§4. Suppose that the monoidal category $(\mathcal{A}, \otimes, 1)$ is cartesian, in the sense that the tensor product $\otimes$ coincides with its cartesian product. Show that every object C is equipped in that case with a comonoid structure (C, d, e).

§5. We recall that a coalgebra of a comonad (K, δ, ε) is a pair (A, k) consisting of an object A of the category $\mathcal{A}$ and a morphism $k : A \to TA$ such that the two diagrams below commute :



Once again, suppose that the monoidal category $(\mathcal{A}, \otimes, 1)$ is cartesian in the sense that the tensor product $\otimes$ coincides with the cartesian product. Suppose moreover that the category $\mathcal{A}$ contains an object C. Now, consider the comonad (K, δ, ε) associated in §3. to the comonoid (C, d, u) defined in §4. on the object C. Show that there is a one-to-one correspondence between these two notions :

— the coalgebras (A, k) of the comonad (K, δ, ε),
— the morphisms $A \to C$ in the cartesian category $\mathcal{A}$.

Exercise B. In this exercise, we want to emulate the interaction between a deterministic automaton $\mathcal{A}$ and a finite word $w = a_1 \cdots a_n$ by interpreting both of them in the category **Coh** of coherence spaces. To that purpose, and in order to simplify things, we work with an alphabet $\Sigma = \{a, b\}$ consisting of only two letters. Recall that a deterministic automaton $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$ on such an alphabet $\Sigma = \{a, b\}$ is defined as a finite set of states Q, two transition functions

$$\delta_a : Q \to Q \qquad\qquad \delta_b : Q \to Q$$

an initial state $q_0 \in Q$ and a subset $F \subseteq Q$ of final states. One says that a finite word $w = a_1 \dots a_n$ is accepted by the automaton $\mathcal{A}$ when there exists a run of the automaton labelled by w from the initial state q_0 to a final state $q_f \in F$ of the automaton.

§1. Show that a finite word $w = a_1 \dots a_n$ is accepted by a deterministic automaton $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$ if and only if

$$\delta_{a_n} \circ \cdots \circ \delta_{a_1} (q_0) \quad \in \quad F.$$

Note in particular the order in which the letters are written.

§2. Show that every finite word $a_1 \cdots a_n$ on the alphabet $\Sigma = \{a, b\}$ may be faithfully encoded as a simply-typed λ-term

$$[a_1, \cdots, a_n] \quad := \quad a_1(a_2 \cdots (a_n(x)))$$

typed by the typing judgment

$$a : \alpha \Rightarrow \alpha \, , \, b : \alpha \Rightarrow \alpha \, , \, x : \alpha \quad \vdash \quad [a_1, \cdots, a_n] \; : \; \alpha$$

where α is a type variable, and a, b, x are term variables. This encoding of the word $w = a_1 \dots a_n$ as a simply-typed λ-term is called the Church encoding of w.

§3. Conversely, show that every normal form M typed by the judgement

$$a : \alpha \Rightarrow \alpha \, , \, b : \alpha \Rightarrow \alpha \, , \, x : \alpha \quad \vdash \quad M \; : \; \alpha$$

is the Church encoding $M = [a_1, \dots, a_n]$ of a finite word $a_1 \cdots a_n$ on the alphabet Σ.

§4. We suppose from now on that the finite set of states Q of the automaton $\mathcal{A}$ is fixed. We associate to this finite set Q of states the coherence space $\bot_Q$ with web Q and with every two elements $q, q' \in Q$ coherent :

$$\forall q, q' \in Q, \qquad q \frown_{\bot_Q} q'.$$

Explain briefly why the coherence space $\bot_Q$ is equal to

$$\bot_Q \quad = \quad \underbrace{\bot \,\&\, \cdots \,\&\, \bot}_{\text{as many formulas } \bot \text{ as there are elements in } Q.}$$

and may be seen as the dual of the coherence space

$$1_Q \quad = \quad \underbrace{1 \oplus \cdots \oplus 1}_{\text{as many formulas 1 as there are elements in } Q.}$$

§5. We associate to the transition functions δ_a and δ_b the binary relations

$$[\delta_a] = \{(q_{out}, q_{in}) \mid \delta_a(q_{in}) = q_{out}\} \qquad [\delta_b] = \{(q_{out}, q_{in}) \mid \delta_b(q_{in}) = q_{out}\}$$

on the set of states Q. Note that the two relations $[\delta_a]$ and $[\delta_b]$ may be seen as the functions δ_a and δ_b "taken in the opposite direction". Show that $[\delta_a]$ and $[\delta_b]$ define two morphisms

$$[\delta_a] : \bot_Q \longrightarrow \bot_Q \qquad [\delta_b] : \bot_Q \longrightarrow \bot_Q$$

in the category of coherence spaces.

§6. We associate to the set of final states F the set

$$[F] \;=\; \{(*, q) \,|\, q \in F\}$$

where $*$ denotes the only element of the web of the coherence space 1. Show that the binary relation $[F]$ defines a morphism

$$[F] \;:\; 1 \;\rightarrow\; \bot_Q$$

in the category of coherence spaces.

§7. Show that a word $w = a_1 \ldots a_n$ is accepted by the automaton $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$ if and only if the initial state q_0 is an element of the clique associated to the morphism

$$1 \xrightarrow{[F]} \bot_Q \xrightarrow{[\delta_{a_n}]} \bot_Q \xrightarrow{[\delta_{a_{n-1}}]} \quad \ldots \quad \xrightarrow{[\delta_{a_2}]} \bot_Q \xrightarrow{[\delta_{a_1}]} \bot_Q$$

in the category **Coh** of coherence spaces.

Remark : this justifies to use the alternative notation $[\delta_x]$ for the clique $[F]$.

§8. From now on, we write $\langle f_1, f_2, \ldots, f_n, x \rangle$ for the linear λ-term :

$$\langle f_1, f_2, \ldots, f_n, x \rangle \quad := \quad f_1(f_2 \ldots f_n(x)))$$

and observe that it is typed by the following judgment of intuitionistic linear logic :

$$f_1 : \alpha \multimap \alpha, \ldots, f_n : \alpha \multimap \alpha, x : \alpha \ \vdash \ \langle f_1, f_2, \ldots, f_n, x \rangle : \alpha$$

For that reason, it is interpreted as a clique of the coherence space

$$(\ (\bot_Q \multimap \bot_Q) \ \otimes \ \cdots \ \otimes \ (\bot_Q \multimap \bot_Q) \ \otimes \ \bot_Q \ ) \ \multimap \ \bot_Q.$$

when the type variable α is interpreted as $\bot_Q$. Give a direct and explicit description of this clique in the case $n = 2$ and then explain briefly how to generalize to $n = 3$ and beyond.

§9. Deduce from §7. and §8. that the interpretation of the linear λ-term

$$f_1 : \alpha \multimap \alpha, \ldots, f_n : \alpha \multimap \alpha, x : \alpha \ \vdash \ \langle f_1, f_2, \ldots, f_n, x \rangle : \alpha$$

in the category **Coh** of coherence spaces where

1. the type variable α is interpreted as the coherence space $\bot_Q$,
2. each variable $f_i : \alpha \multimap \alpha$ is substituted by the clique $[\delta_{a_i}] : 1 \to \bot_Q \multimap \bot_Q$
3. the variable $x : \alpha$ is substituted by the clique $[\delta_x] : 1 \to \bot_Q$

is equal to set of states $q_0 \in Q$ such that the word $w = a_1, \ldots, a_n$ is accepted by the automaton $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$.

§10. Show that for every word $w = a_1 \ldots a_n$ on the alphabet $\Sigma = \{a, b\}$, the λ-term

$$[a_1, \ldots, a_n] \ = \ a_1(a_2 \cdots a_n(x)))$$

is equal to the linear λ-term

$$\langle f_1, f_2, \ldots, f_n, x \rangle \quad := \quad f_1(f_2 \ldots f_n(x)))$$

where each variable f_i is substituted by the variable $a_i \in \{a, b\}$. From this, deduce that the original typing of the Church encoding $[a_1, \ldots, a_n]$ in the simply-typed λ-calculus as seen in §2 may be refined into the following typing judgment of intuitionistic linear logic :

$$a : !(\alpha \multimap \alpha), b : !(\alpha \multimap \alpha), x : \alpha \ \vdash \ [a_1, \cdots, a_n] : \alpha$$

§11. Adapt very mildly this typing judgement in order to establish that the Church encoding of the word $w = a_1 \dots a_n$ may be interpreted as a morphism

$$[a_1, \cdots, a_n] \quad : \quad (\bot_Q \multimap \bot_Q) \quad \& \quad (\bot_Q \multimap \bot_Q) \quad \& \quad \bot_Q \qquad \longrightarrow \qquad \bot_Q.$$

in the category **Stab** of coherence spaces and stable maps [Hint : use the course here].

§12. Use §9. to show that the composite

$$\top \xrightarrow{[\delta_a]\,\&\,[\delta_b]\,\&\,[\delta_x]} (\bot_Q \multimap \bot_Q)\,\&\,(\bot_Q \multimap \bot_Q)\,\&\,\bot_Q \xrightarrow{[a_1,\dots,a_n]} \bot_Q$$

computed in the category **Stab** of coherence spaces and stable maps defines the set of states q_0 such that the word $w = u_1 \dots u_n$ is accepted by the automaton $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$.

§13. Conclude and define what would deserve the name of Church encoding of a deterministic automaton $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$ in the category **Coh** of coherence spaces.