Examen partiel en ligne 2020-2021

Modèles des langages de programmation
Master Parisien de Recherche en Informatique

Le mardi 1er décembre 2020 de 9h30 à 11h30

Les questions de l'examen sont rédigées en anglais
mais les réponses pourront bien entendu être écrites en français.

Exercise A. Consider the derivation tree of the simply-typed λ-calculus

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x : A \vdash x : A}}{f : A \Rightarrow A, x : A \vdash x : A}\ \text{Weakening}}{f : A \Rightarrow A \vdash \lambda x.x : A \Rightarrow A}\ \text{Abstraction}}{\vdash \lambda f.\lambda x.x : (A \Rightarrow A) \Rightarrow (A \Rightarrow A)}\ \text{Abstraction}$$

in the typing system studied in the course.

§A1. Describe another derivation tree leading to the same typing judgment :

$$\vdash \quad \lambda f.\lambda x.x \quad : \quad (A \Rightarrow A) \Rightarrow (A \Rightarrow A)$$

§A2. Explain with a categorical diagram why the interpretations of the two derivation trees are the same in any cartesian closed category.

Exercise B. In this exercise, we study a categorical formulation of induction and coinduction based on the notions of T-algebra for induction and of T-coalgebra for coinduction. To that purpose, we suppose given a category $\mathcal{C}$ and a functor $T : \mathcal{C} \to \mathcal{C}$. A T-*algebra* is defined as a pair (A, a) consisting of an object A of the category $\mathcal{C}$ and of a map

$$a \quad : \quad TA \longrightarrow A.$$

A T-homomorphism between T-algebras

$$f \quad : \quad (A, a) \longrightarrow (B, b)$$

is defined as a map $f : A \to B$ of the category $\mathcal{C}$ which makes the diagram below commute :

$$\begin{array}{ccc} TA & \xrightarrow{\ Tf\ } & TB \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ A & \xrightarrow{\ f\ } & B \end{array}$$

One obtains in this way a category $\mathbf{Alg}_T$ whose objects are the T-algebras and whose maps are the T-homomorphisms.

§B1. A T-algebra (A, a) is called *initial* when for every map $b : TB \to B$ there exists a unique map $f : A \to B$ such that the following diagram commutes :

$$\begin{array}{ccc} TA & \xrightarrow{\quad Tf \quad} & TB \\ {\scriptstyle a}\big\downarrow & & \big\downarrow{\scriptstyle b} \\ A & \xrightarrow{\quad f \quad} & B \end{array}$$

Explain why a T-algebra (A, a) is initial in that sense if and only if it is initial as an object in the category $\mathbf{Alg}_T$.

§B2. Consider the functor $T : \mathbf{Set} \to \mathbf{Set}$ which transports every set A to the set $TA = 1 \uplus A$ where $\uplus$ denotes the disjoint union of sets, and $1 = \{*\}$. Show that the set of natural numbers $\mathbb{N}$ equipped with the bijection

$$\begin{array}{ccccc} succ & : & 1 \uplus \mathbb{N} & \longrightarrow & \mathbb{N} \\ & & n & \mapsto & n+1 \\ & & * & \mapsto & 0 \end{array}$$

defines the initial algebra of the functor T. Explain in particular how this property of $(\mathbb{N}, succ)$ enables one to define in a unique way a function $\mathbb{N} \to A$ by induction on the natural numbers.

§B3. In a dual way, one defines a T-coalgebra in a category $\mathcal{C}$ as a pair (A, a) consisting of an object A and of a map

$$a \quad : \quad A \longrightarrow TA$$

of the category $\mathcal{C}$. A T-homomorphism between T-coalgebras

$$f \quad : \quad (A, a) \longrightarrow (B, b)$$

is defined as a map $f : A \to B$ of the category $\mathcal{C}$ which makes the diagram below commute :

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ {\scriptstyle a}\big\downarrow & & \big\downarrow{\scriptstyle b} \\ TA & \xrightarrow{\quad Tf \quad} & TB \end{array}$$

One defines in this way a category $\mathbf{CoAlg}_T$ whose objects are the T-coalgebras and whose maps are the T-homomorphisms. A T-coalgebra (B, b) is called *terminal* when for every map

$a : A \to TA$ there exists a unique map $f : A \to B$ such that the diagram below commutes :

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ TA & \xrightarrow{Tf} & TB \end{array}$$

Explain why a T-coalgebra (A, a) is terminal in that sense precisely when it is terminal as an object in the category $\mathbf{CoAlg}_T$.

§B4. Consider the same functor $T : \mathbf{Set} \to \mathbf{Set}$ as in §B2. which transports every set A to the set $TA = 1 \uplus A$. Show that the set of completed natural numbers $\mathbb{N} \uplus \{\infty\}$ equipped with the function

$$\begin{array}{ccccc} pred & : & \mathbb{N} \uplus \{\infty\} & \longrightarrow & 1 \uplus \mathbb{N} \uplus \{\infty\} \\ & & 0 & \mapsto & * \\ & & n+1 & \mapsto & n \\ & & \infty & \mapsto & \infty \end{array}$$

defines the terminal T-coalgebra. Explain that it enables one to define a function by coinduction on the natural numbers (if you happen to know the terminology).

Problem. The purpose of this series of exercises is to describe coherence spaces as objects X isomorphic to their double negation $\sim\sim X$ in a larger category **Conf** of configuration spaces. We will then see how the various constructions on coherence spaces studied during the course : the classical duality, the tensor product, the sum, can be derived from more primitive constructions on configuration spaces. A **configuration space** is defined as a pair

$$X \quad = \quad (|X|, \mathrm{Config}(X))$$

consisting of a countable set $|X|$ called the web of X and of a set

$$\mathrm{Config}(X) \quad \subseteq \quad \wp(|X|)$$

of subsets of $|X|$. The elements of $\mathrm{Config}(X)$ are called the **configurations** of X. One asks moreover that every element $x \in |X|$ of the web is an element of a configuration $u \in \mathrm{Config}(X)$:

$$\forall x \in |X|, \quad \exists u \in \mathrm{Config}(X), \qquad x \in u.$$

It is immediate that every coherence space A defines a configuration space $U(A)$ with same web :

$$|U(A)| \quad = \quad |A|$$

and whose configurations are the cliques of A.

Now, one defines the **negation** of a configuration space X as the configuration space $\sim X$ with same web

$$|\sim X| \quad = \quad |X|$$

and with set of configurations defined as :

$$\mathrm{Config}(\sim X) \quad = \quad \{ \quad u \subseteq |X| \quad | \quad \forall v \in \mathrm{Config}(X), \quad u \perp v \quad \}$$

where

$$u \perp v$$

means that the intersection $u \cap v$ contains an element at most.

§P1. Show that

$$\mathrm{Config}(X) \quad \subseteq \quad \mathrm{Config}(\sim\sim X)$$

for every configuration space X.

§P2. Show that

$$\sim\sim\sim X \quad = \quad \sim X$$

for every configuration space X.

§P3. Show that for every configuration space X, the configuration space

$$\sim X$$

is of the form

$$\sim X \quad = \quad U(A)$$

for a given coherence space A that will be described.

§P4. Show in particular that for every coherence space A, one has

$$\sim U(A) \quad = \quad U(A^{\perp})$$

where $A^{\perp}$ is the coherence space dual to A defined in the course.

§P5. Deduce the following equation

$$U(A) \quad = \quad \sim\sim U(A)$$

for every coherence space A.

§P6. Deduce from the previous questions the existence of a bijection between the coherence spaces A and the configuration spaces X such that

$$X \quad = \quad \sim\sim X.$$

§P7. The category **Conf** is defined as the category whose objects are the configuration spaces, and whose morphisms

$$f \quad : \quad X \quad \longrightarrow \quad Y$$

are the binary relations

$$f \quad \subseteq \quad |X| \times |Y|$$

such that :

— the relation f transports the configurations in the forward direction :

$$\forall u \in \mathrm{Config}(X), \quad f(u) \quad \in \quad \mathrm{Config}(Y)$$

where we write

$$f(u) \quad = \quad \{ \quad y \in |Y| \quad | \quad \exists x \in u \text{ tel que } (x,y) \in f \quad \}$$

— the relation f is locally injective in the sense that for every configuration $u \in \mathrm{Config}(X)$, one has :

$$\forall x_1, x_2 \in u, \qquad \big(\, \exists y \in |Y|, \quad (x_1, y) \in f \text{ et } (x_2, y) \in f \, \big) \quad \Rightarrow \quad x_1 = x_2.$$

The identity on the configuration space X is the morphism defined as follows :

$$\mathrm{id}_X \quad = \quad \{ \quad (x,x) \quad | \quad x \in |X| \quad \}.$$

Show that these data define a category, where morphisms are composed as relations. [note : you are welcome to use the fact that binary relations between sets define a category noted **Rel**].

§P8. Show that the construction $U(-)$ defines a fully faithful functor

$$U(-) \quad : \quad \mathbf{Coh} \quad \longrightarrow \quad \mathbf{Conf}$$

Reminder : a functor U is called fully faithful when the function

$$\begin{array}{ccc} \mathbf{Coh}(A,B) & \rightarrow & \mathbf{Conf}(U(A),U(B)) \\ f & \mapsto & U(f) \end{array}$$

is bijective for every pair of coherence spaces A and B.

This result enables us to see the category **Coh** as the full subcategory of **Conf** consisting of the configuration spaces X such that $X = \sim\sim X$. Reminder : a category $\mathcal{B}$ is a full subcategory of a category $\mathcal{C}$ when the class of objects of $\mathcal{B}$ is included in the class of objects of $\mathcal{C}$, and the sets of morphisms $\mathcal{B}(B_1, B_2)$ and $\mathcal{C}(B_1, B_2)$ between two objects B_1 and B_2 of $\mathcal{B}$ (and thus of $\mathcal{C}$) are the same in the categories $\mathcal{B}$ and $\mathcal{C}$:

$$\mathcal{B}(B_1, B_2) = \mathcal{C}(B_1, B_2)$$

with the same composition and identity laws in $\mathcal{B}$ and in $\mathcal{C}$.

§P9. Show that the construction $\sim$ defines a functor

$$\sim \quad : \quad \mathbf{Conf} \quad \longrightarrow \quad \mathbf{Conf}^{\,op}$$

which transports a morphism $f : X \to Y$ in the morphism

$$\sim f \quad : \quad \sim Y \quad \longrightarrow \quad \sim X$$

defined as

$$(y, x) \in \sim f \qquad \Longleftrightarrow \qquad (x, y) \in f$$

for all $x \in |X|$ and $y \in |Y|$.

§P10. Deduce from the previous question that the double négation $\sim\sim$ defines a functor

$$\sim\sim \quad : \quad \mathbf{Conf} \quad \longrightarrow \quad \mathbf{Conf}.$$

§P11. Describe in a simple way the coherence space $\sim\sim X$ associated to a configuration space X, as well as the morphism

$$\sim\sim f \quad : \quad \sim\sim X \quad \longrightarrow \quad \sim\sim Y$$

associated to a morphism $f : X \to Y$.

§P12. Show that the relation

$$\eta_X \quad = \quad \{ \quad (x, x) \quad | \quad x \in |X| \quad \}$$

defines for every configuration space X a morphism

$$\eta_X \quad : \quad X \quad \longrightarrow \quad \sim\sim X$$

of the category **Conf**.

§P13. Show that the family of morphisms $\eta_X : X \to \sim\sim X$ is natural in X.
We find convenient write

$$F \quad : \quad \textbf{Conf} \quad \longrightarrow \quad \textbf{Coh}.$$

for that double negation functor.

§P14. Give the example of a configuration space X such that the inclusion

$$\mathrm{Config}(X) \subseteq \mathrm{Config}(UF(X))$$

is strict [note : one can take a configuration space X with three points in its web $|X|$].

§P15. Suppose given a configuration space X and a coherence space A. Show that a binary relation

$$f \quad \subseteq \quad |X| \times |A|$$

is an element of

$$\textbf{Conf}(X, U(A))$$

if and only if the binary relation f is an element of

$$\textbf{Coh}(F(X), A).$$

§P16. From this, deduce that the functor

$$F \quad : \quad \textbf{Conf} \quad \longrightarrow \quad \textbf{Coh}$$

is left adjoint to the functor

$$U \quad : \quad \textbf{Coh} \quad \longrightarrow \quad \textbf{Conf}$$

§P16. Describe the unit

$$\eta_X \quad : \quad X \quad \longrightarrow \quad UF(X)$$

and the counit

$$\varepsilon_A \quad : \quad FU(A) \quad \longrightarrow \quad A$$

of the adjunction $F \dashv U$ for a configuraton space X and for a coherence space A.

§P17. Given two configuration spaces X and Y, one defines the configuration space $X \bullet Y$ as follows :

$$|X \bullet Y| \quad = \quad |X| \times |Y|$$

$$\mathrm{Config}(X \bullet Y) \quad = \quad \{ \;\; u \times v \quad | \quad u \in \mathrm{Config}(X) \;\text{ et }\; v \in \mathrm{Config}(Y) \;\; \}$$

We admit that the tensor product defines a structure of symmetric monoidal category on the category **Conf**. Show that

$$F(X) \otimes F(Y) \quad = \quad F(X \bullet Y)$$

where $\otimes$ denotes the tensor product of coherence spaces defined in the course. Explain in what sense this equation enables one to deduce the tensor product $\otimes$ of coherence spaces from the tensor product $\bullet$ on configuration spaces.

§P18. Given two configuration spaces X and Y, the configuration space $X+Y$ is defined as follows :

$$\begin{array}{rcl} |X+Y| & = & |X|+|Y| \\ \mathrm{Config}(X+Y) & = & \{ \ \mathtt{inl}(u) \mid u \in \mathrm{Config}(X) \ \} \\ & & \cup \ \{ \ \mathtt{inr}(v) \mid v \in \mathrm{Config}(Y) \ \} \end{array}$$

Show that $X+Y$ defined a cartesian sum of X and of Y in the category **Conf**. [Note : a cartesian sum is the dual of a cartesian product, that is, a cartesian product in the opposite category $\mathbf{Conf}^{op}$.]

§P19. Show that

$$F(X) \oplus F(Y) \quad = \quad F(X+Y).$$

Explain in what sense this equation enables one to deduce the cartesian sum $\otimes$ of coherence spaces from the cartesian sum $\bullet$ on configuration spaces.

§P20. Suppose given two configuration spaces X and Y. Show that a binary relation

$$f \quad \subseteq \quad |X| \times |Y|$$

is an element of

$$\mathbf{Conf}(X, \sim Y)$$

if and only if

$$\forall (u,v) \in \mathrm{Config}(X) \times \mathrm{Config}(Y), \qquad f \quad \perp \quad u \times v.$$

[Note : we use here the fact that every element y of the web of Y appears as a specific configuration v of the configuration space Y.]

§P21. Deduce a bijection

$$\mathbf{Conf}(X, \sim Y) \quad \cong \quad \mathbf{Conf}(X \bullet Y, \perp)$$

where $\perp$ denotes the configuration space whose web is a singleton $\{*\}$ and whose two configurations are the empty set and the singleton set $\{*\}$.