Lecture notes

A survival guide on coherence spaces

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1 Coherence spaces

1.1 Definition

Definition 1.1 (coherence spaces) *A coherence space is a pair*

$$A \quad = \quad (\,|A|\,,\, \mathrel{\substack{\frown\\\smile}}_A$$

consisting of a set $|A|$ called the web of A and of a reflexive and symmetric relation

$$\mathrel{\substack{\frown\\\smile}}_A \quad \subseteq \quad |A| \times |A|$$

called the coherence.

So, "coherence space" is a somewhat pedantic word for "reflexive graph".

1.2 Notations

Our purpose in this note is to construct a model of linear logic where the proofs of a given formula A are interpreted as *cliques* of the associated coherence space $[\![A]\!]$. As we will see, the interpretation is based to a large extent on the operation of negating a coherence space A into its dual coherence space $A^{\perp}$. Since the dual graph of a reflexive graph is not reflexive, one finds convenient to introduce the notations below:

- $a \frown_A a'$ means that $a \mathrel{\substack{\frown\\\smile}}_A a'$ and $a \neq a'$.
- $a \mathrel{\substack{\smile\\\frown}}_A a'$ means that $\neg\,(a \mathrel{\substack{\frown\\\smile}}_A a')$ or $a = a'$.

1.3 Basic constructions

Negation. The dual

$$A^{\perp}$$

of a coherence space A has the same web as the original coherence space

$$|A^{\perp}| \quad = \quad |A|$$

and coherence defined as

$$\forall a, a' \in |A| \qquad a \mathrel{\substack{\frown \\ \smile}}_{A^{\perp}} a' \quad \Longleftrightarrow \quad a \mathrel{\substack{\smile \\ \frown}}_{A} a'$$

In other words, the coherence space $A^{\perp}$ is the dual graph of A where every point of the web is moreover made coherent to itself, in order to obtain a reflexive graph.

Sum. The sum

$$A \oplus B$$

of two coherence spaces is the coherence space with web the disjoint sum of the two webs:

$$|A \oplus B| \quad = \quad |A| + |B|$$

with coherence defined as

$$\begin{array}{ll} \forall a, a' \in |A| & \mathbf{inl}(a) \mathrel{\substack{\frown \\ \smile}}_{A \oplus B} \mathbf{inl}(a') \iff a \mathrel{\substack{\frown \\ \smile}}_{A} a' \\ \forall b, b' \in |B| & \mathbf{inr}(b) \mathrel{\substack{\frown \\ \smile}}_{A \oplus B} \mathbf{inr}(b') \iff b \mathrel{\substack{\frown \\ \smile}}_{B} b' \\ \forall a \in |A| \quad \forall b \in |B| & \mathbf{inl}(a) \mathrel{\substack{\smile \\ \frown}}_{A \oplus B} \mathbf{inr}(b) \end{array}$$

In other words, the coherence space $A \oplus B$ is simply defined as the disjoint sum of the two reflexive graphs A and B.

Tensor product. The tensor product

$$A \otimes B$$

of two coherence spaces is the coherence space with web the product of the two webs:

$$|A \otimes B| \quad = \quad |A| \times |B|$$

with coherence defined as

$$\begin{array}{l}\forall a, a' \in |A| \\ \forall b, b' \in |B|\end{array} \qquad a \otimes b \frown_{A \otimes B} a' \otimes b' \quad \iff \quad a \frown_A a' \quad \text{and} \quad b \frown_B b'.$$

In other words, the coherence space $A \otimes B$ is simply defined as the product of the two reflexive graphs A and B. Note that for convenience, we write the pair (a, b) as $a \otimes b$ when it is an element of the web of $A \otimes B$. We will do it in a similar way for the elements $a \parr b$ and $a \multimap b$ of the web $A \parr B$ or $A \multimap B$. This is only a convention which we choose to make the constructions more readable. The anxious reader may replace all of them mentally: $a \otimes b$, $a \parr b$, $a \multimap b$ by a pair (a, b).

Product. The product

$$A \,\&\, B$$

of two coherence spaces is defined as

$$A \,\&\, B \quad = \quad (A^\perp \oplus B^\perp)^\perp$$

Parallel product. The parallel product

$$A \parr B$$

of two coherence spaces is defined as

$$A \parr B \quad = \quad (A^\perp \otimes B^\perp)^\perp$$

Linear implication. The linear implication

$$A \multimap B$$

of two coherence spaces A and B is defined as

$$A \multimap B \quad = \quad (A \otimes B^\perp)^\perp$$

Note that

$$A \multimap B \quad = \quad B^\perp \multimap A^\perp$$

Exercise. Given two pairs of elements $(a, b) = a \multimap b$ and $(a', b') = a' \multimap b'$ of the web of $A \multimap B$, show that

$$a \multimap b \frown_{A \multimap B} a' \multimap b' \quad \Longleftrightarrow \quad \begin{cases} a \frown_A a' \text{ implies } b \frown_B b' \\ \text{and} \\ b \frown_{B^\perp} b' \text{ implies } a \frown_{A^\perp} a' \end{cases}$$

Show that this definition is equivalent to the following one

$$a \multimap b \frown_{A \multimap B} a' \multimap b' \quad \Longleftrightarrow \quad \begin{cases} a \frown_A a' \text{ implies } b \frown_B b' \\ \text{and} \\ b = b' \text{ implies } a \stackrel{\smile}{\frown}_A a' \end{cases}$$

2 The monoidal category of coherence spaces

2.1 Definition

The category **Coh** has

- coherence spaces as objects
- cliques of $A \multimap B$ as morphisms $A \to B$.

2.2 The monoidal structure

Given two morphisms

$$f : A \longrightarrow B \qquad\qquad g : A' \longrightarrow B'$$

the morphism

$$f \otimes g : A \otimes A' \longrightarrow B \otimes B'$$

is defined as the clique

$$f \otimes g \quad = \quad \{ \ (a \otimes a') \multimap (b \otimes b') \ \mid \ a \multimap b \in f \quad , \quad a' \multimap b' \in g \ \}$$

Exercise. Show that this defines a clique of $(A \otimes A') \multimap (B \otimes B')$

Exercise. Show that this defines a functor

$$\otimes \quad : \quad \mathbf{Coh} \times \mathbf{Coh} \quad \longrightarrow \quad \mathbf{Coh}$$

2.3 Associativity map

Given three coherence spaces A, B, C, the associativity map

$$\alpha_{A,B,C} \quad : \quad (A \otimes B) \otimes C \quad \longrightarrow \quad A \otimes (B \otimes C)$$

is defined as the clique

$$\alpha_{A,B,C} \quad = \quad \{ \quad ((a \otimes b) \otimes c) \multimap (a \otimes (b \otimes c)) \quad | \quad a \in |A|, b \in |B|, c \in |C| \quad \}$$

2.4 Unit maps

Given a coherence space A, the unit maps

$$\lambda_A \quad : \quad I \otimes A \quad \longrightarrow \quad A$$

$$\rho_A \quad : \quad A \otimes I \quad \longrightarrow \quad A$$

are defined as the cliques

$$\lambda_A \quad = \quad \{ \quad (* \otimes a) \multimap a \quad | \quad a \in |A| \quad \}$$

$$\rho_A \quad = \quad \{ \quad (a \otimes *) \multimap a \quad | \quad a \in |A| \quad \}$$

2.5 Symmetry map

Given two coherence spaces A, B, the symmetry map

$$\gamma_{A,B} \quad : \quad A \otimes B \quad \longrightarrow \quad B \otimes A$$

is defined as the clique

$$\gamma_{A,B} \quad = \quad \{ \quad (a \otimes b) \multimap (b \otimes a) \quad | \quad a \in |A|, \quad b \in |B| \quad \}$$

Exercise. Show that γ is natural in A and B

Exercise. Show that γ defines a symmetry in the monoidal category **Coh**.

2.6 Evaluation map

Given two coherence spaces A, B, the evaluation map

$$\mathbf{eval}_{A,B} \quad : \quad A \otimes (A \multimap B) \quad \longrightarrow \quad B$$

is defined as the clique

$$\mathbf{eval}_{A,B} \quad = \quad \{ \quad (a \otimes (a \multimap b)) \multimap b \quad | \quad a \in |A|, \quad b \in |B| \quad \}$$

Exercise. Show that $\mathrm{eval}_{A,B}$ defines a clique of the coherence space

$$(A \otimes (A \multimap B)) \multimap B$$

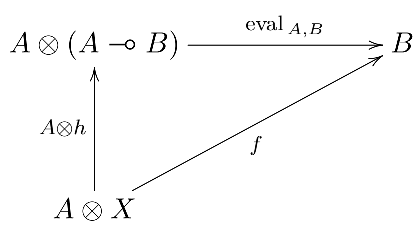
Exercise. Show that $\mathrm{eval}_{A,B}$ satisfies the following universality property: for every coherence space X and every morphism

$$f \quad : \quad A \otimes X \quad \longrightarrow \quad B$$

there exists a unique morphism

$$h \quad : \quad X \quad \longrightarrow \quad A \multimap B$$

making the diagram



commute. Deduce that the category **Coh** is symmetric monoidal closed.

2.7 Cartesian product

Given two coherences spaces A, B define the projection maps

$$\pi_1 \quad : \quad A\&B \quad \longrightarrow \quad A$$

$$\pi_2 \quad : \quad A\&B \quad \longrightarrow \quad B$$

as the coherence spaces

$$\pi_1 \quad = \quad \{ \quad \mathbf{inl}(a) \multimap a \quad | \quad a \in |A| \quad \}$$

$$\pi_2 \quad = \quad \{ \quad \mathbf{inr}(b) \multimap b \quad | \quad b \in |B| \quad \}$$

Exercise. Show that the coherence space $A\&B$ equipped with the two projection maps π_1 and π_2 defines a cartesian product of A and B in the category **Coh**.