Modèles des langages de programmation
Domaines, catégories, jeux

Programme de cette seconde séance:

Modèle ensembliste du lambda-calcul ;
Catégories cartésiennes fermées

Synopsis

1 — the simply-typed λ-calculus,

2 — the set-theoretic model of the λ-calculus,

3 — categories and functors

4 — cartesian categories

5 — cartesian closed categories

6 — interprétation of the simply-typed λ-calculus in a ccc

The simply-typed λ-calculus

The pure λ-calculus

Terms $\quad M \quad ::= \quad x \quad | \quad M\,N \quad | \quad \text{ł}x.\,M$

The β-reduction:

$$(\lambda x.M)\,N \quad \longrightarrow \quad M\,[x := N]$$

The η-expansion:

$$M \quad \longrightarrow \quad \lambda x.\,(M\,x)$$

Remark: every term is considered up to renaming $\equiv_\alpha$ of the bound variables, typically:

$$\lambda x.\lambda y.x \quad \equiv_\alpha \quad \lambda z.\lambda y.z$$

The simply-typed λ-calculus

The simple types A, B are constructed by the grammar:

$$A, B \ ::= \ \alpha \mid A \Rightarrow B.$$

A typing context Γ is a finite sequence

$$\Gamma = (x_1 : A_1, \ldots, x_n : A_n)$$

where each x_i is a variable and each A_i is a simple type.

A sequent is a triple

$$x_1 : A_1, \ldots, x_n : A_n \vdash P : B$$

where

$$x_1 : A_1, \ldots, x_n : A_n$$

is a typing context, P is a λ-term and B is a simple type.

The simply-typed λ-calculus

Variable

$$\frac{}{x : A \vdash x : A}$$

Abstraction

$$\frac{\Gamma, x : A \vdash P : B}{\Gamma \vdash \lambda x.P : A \Rightarrow B}$$

Application

$$\frac{\Gamma \vdash P : A \Rightarrow B \qquad \Delta \vdash Q : A}{\Gamma, \Delta \vdash PQ : B}$$

Weakening

$$\frac{\Gamma \vdash P : B}{\Gamma, x : A \vdash P : B}$$

Contraction

$$\frac{\Gamma, x : A, y : A \vdash P : B}{\Gamma, z : A \vdash P[x, y \leftarrow z] : B}$$

Exchange

$$\frac{\Gamma, x : A, y : B, \Delta \vdash P : C}{\Gamma, y : B, x : A, \Delta \vdash P : C}$$

Subject reduction

A λ-term P is simply typed when there exists a sequent

$$\Gamma \quad \vdash \quad P : A$$

which may be obtained by a derivation tree.

One establishes that the set of simply typed λ-terms is closed under β-réduction:

Subject Reduction:

If $\Gamma \vdash P : A$ and $P \longrightarrow_{\beta} Q$, then $\Gamma \vdash Q : A$.

The set-theoretic interpretation of the λ-calculus

Interprétation ensembliste

To each atomic type α is associated a set X_α

Then, one extends the interpretation to every type:

$$[\![\alpha]\!] = X_\alpha \qquad [\![A \Rightarrow B]\!] \;=\; [\![B]\!]^{[\![A]\!]} \;=\; \mathbf{Set}\,([\![A]\!], [\![B]\!])$$

A sequent $x_1 : A_1, \ldots, x_n : A_n \;\vdash\; M : B$

is interpreted as a function $[\![A_1]\!] \times \cdots \times [\![A_n]\!] \longrightarrow [\![B]\!]$

Soundness theorem

Theorem.

The interpretation provides an invariant of λ-terms modulo β and η.

- ▷ if $\Gamma \vdash (\lambda x.M) : A \Rightarrow B$ and $\Delta \vdash N : A$, then

$$[\![\Gamma, \Delta \vdash (\lambda x.M)N : B]\!] \quad = \quad [\![\Gamma, \Delta \vdash M[x := N] : B]\!]$$

- ▷ if $\Gamma \vdash M : A \Rightarrow B$ then

$$[\![\Gamma \vdash (\lambda x.Mx) : A \Rightarrow B]\!] \quad = \quad [\![\Gamma \vdash M : A \Rightarrow B]\!]$$

Categories and functors

Categories

A category $\mathcal{C}$ is given by

$[0]$ a class of objects

$[1]$ a set $\mathbf{Hom}(A,B)$ of morphisms

$$f \quad : \quad A \quad \longrightarrow \quad B$$

for every pair of objects (A,B)

$[2]$ a composition law $\circ : \mathbf{Hom}(B,C) \times \mathbf{Hom}(A,B) \longrightarrow \mathbf{Hom}(A,C)$

$[2]$ an identity morphism

$$id_A \quad : \quad A \quad \longrightarrow \quad A$$

for every object A,

Categories

satisfying the following properties:

[3] the composition law $\circ$ is associative:

$$\begin{array}{ll} \forall f \in \mathbf{Hom}(A,B) & \\ \forall g \in \mathbf{Hom}(B,C) & f \circ (g \circ h) = (f \circ g) \circ h \\ \forall h \in \mathbf{Hom}(C,D) & \end{array}$$

[3] the morphisms id are neutral elements

$$\forall f \in \mathbf{Hom}(A,B) \qquad f \circ id_A \ = \ f \ = \ id_B \circ f$$

Examples

1. the category **Set** of sets and functions
2. the category **Ord** of partial orders and monotone functions
3. the category **Dom** of domains and continuous functions
4. the category **Coh** of coherence spaces and linear maps
5. every partial order
6. every monoid

Opposite category

The opposite of a category

$$\mathcal{C}$$

is the category noted

$$\mathcal{C}^{op}$$

whose morphisms

$$f \quad : \quad A \quad \longrightarrow \quad B$$

are defined as the morphisms

$$f \quad : \quad B \quad \longrightarrow \quad A$$

of the original category $\mathcal{C}$.

Product category

The product

$$\mathcal{A} \times \mathcal{B}$$

of two categories

$$\mathcal{A} \qquad\qquad \mathcal{B}$$

is the category

- ▷ whose objects are the pairs (A, B) of objects of $\mathcal{A}$ and $\mathcal{B}$,
- ▷ whose morphisms

$$(A, B) \quad \longrightarrow \quad (A', B')$$

are the pairs of morphisms

$$f : A \longrightarrow A' \qquad\qquad g : B \longrightarrow B'$$

Product category

with composition and identities defined as expected:

$$(A,B)\xrightarrow{id_{(A,B)}}(A,B) \quad = \quad (A,B)\xrightarrow{(id_A,id_B)}(A,B)$$

$$(A,B)\xrightarrow{(f,g)}(A',B')\xrightarrow{(f',g')}(A'',B'') \quad = \quad (A,B)\xrightarrow{(f'\circ f,g'\circ g)}(A'',B'')$$

Functors

A functor between categories

$$F \quad : \quad \mathcal{C} \quad \longrightarrow \quad \mathcal{D}$$

is defined as the following data:

$[0]$ an object FA of $\mathcal{D}$ for every object A of $\mathcal{C}$,

$[1]$ a function

$$F_{A,B} \quad : \quad \mathbf{Hom}_{\mathcal{C}}(A, B) \quad \longrightarrow \quad \mathbf{Hom}_{\mathcal{D}}(FA, FB)$$

for every pair of objects (A, B) of the category $\mathcal{C}$.

Functors

One requires moreover

[2] that F preserves composition

$$FA \xrightarrow{Ff} FB \xrightarrow{Fg} FC \qquad = \qquad FA \xrightarrow{F(g \circ f)} FC$$

[2] that F preserves the identities

$$FA \xrightarrow{Fid_A} FA \qquad = \qquad FA \xrightarrow{id_{FA}} FA$$

Illustration $[\,\textit{orders}\,]$

Every ordered set

$$(X, \leq)$$

defines a category

$$[X, \leq]$$

- ▷ whose objects are the elements of X
- ▷ whose hom-sets are defined as

$$\mathbf{Hom}(x, y) \quad = \quad \begin{cases} \{*\} & \text{if } \; x \leq y \\ \varnothing & \text{otherwise} \end{cases}$$

In this category, there exists at most one map between two objects

Illustration [*orders*]

Exercise: given two ordered sets

$$(X, \leq) \qquad (Y, \leq)$$

a functor

$$F \quad : \quad [X, \leq] \quad \longrightarrow \quad [Y, \leq]$$

is the same thing as a monotonic function

$$F \quad : \quad (X, \leq) \quad \longrightarrow \quad (Y, \leq)$$

between the underlying ordered sets.

Illustration [*monoids*]

A monoid $(M, \cdot, e)$ is a set M equipped with a binary operation

$$\cdot \quad : \quad M \times M \quad \longrightarrow \quad M$$

and a neutral element

$$e \quad : \quad \{*\} \quad \longrightarrow M$$

satisfying the two properties below:

Associativity law $\quad \forall x, y, z \in M, \quad (x \cdot y) \cdot z = x \cdot (y \cdot z)$

Unit law $\quad \forall x \in M, \quad x \cdot e = x = e \cdot x.$

Illustration [*monoids*]

Key observation: there is a one-to-one relationship

$$(M, \cdot, e) \quad \mapsto \quad \Sigma\,(M, \cdot, e)$$

between

- ▷ monoids
- ▷ categories with one object $*$

obtained by defining $\Sigma(M, \cdot, e)$ as the category with unique hom-set

$$\Sigma(M, \cdot, e)\ (*, *) \quad = \quad M$$

and composition law and unit defined as

$$g \circ f \quad = \quad g \cdot f \qquad\qquad id_{\,*} = e$$

Illustration [*monoids*]

Key observation: given two monoids

$$(M, \cdot, e) \qquad\qquad (N, \bullet, u)$$

a functor

$$F \quad : \quad \Sigma\,(M, \cdot, e) \quad \longrightarrow \quad \Sigma\,(N, \bullet, u)$$

is the same thing as a homomorphism

$$f \quad : \quad (M, \cdot, e) \quad \longrightarrow \quad (N, \bullet, u)$$

between the underlying monoids.

Recall that a homomorphism is a function f such that

$$\forall x, y \in M, \quad f(x \cdot y) \; = \; f(x) \bullet f(y) \qquad\qquad f(e) \; = \; u$$

Transformations

A transformation

$$\theta \quad : \quad F \overset{\cdot}{\longrightarrow} G$$

between two functors

$$F, G \quad : \quad \mathcal{A} \longrightarrow \mathcal{B}$$

is a family of morphisms

$$(\theta_A : FA \longrightarrow GA)_{A \in Obj(\mathcal{A})}$$

of the category $\mathcal{B}$ indexed by the objects of the category $\mathcal{A}$.

Natural transformations

A transformation $\theta \quad : \quad F \Rightarrow G \quad : \quad \mathcal{A} \longrightarrow \mathcal{B}$

is natural when the diagram

$$\begin{tikzcd}
FA \arrow[r, "\theta_A"] \arrow[d, "Ff"'] & GA \arrow[d, "Gf"] \\
FB \arrow[r, "\theta_B"] & GB
\end{tikzcd}$$

commutes for every morphism $f : A \longrightarrow B$.

Isomorphism

In a category $\mathcal{C}$, a morphism

$$f \quad : \quad A \quad \longrightarrow \quad B$$

is called an isomorphism when there exists a morphism

$$g \quad : \quad B \quad \longrightarrow \quad A$$

satisfying

$$g \circ f = id_A \qquad \text{et} \qquad f \circ g = id_B.$$

Exercise. Show that $g \circ f : A \longrightarrow C$ is an isomorphism when $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are isomorphisms.

Exercise. Show that every functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ transports an isomorphism of $\mathcal{C}$ into an isomorphism of $\mathcal{D}$.

Bifunctors

A bifunctor F from $\mathcal{C}$ and $\mathcal{D}$ to $\mathcal{E}$ is given by:

- ▷ a functor $F(A,-) \quad : \quad \mathcal{D} \longrightarrow \quad \mathcal{E}$

  for every object A of the category $\mathcal{C}$

- ▷ a functor $F(-,B) \quad : \quad \mathcal{C} \longrightarrow \quad \mathcal{E}$

  for every object B of the category $\mathcal{D}$

such that the diagram

$$\begin{tikzcd}
F(A,B) \arrow[r, "F(A,g)"] \arrow[d, "F(f,B)"'] & F(A,B') \arrow[d, "F(f,B')"] \\
F(A',B) \arrow[r, "F(A',g)"] & F(A',B')
\end{tikzcd}$$

commutes for all morphisms $f : A \longrightarrow A'$ in $\mathcal{C}$ and $g : B \longrightarrow B'$ in $\mathcal{D}$.

Cartesian categories

Products

The product of two objects A and B in a category $\mathcal{C}$ is an object $A \times B$ equipped with two morphisms

$$\pi_1 : A \times B \longrightarrow A \qquad \pi_2 : A \times B \longrightarrow B$$

such that for every diagram

$$\begin{tikzcd} & & A \\ X \arrow[urr, bend left, "f"] \arrow[drr, bend right, "g"'] & A \times B \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"] & \\ & & B \end{tikzcd}$$

there exists a unique morphism $h : X \longrightarrow A \times B$ making the diagram

$$\begin{tikzcd} & & A \\ X \arrow[r, "h"] \arrow[urr, bend left, "f"] \arrow[drr, bend right, "g"'] & A \times B \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"] & \\ & & B \end{tikzcd}$$

commute.

Illustrations

1. The cartesian product in the category **Set**,
2. The lub $a \wedge b$ of two elements a and b in an ordered set $(X, \leq)$,
3. The cartesian product in the category **Dom**,

Terminal object

An object $\mathbf{1}$ is terminal in a category $\mathcal{C}$ when $\mathbf{Hom}(A, \mathbf{1})$ is a singleton for all objects A.

One may consider $\mathbf{1}$ as the nullary product in $\mathcal{C}$.

Example 1. the singleton $\{*\}$ in the categories $\mathbf{Set}$ and $\mathbf{Dom}$,

Example 2. the maximum of an ordered set $(X, \leq)$

Cartesian category

A cartesian category is a category $\mathcal{C}$ equipped with a product

$$A \times B$$

for all pairs A, B of objects, and of a terminal object

$$\mathbf{1}$$

Cartesian categories

In every cartesian category, one finds

- weakening maps $\epsilon_A : A \longrightarrow \mathbf{1}$,
- diagonal maps $\delta_A : A \longrightarrow A \times A$ obtained as

$$\begin{tikzcd}
 & & A \\
A \arrow[r, "\delta_A"] \arrow[urr, bend left, "id_A"] \arrow[drr, bend right, "id_A"'] & A \times A \arrow[ur, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
 & & A
\end{tikzcd}$$

- symmetry maps $\gamma_{A,B} : A \times B \longrightarrow B \times A$ obtained as

$$\begin{tikzcd}
 & & B \\
A \times B \arrow[r, "\gamma_{A,B}"] \arrow[urr, bend left, "\pi_2"] \arrow[drr, bend right, "\pi_1"'] & B \times A \arrow[ur, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
 & & A
\end{tikzcd}$$

Exercise: Show that $(- \times -)$ defines a bifunctor $\mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$.

Cartesian closed categories

First definition

Ccc

A cartesian closed category is a cartesian category

$$(\mathcal{C}, \times, 1)$$

together with the following data for all objects A and B:

- ▷ of an object $A \Rightarrow B$
- ▷ of a morphism $eval_{A,B} : A \times (A \Rightarrow B) \longrightarrow B$

such that for every object X and morphism

$$f : A \times X \longrightarrow B$$

there exists a unique morphism $h : X \longrightarrow A \Rightarrow B$ making the diagram

$$\begin{tikzcd}
A \times (A \Rightarrow B) \arrow[r, "eval_{A,B}"] & B \\
A \times X \arrow[u, "A \times h"] \arrow[ur, "f"'] &
\end{tikzcd}$$

commute.

Cartesian closed category

Second definition

Adjunction

An adjunction is a triple consisting of two functors

$$L : \mathcal{A} \longrightarrow \mathcal{B} \qquad\qquad R : \mathcal{B} \longrightarrow \mathcal{A}$$

and a family of bijections

$$\phi_{A,B} \quad : \quad \mathbf{Hom}_{\mathcal{B}}(LA, B) \;\cong\; \mathbf{Hom}_{\mathcal{A}}(A, RB)$$

natural in A and B, for all pairs of objects A, B of $\mathcal{A}$ and $\mathcal{B}$.

$$\boxed{\quad \frac{LA \xrightarrow{\mathcal{B}} B}{A \xrightarrow{\mathcal{A}} RB} \qquad \phi_{A,B} \quad}$$

One writes $L \dashv R$ and one says that L is left adjoint to R.

Example: the free vector space

$$\begin{tikzcd}
\mathbf{Set} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathbf{Vect} \arrow[l, bend left, "R"]
\end{tikzcd}$$

where

$\mathcal{A} = \mathbf{Set}$: the category of sets and functions
$\mathcal{B} = \mathbf{Vect}$: the category of vector spaces on a field k

R : the « forgetful » functor $V \mapsto U(V)$
L : the « free vector space » functor $X \mapsto kX$

$$kX \; := \; \left\{ \; \sum_{x \in X} \lambda_x \; x \;\; \middle| \;\; \lambda_x \in k \text{ null almost everywhere.} \; \right\}$$

Illustration: the free monoid

$$\begin{tikzcd}
\mathbf{Set} \arrow[rr, bend left, "L"] \arrow[rr, phantom, "\perp"] & & \mathbf{Mon} \arrow[ll, bend left, "R"]
\end{tikzcd}$$

where

$\mathcal{A} = \mathbf{Set}$: the category of sets and functions
$\mathcal{B} = \mathbf{Mon}$: the category of monoids and homomorphisms,

R : the « forgetful » functor $M \mapsto U(M)$.
L : the « free monoid » functor $A \mapsto A^*$.

$$A \quad := \quad \coprod_{n\in\mathbb{N}} A^n$$

What does natural bijection ϕ exactly mean?

By ϕ natural, one means that the two families of sets

$$\mathbf{Hom}_{\mathcal{B}}(LA, B) \qquad\qquad \mathbf{Hom}_{\mathcal{A}}(A, RB)$$

define functors

$$\mathbf{Hom}_{\mathcal{B}}(L-, -) \quad : \quad \mathcal{A}^{op} \times \mathcal{B} \quad \longrightarrow \quad \mathbf{Set}$$

$$\mathbf{Hom}_{\mathcal{A}}(-, R-) \quad : \quad \mathcal{A}^{op} \times \mathcal{B} \quad \longrightarrow \quad \mathbf{Set}$$

and that the family of bijections ϕ defines a **natural transformation**

$$\phi \ : \ \mathbf{Hom}_{\mathcal{B}}(L-, -) \cong \mathbf{Hom}_{\mathcal{A}}(-, R-) \ : \ \mathcal{A}^{op} \times \mathcal{B} \ \longrightarrow \ \mathbf{Set}$$

between them.

What does natural bijection ϕ exactly mean?

Natural in A and B thus means that every commutative diagram

$$\begin{tikzcd}
LA \arrow[r, "g"] & B \arrow[d, "h_B"] \\
LA' \arrow[u, "Lh_A"] \arrow[r, "f"'] & B'
\end{tikzcd}$$

is transformed into a commutative diagram

$$\begin{tikzcd}
A \arrow[r, "\phi_{A,B}(g)"] & RB \arrow[d, "Rh_B"] \\
A' \arrow[u, "h_A"] \arrow[r, "\phi_{A',B'}(f)"'] & RB'
\end{tikzcd}$$

Cartesian exponentiation

Consider an object A in a cartesian category$(\mathcal{C}, \times, \mathbf{1})$.

A cartesian exponentiation of A is a pair consisting of a functor

$$(A \Rightarrow -) : \mathcal{C} \longrightarrow \mathcal{C}$$

and of a family of bijections

$$\phi_{A,B,C} \quad : \quad \mathbf{Hom}(A \times B, C) \longrightarrow \mathbf{Hom}(B, A \Rightarrow C)$$

natural in B and C.

In other words, it is an **adjunction** between the functors

$$A \times - \quad \dashv \quad A \Rightarrow -$$

What natural bijection means in that case

Naturality in B and C means that the family of bijections

$$\phi_{A,B,C} \quad : \quad \mathbf{Hom}(A \times B, C) \longrightarrow \mathbf{Hom}(B, A \Rightarrow C)$$

transforms every commutative diagram

$$\begin{tikzcd}
A \times B \arrow[r, "g"] & C \arrow[d, "h_C"] \\
A \times B' \arrow[u, "A\times h_B"] \arrow[r, "f"'] & C'
\end{tikzcd}$$

into a commutative diagram

$$\begin{tikzcd}
B \arrow[r, "\phi_{A,B,C}(g)"] & A \Rightarrow C \arrow[d, "A\Rightarrow h_C"] \\
B' \arrow[u, "h_B"] \arrow[r, "\phi_{A,B',C'}(f)"'] & A \Rightarrow C'
\end{tikzcd}$$

Cartesian closed category

Definition.

A cartesian closed category (ccc) is a cartesian category

$$(\mathcal{C}, \times, \mathbf{1})$$

equipped with a cartesian exponentation

$$\frac{A \times B \longrightarrow C}{B \longrightarrow A \Rightarrow C} \quad \phi_{A,B,C}$$

for every object A of the category.

Parameter theorem

We have seen that the cartesian product defines a bifoncteur

$$A, B \quad \mapsto \quad A \times B \quad : \quad \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$$

in every cartesian category. In the same way,

Parameter theorem (MacLane)

The family of cartesian exponentiations

$$(A \Rightarrow -)_A \quad : \quad \mathcal{C} \quad \longrightarrow \mathcal{C}$$

defines a unique bifunctor

$$A, B \quad \mapsto \quad A \Rightarrow B \quad : \quad \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathcal{C}$$

such that the bijections $\phi_{A,B,C}$ are natural in A, B, C.

Parameter theorem

Here, natural in A, B, C means that the family of bijections

$$(\phi_{A,B,C})_{A,B,C} \quad : \quad \mathbf{Hom}(A \times B, C) \longrightarrow \mathbf{Hom}(B, A \Rightarrow C)$$

transforms every commutative diagram

$$\begin{tikzcd}
A \times B \arrow[r, "g"] \arrow[d, "h_C", phantom] & C \arrow[d, "h_C"] \\
A' \times B' \arrow[u, "h_A \times h_B"] \arrow[r, "f"'] & C'
\end{tikzcd}$$

in a commutative diagram:

$$\begin{tikzcd}
B \arrow[r, "\phi_{A,B,C}(g)"] & A \Rightarrow C \arrow[d, "h_A \Rightarrow h_C"] \\
B' \arrow[u, "h_B"] \arrow[r, "\phi_{A',B',C'}(f)"'] & A' \Rightarrow C'
\end{tikzcd}$$

Interpretation of the simply-typed λ-calculus in a CCC

The simply-typed λ-calculus

The simple types A, B are constructed by the grammar:

$$A, B \quad ::= \quad \alpha \mid A \Rightarrow B.$$

A typing context Γ is a finite sequence

$$\Gamma = (x_1 : A_1, \ldots, x_n : A_n)$$

where each x_i is a variable and each A_i is a simple type.

A sequent is a triple

$$x_1 : A_1, \ldots, x_n : A_n \vdash P : B$$

where

$$x_1 : A_1, \ldots, x_n : A_n$$

is a typing context, P is a λ-term and B is a simple type.

The simply-typed λ-calculus

Variable	$\dfrac{}{x : A \vdash x : A}$
Abstraction	$\dfrac{\Gamma, x : A \vdash P : B}{\Gamma \vdash \lambda x.P : A \Rightarrow B}$
Application	$\dfrac{\Gamma \vdash P : A \Rightarrow B \qquad \Delta \vdash Q : A}{\Gamma, \Delta \vdash PQ : B}$
Weakening	$\dfrac{\Gamma \vdash P : B}{\Gamma, x : A \vdash P : B}$
Contraction	$\dfrac{\Gamma, x : A, y : A \vdash P : B}{\Gamma, z : A \vdash P[x, y \leftarrow z] : B}$
Exchange	$\dfrac{\Gamma, x : A, y : B, \Delta \vdash P : C}{\Gamma, y : B, x : A, \Delta \vdash P : C}$

Interpretation of the λ-calculus

Step 1. We suppose given a function

$$\xi \quad : \quad \alpha \quad \mapsto \quad \xi(\alpha)$$

which associates an object $\xi(\alpha)$ to every type variable α.

Step 2. Every type A is then interpreted as an object

$$[\![A]\!]$$

of the cartesian closed category by structural induction:

$$\begin{array}{rcl} [\![\alpha]\!] & = & \xi(\alpha) \\ [\![A \times B]\!] & = & [\![A]\!] \times [\![B]\!] \\ [\![A \Rightarrow B]\!] & = & [\![A]\!] \Rightarrow [\![B]\!] \end{array}$$

Interpretation of the λ-calculus

Step 3. Every sequent

$$x_1 : A_1 , \ldots , x_n : A_n \quad \vdash \quad t \, : \, B$$

is interpreted as a morphism

$$[\![t]\!] \quad : \quad [\![A_1]\!] \times \cdots \times [\![A_n]\!] \quad \longrightarrow \quad [\![B]\!]$$

by structural induction on the derivation tree which produced it.

The logical rules

- ▷ Variable: $[\![A]\!] \xrightarrow{id_{[\![A]\!]}} [\![A]\!]$
- ▷ Lambda:

$$A \times \Gamma \xrightarrow{f} B$$

becomes

$$\Gamma \xrightarrow{\phi_{A,\Gamma,B}(f)} A \Rightarrow B$$

- ▷ Application:

$$\Gamma \xrightarrow{f} A \quad \text{and} \quad \Delta \xrightarrow{g} A \Rightarrow B$$

become

$$\Gamma \times \Delta \xrightarrow{f \times g} A \times (A \Rightarrow B) \xrightarrow{eval_{A,B}} B$$

The structural rules

- Contraction:

$$A \times A \times \Gamma \xrightarrow{f} B$$

becomes

$$A \times \Gamma \xrightarrow{\delta_A \times \Gamma} A \times A \times \Gamma \xrightarrow{f} B$$

- Weakening:

$$\Gamma \xrightarrow{f} B$$

becomes

$$A \times \Gamma \xrightarrow{\varepsilon_A \times \Gamma} 1 \times \Gamma \xrightarrow{\sim} \Gamma \xrightarrow{f} B$$

- Permutation:

$$\Gamma \times A \times B \times \Delta \xrightarrow{f} B$$

becomes

$$\Gamma \times B \times A \times \Delta \xrightarrow{\Gamma \times \gamma_{A,B} \times \Delta} \Gamma \times A \times B \times \Delta \xrightarrow{f} B$$

Soundness theorem

Theorem.

In every cartesian closed category $\mathcal{C}$, the interpretation $[\![-]\!]$ is an invariant modulo β, η.

▷ If $\Gamma \vdash (\lambda x.M) : A \Rightarrow B$ and $\Delta \vdash N : A$, then

$$[\![\Gamma, \Delta \vdash (\lambda x.M)N : B]\!] \quad = \quad [\![\Gamma, \Delta \vdash M[x := N] : B]\!]$$

▷ If $\Gamma \vdash M : A \Rightarrow B$ then

$$[\![\Gamma \vdash (\lambda x.Mx) : A \Rightarrow B]\!] \quad = \quad [\![\Gamma \vdash M : A \Rightarrow B]\!]$$

Exercise. Establish the soundness theorem.