

# **Modèles des langages de programmation**

## **Domaines, catégories, jeux**

Programme de cette seconde séance:

Modèle ensembliste du lambda-calcul ;

Catégories cartésiennes fermées

# Synopsis

- 1 — the simply-typed  $\lambda$ -calculus,
- 2 — the set-theoretic model of the  $\lambda$ -calculus,
- 3 — categories and functors
- 4 — cartesian categories
- 5 — cartesian closed categories
- 6 — interprétation of the simply-typed  $\lambda$ -calculus in a ccc

# The simply-typed $\lambda$ -calculus

# The pure $\lambda$ -calculus

**Terms**       $M ::= x \mid MN \mid \lambda x.M$

The  $\beta$ -reduction:

$$(\lambda x.M)N \longrightarrow M[x := N]$$

The  $\eta$ -expansion:

$$M \longrightarrow \lambda x.(Mx)$$

Remark: every term is considered up to renaming  $\equiv_\alpha$  of the bound variables, typically:

$$\lambda x.\lambda y.x \equiv_\alpha \lambda z.\lambda y.z$$

# The simply-typed $\lambda$ -calculus

The simple types  $A, B$  are constructed by the grammar:

$$A, B ::= \alpha \mid A \Rightarrow B.$$

A **typing context**  $\Gamma$  is a finite sequence

$$\Gamma = (x_1 : A_1, \dots, x_n : A_n)$$

where each  $x_i$  is a variable and each  $A_i$  is a simple type.

A **sequent** is a triple

$$x_1 : A_1, \dots, x_n : A_n \vdash P : B$$

where

$$x_1 : A_1, \dots, x_n : A_n$$

is a typing context,  $P$  is a  $\lambda$ -term and  $B$  is a simple type.

# The simply-typed $\lambda$ -calculus

Variable

$$\frac{}{x : A \vdash x : A}$$

Abstraction

$$\frac{\Gamma, x : A \vdash P : B}{\Gamma \vdash \lambda x. P : A \Rightarrow B}$$

Application

$$\frac{\Gamma \vdash P : A \Rightarrow B \quad \Delta \vdash Q : A}{\Gamma, \Delta \vdash PQ : B}$$

Weakening

$$\frac{\Gamma \vdash P : B}{\Gamma, x : A \vdash P : B}$$

Contraction

$$\frac{\Gamma, x : A, y : A \vdash P : B}{\Gamma, z : A \vdash P[x, y \leftarrow z] : B}$$

Exchange

$$\frac{\Gamma, x : A, y : B, \Delta \vdash P : C}{\Gamma, y : B, x : A, \Delta \vdash P : C}$$

## Subject reduction

A  $\lambda$ -term  $P$  is **simply typed** when there exists a sequent

$$\Gamma \vdash P : A$$

which may be obtained by a derivation tree.

One establishes that the set of simply typed  $\lambda$ -terms is closed under  $\beta$ -réduction:

### **Subject Reduction:**

If  $\Gamma \vdash P : A$  and  $P \longrightarrow_{\beta} Q$ , then  $\Gamma \vdash Q : A$ .

# The set-theoretic interpretation of the $\lambda$ -calculus



## Interprétation ensembliste

To each atomic type  $\alpha$  is associated a set  $X_\alpha$

Then, one extends the interpretation to every type:

$$\llbracket \alpha \rrbracket = X_\alpha \quad \llbracket A \Rightarrow B \rrbracket = \llbracket B \rrbracket^{\llbracket A \rrbracket} = \mathbf{Set}(\llbracket A \rrbracket, \llbracket B \rrbracket)$$

A sequent

$$x_1 : A_1, \dots, x_n : A_n \vdash M : B$$

is interpreted as a function

$$\llbracket A_1 \rrbracket \times \dots \times \llbracket A_n \rrbracket \longrightarrow \llbracket B \rrbracket$$

# Soundness theorem

## Theorem.

The interpretation provides an invariant of  $\lambda$ -terms modulo  $\beta$  and  $\eta$ .

▷ if  $\Gamma \vdash (\lambda x.M) : A \Rightarrow B$  and  $\Delta \vdash N : A$ , then

$$\llbracket \Gamma, \Delta \vdash (\lambda x.M)N : B \rrbracket = \llbracket \Gamma, \Delta \vdash M[x := N] : B \rrbracket$$

▷ if  $\Gamma \vdash M : A \Rightarrow B$  then

$$\llbracket \Gamma \vdash (\lambda x.Mx) : A \Rightarrow B \rrbracket = \llbracket \Gamma \vdash M : A \Rightarrow B \rrbracket$$

# Categories and functors

# Categories

A category  $\mathcal{C}$  is given by

[0] a class of **objects**

[1] a set  $\mathbf{Hom}(A, B)$  of **morphisms**

$$f : A \longrightarrow B$$

for every pair of objects  $(A, B)$

[2] a **composition law**  $\circ : \mathbf{Hom}(B, C) \times \mathbf{Hom}(A, B) \longrightarrow \mathbf{Hom}(A, C)$

[2] an **identity** morphism

$$id_A : A \longrightarrow A$$

for every object  $A$ ,

# Categories

satisfying the following properties:

[3] the composition law  $\circ$  is associative:

$$\begin{aligned} \forall f \in \mathbf{Hom}(A, B) \\ \forall g \in \mathbf{Hom}(B, C) \\ \forall h \in \mathbf{Hom}(C, D) \end{aligned} \quad f \circ (g \circ h) = (f \circ g) \circ h$$

[3] the morphisms  $id$  are neutral elements

$$\forall f \in \mathbf{Hom}(A, B) \quad f \circ id_A = f = id_B \circ f$$

# Examples

1. the category **Set** of sets and functions
2. the category **Ord** of partial orders and monotone functions
3. the category **Dom** of domains and continuous functions
4. the category **Coh** of coherence spaces and linear maps
5. every partial order
6. every monoid

# Opposite category

The opposite of a category

$\mathcal{C}$

is the category noted

$\mathcal{C}^{op}$

whose morphisms

$$f : A \longrightarrow B$$

are defined as the morphisms

$$f : B \longrightarrow A$$

of the original category  $\mathcal{C}$ .

# Product category

The **product**

$$\mathcal{A} \times \mathcal{B}$$

of two categories

$$\mathcal{A} \quad \mathcal{B}$$

is the category

- ▷ whose objects are the pairs  $(A, B)$  of objects of  $\mathcal{A}$  and  $\mathcal{B}$ ,
- ▷ whose morphisms

$$(A, B) \longrightarrow (A', B')$$

are the pairs of morphisms

$$f : A \longrightarrow A' \quad g : B \longrightarrow B'$$



## Product category

with composition and identities defined as expected:

$$(A, B) \xrightarrow{id_{(A,B)}} (A, B) = (A, B) \xrightarrow{(id_A, id_B)} (A, B)$$

$$(A, B) \xrightarrow{(f,g)} (A', B') \xrightarrow{(f',g')} (A'', B'') = (A, B) \xrightarrow{(f' \circ f, g' \circ g)} (A'', B'')$$

# Functors

A **functor** between categories

$$F : \mathcal{C} \longrightarrow \mathcal{D}$$

is defined as the following data:

[0] an object  $FA$  of  $\mathcal{D}$  for every object  $A$  of  $\mathcal{C}$ ,

[1] a function

$$F_{A,B} : \mathbf{Hom}_{\mathcal{C}}(A,B) \longrightarrow \mathbf{Hom}_{\mathcal{D}}(FA,FB)$$

for every pair of objects  $(A,B)$  of the category  $\mathcal{C}$ .

# Functors

One requires moreover

[2] that  $F$  preserves composition

$$FA \xrightarrow{Ff} FB \xrightarrow{Fg} FC = FA \xrightarrow{F(g \circ f)} FC$$

[2] that  $F$  preserves the identities

$$FA \xrightarrow{Fid_A} FA = FA \xrightarrow{id_{FA}} FA$$

## Illustration [orders]

Every ordered set

$$(X, \leq)$$

defines a category

$$[X, \leq]$$

- ▷ whose objects are the elements of  $X$
- ▷ whose hom-sets are defined as

$$\mathbf{Hom}(x, y) = \begin{cases} \{*\} & \text{if } x \leq y \\ \emptyset & \text{otherwise} \end{cases}$$

In this category, there exists at most one map between two objects

## Illustration [orders]

**Exercise:** given two ordered sets

$$(X, \leq) \quad (Y, \leq)$$

a functor

$$F : [X, \leq] \longrightarrow [Y, \leq]$$

is the same thing as a monotonic function

$$F : (X, \leq) \longrightarrow (Y, \leq)$$

between the underlying ordered sets.

## Illustration [ *monoids* ]

A monoid  $(M, \cdot, e)$  is a set  $M$  equipped with a binary operation

$$\cdot : M \times M \longrightarrow M$$

and a neutral element

$$e : \{*\} \longrightarrow M$$

satisfying the two properties below:

Associativity law  $\forall x, y, z \in M, (x \cdot y) \cdot z = x \cdot (y \cdot z)$

Unit law  $\forall x \in M, x \cdot e = x = e \cdot x.$

## Illustration [ *monoids* ]

**Key observation:** there is a one-to-one relationship

$$(M, \cdot, e) \mapsto \Sigma(M, \cdot, e)$$

between

- ▷ monoids
- ▷ categories with one object \*

obtained by defining  $\Sigma(M, \cdot, e)$  as the category with unique hom-set

$$\Sigma(M, \cdot, e) (*, *) = M$$

and composition law and unit defined as

$$g \circ f = g \cdot f \qquad id_* = e$$

## Illustration [ *monoids* ]

**Key observation:** given two monoids

$$(M, \cdot, e)$$

$$(N, \bullet, u)$$

a functor

$$F : \Sigma(M, \cdot, e) \longrightarrow \Sigma(N, \bullet, u)$$

is the same thing as a homomorphism

$$f : (M, \cdot, e) \longrightarrow (N, \bullet, u)$$

between the underlying monoids.

Recall that a homomorphism is a function  $f$  such that

$$\forall x, y \in M, \quad f(x \cdot y) = f(x) \bullet f(y) \quad f(e) = u$$



# Transformations

A transformation

$$\theta : F \dot{\longrightarrow} G$$

between two functors

$$F, G : \mathcal{A} \longrightarrow \mathcal{B}$$

is a family of morphisms

$$(\theta_A : FA \longrightarrow GA)_{A \in \text{Obj}(\mathcal{A})}$$

of the category  $\mathcal{B}$  indexed by the objects of the category  $\mathcal{A}$ .

## Natural transformations

A transformation  $\theta : F \Rightarrow G : \mathcal{A} \longrightarrow \mathcal{B}$

is **natural** when the diagram

$$\begin{array}{ccc} FA & \xrightarrow{\theta_A} & GA \\ Ff \downarrow & & \downarrow Gf \\ FB & \xrightarrow{\theta_B} & GB \end{array}$$

commutes for every morphism  $f : A \longrightarrow B$ .

# Isomorphism

In a category  $\mathcal{C}$ , a morphism

$$f : A \longrightarrow B$$

is called an **isomorphism** when there exists a morphism

$$g : B \longrightarrow A$$

satisfying

$$g \circ f = id_A \quad \text{et} \quad f \circ g = id_B.$$

Exercise. Show that  $g \circ f : A \longrightarrow C$  is an isomorphism when  $f : A \longrightarrow B$  and  $g : B \longrightarrow C$  are isomorphisms.

Exercise. Show that every functor  $F : \mathcal{C} \longrightarrow \mathcal{D}$  transports an isomorphism of  $\mathcal{C}$  into an isomorphism of  $\mathcal{D}$ .

# Bifunctors

A **bifunctor**  $F$  from  $\mathcal{C}$  and  $\mathcal{D}$  to  $\mathcal{E}$  is given by:

▷ a functor  $F(A, -) : \mathcal{D} \longrightarrow \mathcal{E}$

for every object  $A$  of the category  $\mathcal{C}$

▷ a functor  $F(-, B) : \mathcal{C} \longrightarrow \mathcal{E}$

for every object  $B$  of the category  $\mathcal{D}$

such that the diagram

$$\begin{array}{ccc} F(A, B) & \xrightarrow{F(A, g)} & F(A, B') \\ F(f, B) \downarrow & & \downarrow F(f, B') \\ F(A', B) & \xrightarrow{F(A', g)} & F(A', B') \end{array}$$

commutes for all morphisms  $f : A \longrightarrow A'$  in  $\mathcal{C}$  and  $g : B \longrightarrow B'$  in  $\mathcal{D}$ .

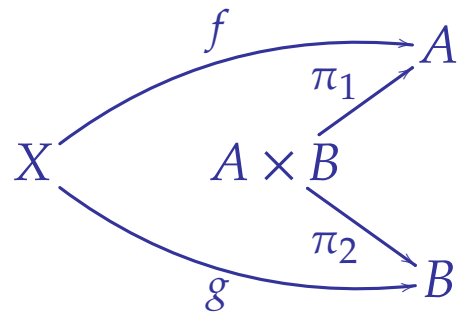
# Cartesian categories

# Products

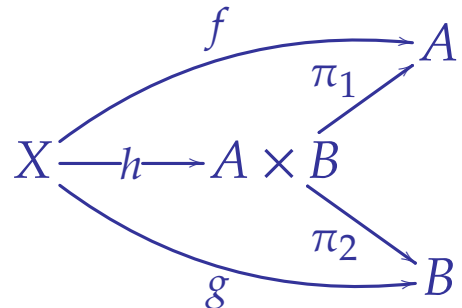
The **product** of two objects  $A$  and  $B$  in a category  $\mathcal{C}$  is an object  $A \times B$  equipped with two morphisms

$$\pi_1 : A \times B \longrightarrow A \qquad \pi_2 : A \times B \longrightarrow B$$

such that for every diagram



there exists a unique morphism  $h : X \longrightarrow A \times B$  making the diagram



commute.

## Illustrations

1. The cartesian product in the category **Set**,
2. The lub  $a \wedge b$  of two elements  $a$  and  $b$  in an ordered set  $(X, \leq)$ ,
3. The cartesian product in the category **Dom**,

## Terminal object

An object  $\mathbf{1}$  is **terminal** in a category  $\mathcal{C}$  when  $\mathbf{Hom}(A, \mathbf{1})$  is a singleton for all objects  $A$ .

One may consider  $\mathbf{1}$  as the nullary product in  $\mathcal{C}$ .

Example 1. the singleton  $\{*\}$  in the categories **Set** and **Dom**,

Example 2. the maximum of an ordered set  $(X, \leq)$



## Cartesian category

A **cartesian category** is a category  $\mathcal{C}$  equipped with a product

$$A \times B$$

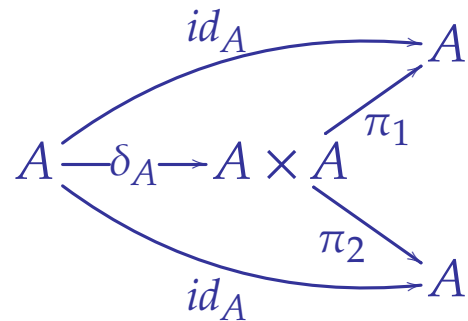
for all pairs  $A, B$  of objects, and of a terminal object

1

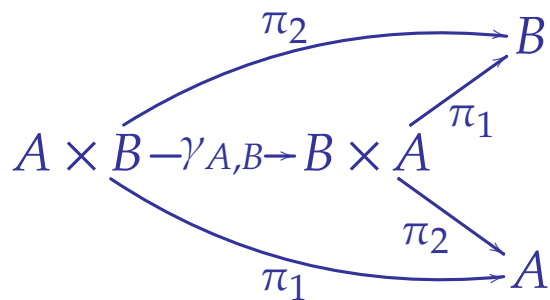
# Cartesian categories

In every cartesian category, one finds

- ▷ weakening maps  $\epsilon_A : A \longrightarrow \mathbf{1}$ ,
- ▷ diagonal maps  $\delta_A : A \longrightarrow A \times A$  obtained as



- ▷ symmetry maps  $\gamma_{A,B} : A \times B \longrightarrow B \times A$  obtained as



Exercise: Show that  $(- \times -)$  defines a bifunctor  $\mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ .

# **Cartesian closed categories**

## **First definition**

# Ccc

A cartesian closed category is a cartesian category

$$(\mathcal{C}, \times, 1)$$

together with the following data for all objects  $A$  and  $B$ :

- ▷ of an object  $A \Rightarrow B$
- ▷ of a morphism  $eval_{A,B} : A \times (A \Rightarrow B) \longrightarrow B$

such that for every object  $X$  and morphism

$$f : A \times X \longrightarrow B$$

there exists a unique morphism  $h : X \longrightarrow A \Rightarrow B$  making the diagram

$$\begin{array}{ccc} A \times (A \Rightarrow B) & \xrightarrow{eval_{A,B}} & B \\ \uparrow A \times h & \nearrow f & \\ A \times X & & \end{array}$$

commute.

# **Cartesian closed category**

## **Second definition**

# Adjunction

An **adjunction** is a triple consisting of two functors

$$L : \mathcal{A} \longrightarrow \mathcal{B} \quad R : \mathcal{B} \longrightarrow \mathcal{A}$$

and a family of bijections

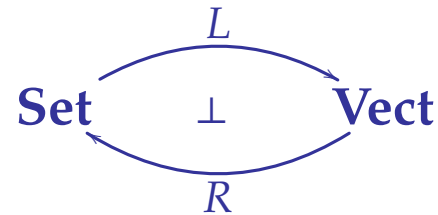
$$\phi_{A,B} : \mathbf{Hom}_{\mathcal{B}}(LA, B) \cong \mathbf{Hom}_{\mathcal{A}}(A, RB)$$

natural in  $A$  and  $B$ , for all pairs of objects  $A, B$  of  $\mathcal{A}$  and  $\mathcal{B}$ .

$$\frac{LA \xrightarrow{\mathcal{B}} B}{A \xrightarrow{\mathcal{A}} RB} \quad \phi_{A,B}$$

One writes  $L \dashv R$  and one says that  $L$  is **left adjoint** to  $R$ .

## Example: the free vector space



where

$\mathcal{A} = \mathbf{Set}$  : the category of sets and functions

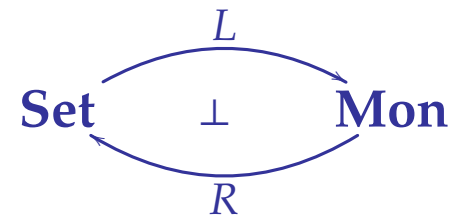
$\mathcal{B} = \mathbf{Vect}$  : the category of vector spaces on a field  $k$

$R$  : the « forgetful » functor  $V \mapsto U(V)$

$L$  : the « free vector space » functor  $X \mapsto kX$

$$kX := \left\{ \sum_{x \in X} \lambda_x x \mid \lambda_x \in k \text{ null almost everywhere.} \right\}$$

## Illustration: the free monoid



where

- $\mathcal{A} = \mathbf{Set}$  : the category of sets and functions
- $\mathcal{B} = \mathbf{Mon}$  : the category of monoids and homomorphisms,
  
- $R$  : the « forgetful » functor  $M \mapsto U(M)$ .
- $L$  : the « free monoid » functor  $A \mapsto A^*$ .

$$A := \coprod_{n \in \mathbb{N}} A^n$$



## What does natural bijection $\phi$ exactly mean?

By  $\phi$  natural, one means that the two families of sets

$$\mathbf{Hom}_{\mathcal{B}}(LA, B)$$

$$\mathbf{Hom}_{\mathcal{A}}(A, RB)$$

define functors

$$\mathbf{Hom}_{\mathcal{B}}(L-, -) : \mathcal{A}^{op} \times \mathcal{B} \longrightarrow \mathbf{Set}$$

$$\mathbf{Hom}_{\mathcal{A}}(-, R-) : \mathcal{A}^{op} \times \mathcal{B} \longrightarrow \mathbf{Set}$$

and that the family of bijections  $\phi$  defines a **natural transformation**

$$\phi : \mathbf{Hom}_{\mathcal{B}}(L-, -) \cong \mathbf{Hom}_{\mathcal{A}}(-, R-) : \mathcal{A}^{op} \times \mathcal{B} \longrightarrow \mathbf{Set}$$

between them.

# What does natural bijection $\phi$ exactly mean?

Natural in  $A$  and  $B$  thus means that every commutative diagram

$$\begin{array}{ccc} LA & \xrightarrow{g} & B \\ \uparrow Lh_A & & \downarrow h_B \\ LA' & \xrightarrow{f} & B' \end{array}$$

is transformed into a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\phi_{A,B}(g)} & RB \\ \uparrow h_A & & \downarrow Rh_B \\ A' & \xrightarrow{\phi_{A',B'}(f)} & RB' \end{array}$$

# Cartesian exponentiation

Consider an object  $A$  in a cartesian category  $(\mathcal{C}, \times, \mathbf{1})$ .

A **cartesian exponentiation** of  $A$  is a pair consisting of a functor

$$(A \Rightarrow -) : \mathcal{C} \longrightarrow \mathcal{C}$$

and of a family of bijections

$$\phi_{A,B,C} : \mathbf{Hom}(A \times B, C) \longrightarrow \mathbf{Hom}(B, A \Rightarrow C)$$

natural in  $B$  and  $C$ .

In other words, it is an **adjunction** between the functors

$$A \times - \quad \dashv \quad A \Rightarrow -$$

## What natural bijection means in that case

Naturality in  $B$  and  $C$  means that the family of bijections

$$\phi_{A,B,C} : \mathbf{Hom}(A \times B, C) \longrightarrow \mathbf{Hom}(B, A \Rightarrow C)$$

transforms every commutative diagram

$$\begin{array}{ccc}
 A \times B & \xrightarrow{g} & C \\
 \uparrow A \times h_B & & \downarrow h_C \\
 A \times B' & \xrightarrow{f} & C'
 \end{array}$$

into a commutative diagram

$$\begin{array}{ccc}
 B & \xrightarrow{\phi_{A,B,C}(g)} & A \Rightarrow C \\
 \uparrow h_B & & \downarrow A \Rightarrow h_C \\
 B' & \xrightarrow{\phi_{A,B',C'}(f)} & A \Rightarrow C'
 \end{array}$$

# Cartesian closed category

## Definition.

A **cartesian closed category** (ccc) is a cartesian category

$$(\mathcal{C}, \times, \mathbf{1})$$

equipped with a cartesian exponentiation

$$\frac{A \times B \longrightarrow C}{B \longrightarrow A \Rightarrow C} \quad \phi_{A,B,C}$$

for every object  $A$  of the category.

# Parameter theorem

We have seen that the cartesian product defines a bifoncteur

$$A, B \mapsto A \times B : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$$

in every cartesian category. In the same way,

Parameter theorem (MacLane)

The family of cartesian exponentiations

$$(A \Rightarrow -)_A : \mathcal{C} \longrightarrow \mathcal{C}$$

defines a unique bifunctor

$$A, B \mapsto A \Rightarrow B : \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathcal{C}$$

such that the bijections  $\phi_{A,B,C}$  are natural in  $A, B, C$ .

# Parameter theorem

Here, natural in  $A, B, C$  means that the family of bijections

$$(\phi_{A,B,C})_{A,B,C} : \mathbf{Hom}(A \times B, C) \longrightarrow \mathbf{Hom}(B, A \Rightarrow C)$$

transforms every commutative diagram

$$\begin{array}{ccc}
 A \times B & \xrightarrow{g} & C \\
 \uparrow h_A \times h_B & & \downarrow h_C \\
 A' \times B' & \xrightarrow{f} & C'
 \end{array}$$

in a commutative diagram:

$$\begin{array}{ccc}
 B & \xrightarrow{\phi_{A,B,C}(g)} & A \Rightarrow C \\
 \uparrow h_B & & \downarrow h_{A \Rightarrow C} \\
 B' & \xrightarrow{\phi_{A',B',C'}(f)} & A' \Rightarrow C'
 \end{array}$$

# Interpretation of the simply-typed $\lambda$ -calculus in a CCC



# The simply-typed $\lambda$ -calculus

The simple types  $A, B$  are constructed by the grammar:

$$A, B ::= \alpha \mid A \Rightarrow B.$$

A **typing context**  $\Gamma$  is a finite sequence

$$\Gamma = (x_1 : A_1, \dots, x_n : A_n)$$

where each  $x_i$  is a variable and each  $A_i$  is a simple type.

A **sequent** is a triple

$$x_1 : A_1, \dots, x_n : A_n \vdash P : B$$

where

$$x_1 : A_1, \dots, x_n : A_n$$

is a typing context,  $P$  is a  $\lambda$ -term and  $B$  is a simple type.

# The simply-typed $\lambda$ -calculus

Variable	$\frac{}{x : A \vdash x : A}$
Abstraction	$\frac{\Gamma, x : A \vdash P : B}{\Gamma \vdash \lambda x.P : A \Rightarrow B}$
Application	$\frac{\Gamma \vdash P : A \Rightarrow B \quad \Delta \vdash Q : A}{\Gamma, \Delta \vdash PQ : B}$
Weakening	$\frac{\Gamma \vdash P : B}{\Gamma, x : A \vdash P : B}$
Contraction	$\frac{\Gamma, x : A, y : A \vdash P : B}{\Gamma, z : A \vdash P[x, y \leftarrow z] : B}$
Exchange	$\frac{\Gamma, x : A, y : B, \Delta \vdash P : C}{\Gamma, y : B, x : A, \Delta \vdash P : C}$

# Interpretation of the $\lambda$ -calculus

**Step 1.** We suppose given a function

$$\xi : \alpha \mapsto \xi(\alpha)$$

which associates an object  $\xi(\alpha)$  to every type variable  $\alpha$ .

**Step 2.** Every type  $A$  is then interpreted as an object

$$\llbracket A \rrbracket$$

of the cartesian closed category by structural induction:

$$\begin{aligned}\llbracket \alpha \rrbracket &= \xi(\alpha) \\ \llbracket A \times B \rrbracket &= \llbracket A \rrbracket \times \llbracket B \rrbracket \\ \llbracket A \Rightarrow B \rrbracket &= \llbracket A \rrbracket \Rightarrow \llbracket B \rrbracket\end{aligned}$$

## Interpretation of the $\lambda$ -calculus

**Step 3.** Every sequent

$$x_1 : A_1, \dots, x_n : A_n \vdash t : B$$

is interpreted as a morphism

$$\llbracket t \rrbracket : \llbracket A_1 \rrbracket \times \dots \times \llbracket A_n \rrbracket \longrightarrow \llbracket B \rrbracket$$

by structural induction on the derivation tree which produced it.

# The logical rules

▷ Variable: 
$$\llbracket A \rrbracket \xrightarrow{id \llbracket A \rrbracket} \llbracket A \rrbracket$$

▷ Lambda:

$$A \times \Gamma \xrightarrow{f} B$$

becomes

$$\Gamma \xrightarrow{\phi_{A,\Gamma,B}(f)} A \Rightarrow B$$

▷ Application:

$$\Gamma \xrightarrow{f} A \quad \text{and} \quad \Delta \xrightarrow{g} A \Rightarrow B$$

become

$$\Gamma \times \Delta \xrightarrow{f \times g} A \times (A \Rightarrow B) \xrightarrow{eval_{A,B}} B$$

# The structural rules

▷ Contraction:

$$A \times A \times \Gamma \xrightarrow{f} B$$

becomes

$$A \times \Gamma \xrightarrow{\delta_{A \times \Gamma}} A \times A \times \Gamma \xrightarrow{f} B$$

▷ Weakening:

$$\Gamma \xrightarrow{f} B$$

becomes

$$A \times \Gamma \xrightarrow{\varepsilon_{A \times \Gamma}} 1 \times \Gamma \xrightarrow{\sim} \Gamma \xrightarrow{f} B$$

▷ Permutation:

$$\Gamma \times A \times B \times \Delta \xrightarrow{f} B$$

becomes

$$\Gamma \times B \times A \times \Delta \xrightarrow{\Gamma \times \gamma_{A, B} \times \Delta} \Gamma \times A \times B \times \Delta \xrightarrow{f} B$$

# Soundness theorem

## Theorem.

In every cartesian closed category  $\mathcal{C}$ , the interpretation  $\llbracket - \rrbracket$  is an invariant modulo  $\beta, \eta$ .

▷ If  $\Gamma \vdash (\lambda x.M) : A \Rightarrow B$  and  $\Delta \vdash N : A$ , then

$$\llbracket \Gamma, \Delta \vdash (\lambda x.M)N : B \rrbracket = \llbracket \Gamma, \Delta \vdash M[x := N] : B \rrbracket$$

▷ If  $\Gamma \vdash M : A \Rightarrow B$  then

$$\llbracket \Gamma \vdash (\lambda x.Mx) : A \Rightarrow B \rrbracket = \llbracket \Gamma \vdash M : A \Rightarrow B \rrbracket$$

**Exercise.** Establish the soundness theorem.