Travaux Dirigés n° 2

Modèles des langages de programmation
Master Parisien de Recherche en Informatique

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Problem. We saw in the course that every coherence space A induces a partial order $(D_A, \leq_A)$ whose elements are the cliques of the coherence space A, ordered by inclusion.

a. Show that the partial order $(D_A, \leq_A)$ defines a domain, that is: (1) D_A contains a least element denoted $\bot_A$, (2) every increasing sequence of elements $(x_n)_{n \in \mathbb{N}}$ in the partial order D_A has a least upper bound

$$x_\infty \quad = \quad \bigvee_{n \in \mathbb{N}} \quad x_n .$$

b. Every clique

$$f \quad : \quad A \& S \quad \multimap \quad B \& S$$

induces a family of unary (sorted) operations

$$\begin{array}{cccc} b & s & t & b \\ | & | & | & | \\ f & f & f & f \\ | & | & | & | \\ a & a & s & s \end{array}$$

one for each pair of elements $a \in |A|$, $b \in |B|$, and $s, t \in |S|$ such that

$$a[f]b \qquad a[f]s \qquad s[f]t \qquad s[f]b$$

respectively. Let **trees** (f) denote the set of linear trees of the form:

$$\begin{array}{c} b \\ | \\ f \\ | \\ s_n \\ \vdots \\ s_1 \\ | \\ f \\ | \\ a \end{array}$$

containing at least one node f. Define the relation

$$\mathbf{trace}\,(f) \quad : \quad A \quad \longrightarrow \quad B$$

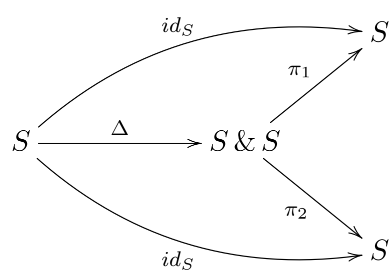
as the set of elements $a \multimap b$ such that there exists a linear tree in $\mathbf{trees}\,(f)$ with input sort a and output sort b. Show that the relation $\mathbf{trace}\,(f)$ defines a clique of the coherence space $A \multimap B$.

Remark: this construction defines a « trace operator » on the category of coherence spaces equipped with the cartesian product.

c. Describe explicitly the diagonal clique

$$\Delta \quad : \quad S \quad \multimap \quad S \,\&\, S$$

characterized as the unique morphism making the diagram



commute in the category of coherence spaces.

d. Every clique

$$f \quad : \quad A \,\&\, S \quad \multimap \quad S$$

induces a clique

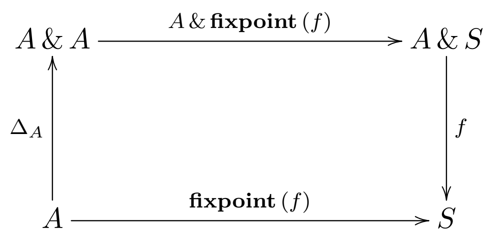
$$\mathbf{fixpoint}\,(f) = \mathbf{trace}\,(\Delta \circ f) \quad : \quad A \quad \multimap \quad S$$

defined as the trace operator applied to the composite

$$A \,\&\, S \xrightarrow{\ f\ } S \xrightarrow{\ \Delta\ } S \,\&\, S.$$

Show that $\mathbf{fixpoint}\,(f)$ coincides with the set of all elements $a \multimap s$ such that there exists a linear tree in $\mathbf{trees}\,(f)$ with input sort a and output sort s.

e. Show that the clique **fixpoint** (f) makes the diagram



commute in the category of coherence spaces.

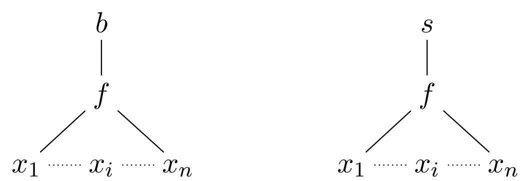
f. At this stage, we are ready to lift to stable functions what we have just achieved for linear functions. We have seen during the course that every stable function

$$f \quad : \quad D_A \times D_S \quad \longrightarrow \quad D_B \times D_S$$

may be seen as a clique

$$f \quad : \quad !\,(\,A \,\&\, S\,) \quad \multimap \quad B \,\&\, S$$

This clique induces in turn an n-ary (sorted) operation



whenever

$$\{x_1, \cdots, x_n\} \quad [f] \quad b \qquad\qquad \{x_1, \cdots, x_n\} \quad [f] \quad s$$

where each x_i is an element of $|A \,\&\, S|$, that is, either an element of $|A|$ or an element of $|S|$. Note that the descendents of a node f are unordered. An element of **trees** (f) is defined as a tree constructed with these n-ary operations, and containing at least a node f. The relation

$$\textbf{trace}\,(f) \quad : \quad !A \quad \longrightarrow \quad B$$

is then defined as the set of elements

$$\{a_1, \ldots, a_n\} \quad \multimap \quad b$$

such that there exists a tree in **trees** (f) with output sort b and with $\{a_1, \cdots, a_n\}$ as set of input sorts. Show that **trace** (f) defines a clique of the coherence space $!A \multimap B$. [Hint: proceed as in question b.]

f/g. Every clique

$$f \quad : \quad !\,(\,A \,\&\, S\,) \quad \multimap \quad S$$

induces a clique

$$\mathbf{fixpoint}\,(f) = \mathbf{trace}\,(\Delta \circ f) \quad : \quad !\,A \quad \multimap \quad S$$

defined as the trace operator applied to the composite

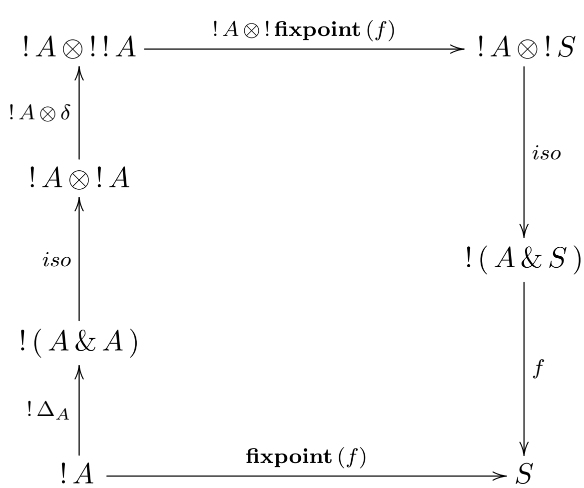
$$!\,(\,A \,\&\, S\,) \xrightarrow{\quad f \quad} S \xrightarrow{\quad \Delta \quad} S \,\&\, S.$$

Show that $\mathbf{fixpoint}\,(f)$ coincides with the set of all elements

$$\{a_1, \ldots, a_n\} \multimap s$$

such that there exists a tree in $\mathbf{trees}\,(f)$ with input sorts $\{a_1, \ldots, a_n\}$ and output sort s, where each a_i is an element of the web $|A|$.

g/h. Show that the clique $\mathbf{fixpoint}\,(f)$ is a parametric fixpoint of f in the sense that the diagram



commutes in the category of coherence spaces.

h/i. Explain how to compute the factorial function

$$\mathbf{factorial} \quad : \quad \mathbb{N} \quad \multimap \quad \mathbb{N}$$

as a fixpoint of the clique

$$\mathbf{fact} \quad : \quad !\,(\mathbb{N} \quad \multimap \quad \mathbb{N}) \quad \multimap \quad \mathbb{N} \quad \multimap \quad \mathbb{N}$$

defined as the set of elements

$$\boxed{\begin{array}{ccccc} !\,(\,\mathbb{N} \;\multimap\; \mathbb{N}\,) & \multimap & \mathbb{N} & \multimap & \mathbb{N} \\ \{\; p \;\mapsto\; q \;\} & \multimap & p+1 & \mapsto & (p+1)\times q \\ \emptyset & \multimap & 1 & \mapsto & 1 \end{array}}$$

where $a \mapsto b$ is an alternative notation for $a \multimap b$. Explain in particular why the tree

$$\begin{array}{c} 5 \;\multimap\; (\,5 \times 4 \times 3 \times 2 \times 1\,) \\ | \\ \mathbf{fact} \\ | \\ 4 \multimap\; (\,4 \times 3 \times 2 \times 1\,) \\ | \\ \mathbf{fact} \\ | \\ 3 \multimap\; (\,3 \times 2 \times 1\,) \\ | \\ \mathbf{fact} \\ | \\ 2 \multimap\; (\,2 \times 1\,) \\ | \\ \mathbf{fact} \\ | \\ 1 \multimap 1 \\ | \\ \mathbf{fact} \end{array}$$

establishes that

$$5 \;\multimap\; (\,5 \times 4 \times 3 \times 2 \times 1\,)$$

is an element of the clique

$$\mathbf{factorial} \quad : \quad \mathbb{N} \quad \multimap \quad \mathbb{N}.$$

i/j. We have seen in the course how to see a pair of morphisms (=cliques) in the category of coherence spaces

$$f \quad : \quad !\,A \quad \longrightarrow \quad B \qquad\qquad g \quad : \quad !\,B \quad \longrightarrow \quad C$$

as a pair of morphisms

$$f \quad : \quad A \quad \xrightarrow{k} \quad B \qquad\qquad g \quad : \quad B \quad \xrightarrow{k} \quad C$$

in the Kleisli category associated to the comonad !. Recall that the composite $g \bullet f$ of the two morphisms in the Kleisli category is defined as follows:

$$!\,A \xrightarrow{\quad \delta_A \quad} !\,!\,A \xrightarrow{\quad !f \quad} !B \xrightarrow{\quad g \quad} C$$

Express the set **trees** $(g \bullet f)$ directly from the sets **trees** (g) and **trees** (f).

j/k. Explain (formally or informally) why the stable function

$$D_A \quad \longrightarrow \quad D_S$$

associated to the clique

$$\textbf{fixpoint}\,(f) \quad : \quad !\,A \quad \multimap \quad S$$

defines the least fixpoint of the stable function

$$D_A \times D_S \quad \longrightarrow \quad D_S$$

associated to the clique

$$f \quad : \quad !\,(\,A \,\&\, S\,) \quad \multimap \quad S.$$